An Exponential Formula for Polynomial Vector Fields *

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Abstract: The well known matrix exponentiation for linear first order systems of ODE is generalized to polynomial vector fields. The substitution process of our Exponential Formula is closely related to the usual iteration of polynomial mappings.

The n-dimensional linear autonomous system of differential equations $\dot{x} = Mx,\ x \in \mathbb{R}^n,\ M \in \mathbb{R}^{n\times n}$ has the flow $\varphi(t) = e^{Mt} := \sum_{i=0}^{\infty} \frac{M^i}{i!} t^i$ defined for all $t \in \mathbb{R}$, where M^i is the i-th power of the coefficient matrix. Application to an initial value $x_0 \in \mathbb{R}^n$ gives the global solution $\varphi(t, x_0) = e^{Mt}x_0$ of the IVP $\dot{x} = Mx,\ x(0) = x_0$. The phase portrait of such a linear system is completely understood, when one knows it in some neighborhood of the origin – in fact this is true for all homogeneous systems. Furthermore in the linear case the global flow is already determined by the infinitesimal, i.e. algebraic, structure of the system and explicit solutions can be calculated using the Jordan normal form of M. All this is very well known (cf. [HS]).

In this paper we describe a simple straightforward generalization of the matrix exponentiation, which gives the local flow for polynomial vector fields $\dot{x} = p(x)$, $x \in \mathbb{R}^n$, $p \in (\mathbb{R}[x])^n$, p of degree $m \geq 2$ in the form $\exp(t\mu) = \sum_{i=0}^{\infty}(t^i/i!)\mu^i$. In this formula μ is a multilinear mapping corresponding to the homogenization of p, and the μ^i are sums of suitably composed multilinear mappings, which generalize the matrix powers M^i. The structure of this composition can be represented in a natural way by graphs, namely by forests of certain rooted trees (see [W1]). Application of $\exp(t\mu)$ to an initial value $x_0 \in \mathbb{R}^n$ then gives the analytic power series solution of the (homogenized form of the) "polynomial IVP" $\dot{x} = p(x),\ x(0) = x_0$. We prove these facts in Section 1 using a fundamental recursion formula for the μ^i; moreover this section includes a proof of the 1-parameter-group property of $\exp(t\mu)$ and remarks on the "history" of

*proposed running head: "An Exponential Formula" the Exponential Formula and on homogeneous polynomial vector fields. Two further recursion formulas for the coefficients of $\exp(t\mu)x_0$, the Simplex and the Convolution Formula, are discussed in Section 2; the latter seems to be the most simple and effective instrument for the practical computation of these coefficients.

In general it is easy to see, that there exist recursion formulas for the solutions $\varphi(t, x_0) \equiv \sum_{i=0}^{\infty} c_i/i!\, t^i$ of polynomial IVP's, which are of the form $c_{n+1} = \Phi_n(c_n, \ldots, c_0)$ with a certain sequence of mappings $\Phi \equiv (\Phi_0, \Phi_1, \Phi_2, \ldots)$ depending on the polynomial vector field; just substitute the above power series $\varphi(t, x_0)$ for x in $\dot{x} = p(x)$, differentiate n times and evaluate at $t = 0$. But of course one would like to have general formulas such as in matrix exponentiation, which are applicable to polynomial vector fields of all degrees and (finite) dimensions, and which give a clear structural understanding of what 'really' happens in building up a sequence Φ. At present we know several such conceptual formulas:

1. The simplest version is to use in dimension $n = 1$ *Faà di Bruno's Formula* for the higher derivatives of a composition of functions (cf. [Co]). The result is rather unenlightening due to the appearance of Bell polynomials; in higher dimensions it is possible in principle to find a generalization of Faà di Bruno's Formula, but the situation is of course even more complicated.

2. *Lie series* (cf. [G]) give a very clean approach for analytic (finite dimensional) vector fields, but it lacks concreteness in the polynomial case due to the use of powers of the linear partial differential operator $p_1(x)\partial_1 + \ldots + p_n(x)\partial_n$, where $p_1, \ldots, p_n$ are the components of $p(x)$.

3. The straightforward generalization of matrix exponentiation introduced in this paper realizes the sequence Φ as a highly symmetric and simply structured substitutional process. It is possible to show the equivalence of the *Exponential Formula* and the Lie series in the case of polynomial vector fields with the help of certain rooted trees; this will be one of the main results contained in [W1]. The *Simplex Formula* in Section 2 can be considered as a variant of the Exponential Formula, which describes the exact contribution of different additive parts of the polynomial vector fields to the solution coefficients.

4. The *Convolution Formula* in Section 2 describes the recursion $c_{n+1} = \Phi_n(c_n, \ldots, c_0)$ as a simply structured convolution with 'structure constants' given by the polynomial vector field. In fact, as we will show in Section 3.3, Faà di Bruno's Formula in dimension $n = 1$ can be seen as a 'desymmetrisation' of the Convolution Formula.

What are the possible uses of such formulas? In contrast to the complete information in the linear case one encounters great difficulties in understanding the dynamics of even the most simple polynomial vector fields:

In dimensions $n \geq 3$ one has chaotic systems like the extensively studied Lorenz-system ($m = 2,\ n = 3$) (cf. [GH]), which is an example for a polynomial approximation of a system of partial differential equations. See [S] for other examples arising from hydrodynamics, plasma physics, chemical reactors and for the genuinely quadratic Lotka-Volterra system, which models an ecosystem of n species.

In two dimensions, where no chaos is possible, Hilbert's 16th problem (dynamical part) asks for the maximum possible number of *limit cycles*, i.e. isolated periodic orbits, for given polynomial degree. The answer to this problem is largely open since its proposal in 1900:

1. The finiteness of the number of limit cycles for every single plane polynomial vector field has been proved rather recently by J. Ecalle and Y.S. Il'yashenko independantely (see [I1,I2,E] and the papers in [Sch]), but it is not known whether there is a uniform bound for given polynomial degree. The methods used in these proofs are very general and valid even for analytic vector fields. Therefore they disregard largely the concrete polynomial information.

2. Although more than 2000 papers exist on *quadratic* plane vector fields, there is not yet a complete classification. So far at most four limit cycles for such vector fields have been observed, but the possibility of arbitrary large numbers of limit cycles has not been excluded (see the surveys [C, CT, Y1, L], the books [Y2, Z], and again [Sch]). Most of the methods used are very special with lots of case divisions; bifurcations are frequently used too – but limit cycles are dynamically stable!

Clearly the situation demands new analytical tools of truely global nature, which avoid the two extremes of too much generality and too much specialization. Although the power series solutions to polynomials vector fields are in general only locally valid, it is our conviction, that the investigation of the sequence of solution coefficients *as a whole* will be – by analyticity – the key ingredient in the understanding of the global dynamics of finite dimensional polynomial vector fields.

In Section 3 we show by a "coalgebraic construction", that the substitution process in the Exponential Formula is related to the group $(\mathbb{R}, +)$ of real numbers under addition in the same way, as the usual iteration of polynomial mappings is related to the group $(\mathbb{R}^*, \cdot)$ of nonzero real numbers under multiplication. In fact from the viewpoint of coalgebras and multilinear algebra the constructions for the Exponential Formula and the iterates of polynomial mappings are the two most simple instances of a general procedure of "iteration", which can be carried out for any affine algebraic group or coassociative comultiplication.

The presentation in this paper is as simple and direct as possible. For example we use only real coefficients, although everything is valid for arbitrary fields of characteristic zero and a major portion even over arbitrary commutative rings with unit.

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1 The Exponential Formula

1.1 The Definition

The matrix exponentiation $\varphi(t) = e^{Mt} := \sum_{i=0}^{\infty} \frac{M^i}{i!} t^i$ for finite dimensional (autonomous) vector fields

$$\dot{x} = Mx, \;\; x = (x_1, \ldots, x_n)^T, \;\; M \in I\!R^{n \times n} \tag{1}$$

will be generalized in this section to finite dimensional polynomial vector fields

$$\dot{x} = p(x), \;\; x = (x_1, \ldots, x_n)^T, \;\; p \in (I\!R[x])^n, \text{ deg } p = m. \tag{2}$$

We write (2) componentwise as:

$$\dot{x}_i = p_i(x_1, \ldots, x_n), \;\; p_i \in I\!R[x], 1 \leq i \leq n, \;\; \max \text{ deg } p_i = m \in I\!N. \tag{3}$$

Applying the technique of homogenization or projectivisation from algebraic geometry, the n-dimensional system becomes an $(n+1)$-dimensional system by introduction of the homogenizing variable z:

$$\dot{x}_i = z^m p_i(\frac{x_1}{z}, \ldots, \frac{x_n}{z}) \;\; (i = 1, \ldots, n), \quad \dot{z} = 0 \; .$$

The original system is easily restored by the substitution $z = 1$. Subsequently we investigate therefore only n-dimensional polynomial vector fields homogeneous of degree m :

$$\dot{x}_i = D_i(x), \;\; i = 1, \ldots, n, \;\; D_i \in I\!R[x] \text{ homogeneous of degree m} \tag{4}$$

We choose once for all the canonical orthonormal base $E = \{e_1, \ldots, e_n\}$ of $I\!R^n$ with $e_i = (\delta_{ij})_{j=1,\ldots,n}$. The matrix $M = ((m_{ij}))_{i,j=1,\ldots,n}$ of the linear system (1) is then simply the representation of an $I\!R$-linear endomorphism $\mu : I\!R^n \to I\!R^n$ with respect to the chosen basis: $\mu(e_k) := \sum_{i=1}^{n} m_{ik} e_i$ on E, i.e. $\mu(e_k)$ is the k-th column vector of M resp. the coefficient vector of x_k in (1). M^i represents the i-fold composition of μ.

We generalize now the construction to the polynomial case $m \geq 2$. Let $\bigoplus^m I\!R^n$ the m-fold direct sum of $I\!R^n$ and $E^m = \{(e_{k_1}, \ldots, e_{k_m}) | k_1, \ldots, k_m \in \{1, \ldots, n\}\}$ its basis. Furthermore let

$$D_i(x_1, \ldots, x_n) := \sum_{k_1,\ldots,k_m=1}^{n} a^i_{k_1 \ldots k_m} x_{k_1} \cdot \ldots \cdot x_{k_m}, \; (i = 1, \ldots, n), \tag{5}$$

then we define μ to be the m-fold multilinear mapping over $I\!R$:

$$\mu : \bigoplus^m I\!R^n \to I\!R^n, \quad \mu(e_{k_1}, \ldots, e_{k_m}) := \sum_{i=1}^{n} a^i_{k_1 \ldots k_m} e_i, \tag{6}$$

i.e. the image of a base vector $(e_{k_1}, \ldots, e_{k_m}) \in E^m$ is the coefficient vector of $x_{k_1} \cdot \ldots \cdot x_{k_m}$ in (4).

By the the universal property of the tensorproduct μ can be represented equivalently as $I\!R$-linear mapping

$$\mu : T_m I\!R^n \equiv \bigotimes^m I\!R^n \to I\!R^n, \quad \mu(e_{k_1} \otimes \ldots \otimes e_{k_m}) := \sum_{i=1}^n a^i_{k_1 \ldots k_m} e_i, \tag{7}$$

which is the definition used subsequently.

μ is called *commutative*, iff

$$\mu(e_{k_1} \otimes \ldots \otimes e_{k_m}) = \mu(e_{k_{\pi(1)}} \otimes \ldots \otimes e_{k_{\pi(m)}})$$

for all permutations π of the numbers $1, \ldots, m$. In general $\mu \in L(T_m I\!R^n, I\!R^n)$ is noncommutative, but commutativety can be easily achieved over $I\!R$ by suitably rewriting the D_i:

Example 1.1 ($m = n = 2$) μ for $\dot{x} = 2xy + y^2$, $\dot{y} = 3x^2 + y^2$ defined by $\mu(e_1 \otimes e_1) = 3e_2$, $\mu(e_2 \otimes e_2) = e_1 + e_2$, $\mu(e_1 \otimes e_2) = 2e_1 \neq 0 = \mu(e_2 \otimes e_1)$. So μ is noncommutative. But writing the same differential equations as $\dot{x} = xy + yx + y^2$, $\dot{y} = 3x^2 + y^2$ we get a commutative μ: $\mu(e_1 \otimes e_2) = e_1 = \mu(e_2 \otimes e_1)$. □

Commutativety of μ will be used in this paper only in Section 2.1 .

So far we have established a bijection between the n-dimensional m-homogenous vector fields in *non*commuting variables $x_1, \ldots, x_n$ and the space of $I\!R$-linear mappings $L(T_m I\!R^n, I\!R^n)$ with respect to the chosen basis. It remains to generalize the i-fold composition of μ from the linear case. μ is for $m \geq 2$ a mapping of degree $-(m-1)$ with respect to the tensorproduct and therefore cannot be composed directly with itself.

First we define for any given μ a family of $I\!R$-linear mappings $d_{\mu,p}$:

$$d_{\mu,p} : T_p I\!R^n \to T_{p-m+1} I\!R^n \tag{8}$$

$$d_{\mu,p} := \begin{cases} 0 & , \text{if } p < m \\ \sum_{\nu=1}^{p-m+1} \otimes^{\nu-1} id \otimes \mu \otimes \otimes^{p-m-\nu+1} id & , \text{if } p \geq m, \end{cases}$$

i.e. μ is for $p \geq m$ shifted componentwise from the left to the right on $T_p I\!R^n$ leaving the other factors unchanged. From the definition it is immediately clear that: $d_{\mu,p}$ is $I\!R$-linear on $T_p I\!R^n$, $d_{\mu,m} = \mu$ and $d_{\mu,p}$ is $I\!R$-linear in μ , i.e.

$$d_{\lambda\mu,p} = \lambda d_{\mu,p}, \; \forall \lambda \in I\!R \text{ and} \tag{9}$$

$$d_{\mu+\mu',p} = d_{\mu,p} + d_{\mu',p}, \quad \forall \mu, \mu' \in L(T_m I\!R^n, I\!R^n).$$

Because p is used only with certain values depending on m, we introduce the abbreviation :

$$[k] := k(m-1) + 1 \text{ for all } k \in I\!N_0 \text{ and fixed } m \in I\!N. \tag{10}$$

Moreover we define

$$(\mu)^i := \begin{cases} id & , \text{if } i = 0 \\ d_{\mu,[1]} \circ \ldots \circ d_{\mu,[i]} & , \text{if } i > 0, \end{cases} \tag{11}$$

$$\delta_p : I\!R^n \to T_p I\!R^n, v \mapsto \otimes^p v \ \text{ if } p > 0 \tag{12}$$

– the latter is the p-fold diagonal mapping – and finally

$$\mu^i := \begin{cases} id & , \text{if } i = 0 \\ (\mu)^i \circ \delta_{[i]} & , \text{if } i > 0. \end{cases} \tag{13}$$

Directly from the definition one sees: $(\mu)^i$ is a $I\!R$-linear mapping from $T_{[i]}I\!R^n$ to $I\!R^n$, but δ_p is linear only in the case $p = 1$. Note that $\mu = (\mu)^1 : T_m I\!R^n \to I\!R^n$ and $\mu^1 : I\!R^n \to I\!R^n$ are different! μ^i for $i > 0$ can be represented as follows:

$$\mu^i : I\!R^n \overset{\delta_{[i]}}{\to} T_{[i]}I\!R^n \overset{d_{\mu,[i]}}{\to} T_{[i-1]}I\!R^n \overset{d_{\mu,[i-1]}}{\to} \ldots \overset{d_{\mu,[2]}}{\to} T_{[1]}I\!R^n = T_m I\!R^n \overset{d_{\mu,[1]}=\mu}{\to} I\!R^n.$$

Definition of the Exponential Formula: *For $m, n \in I\!N$ and μ corresponding to the vector field (4), we define purely formal with the notations (8) and (10)-(13):*

$$\exp(\mu) := \sum_{i=0}^{\infty} \frac{\mu^i}{i!}$$

Remark 1.1: Since $\{t\mu\}^i = (t\mu)^i \circ \delta_{[i]} \overset{(9)}{=} t^i(\mu)^i \circ \delta_{[i]} = t^i \mu^i$ for all $t \in I\!R$, we get:

$$\exp(t\mu) := \sum_{i=0}^{\infty} \frac{t^i}{i!} \mu^i \quad . \tag{14}$$

For all initial values $x_0 \in I\!R^n$ all n power series in the components of $\exp(t\mu)(x_0)$ have a nonvanishing radius of convergence (see Section 1.5).
Remark 1.2: For $m = 1$ one has as a special case of (14) (identifying μ and M): $\exp(t\mu) = \sum_{i=0}^{\infty} \frac{M^i}{i!} t^i = e^{Mt}$.
Remark 1.3: In Remark 1.1 $\exp(t\mu)$ for $t \in I\!R$ has been computed easily with the help of (9); what is the result for $\exp(\mu_1 + \mu_2)$, if $\mu_1, \mu_2 \in L(T_m I\!R^n, I\!R^n)$?

$$\exp(\mu_1 + \mu_2) = \sum_{i=0}^{\infty} \frac{(\mu_1 + \mu_2)^i}{i!} \circ \delta_{[i]} \overset{(8),(11)}{=} \sum_{i=0}^{\infty} \frac{1}{i!} \Big(\sum_{j_1,\ldots,j_i=1}^{2} d_{\mu_{j_1},[1]} \circ \ldots \circ d_{\mu_{j_i},[i]} \Big) \circ \delta_{[i]} \; .$$

The formula shows a strong mixing of μ_1 and μ_2, but in Section 2.1 we will see nevertheless that it is possible to separate different additive parts of μ in transparent way.
Remark 1.4: Application of the Exponential Formula to the system (3) homogenized with the variable z shows for $x^0 = (x_1^0, \ldots, x_n^0, c)^T \in I\!R^{n+1}$, that $\mu^i(x^0) = (*, \ldots, *, 0)^T$ for $i > 0$ and consequently $\exp(t\mu)(x^0) = x^0 + \sum_{i=1}^{\infty} \frac{\mu^i(x^0)}{i!} t^i = (*, \ldots, *, c)^T$. Hence the

different n-dimensional hyperplanes $\{z \equiv c\}$ are invariant under the flow $\exp(t\mu)$, and especially we have on $\{z \equiv 1\}$ the unaltered original system. $\square$

We close this subsection with a simple example:

Example 1.2: Given the IVP $\dot{x} = -x^2$, $x(0) = a \in I\!R$. With $e \equiv e_1 = (1)$ is $\mu(e \otimes e) = -e$, $\mu(a \otimes a) = a^2\mu(e \otimes e) = -a^2 = (-a) \cdot a$ and for $i > 0$: $(\mu)^i(\otimes^{i+1}a) = (\mu)^{i-1} \circ d_{\mu,[i]}(\otimes^{i+1}a) \stackrel{(8)}{=} (\mu)^{i-1} \circ (-a) \sum_{\nu=1}^{i} \otimes^i a = (-a) \cdot i \cdot (\mu)^{i-1}(\otimes^i a) = \ldots = (-a^i) \cdot i! \cdot a$. Hence: $exp(t\mu)(a) = a + \sum_{i=1}^{\infty} \frac{1}{i!}(\mu)^i(\otimes^{i+1}a) = a \cdot \sum_{i=0}^{\infty}(-at^i) = \frac{a}{1+at}$ for $|at| < 1$, which is the well known solution. $\square$

1.2 A recursion formula and the solution property of the Exponential Formula

First of all we formulate and prove a recursion formula of fundamental importance. For $l, k \in I\!N_0$, $l + 1 \geq k \geq 1$ we define the $I\!R$-linear mapping

$$(\mu)_k^l : T_{[l]}I\!R^n \to T_{[k-1]}I\!R^n \tag{15}$$

$$(\mu)_k^l := \begin{cases} id_{T_{[l]}I\!R^n} & \text{, if } k = l+1 \\ d_{\mu,[k]} \circ \ldots \circ d_{\mu,[l]} & \text{, if } k \leq l. \end{cases}$$

Special cases are $(\mu)_l^l = d_{\mu,[l]}$, $(\mu)_1^1 = (\mu)^1 = d_{\mu,[1]} = d_{\mu,m} = \mu$ and $(\mu)_1^l = (\mu)^l = (\mu)^k \circ (\mu)_{k+1}^l$ for $l \geq k \geq 0$.

Theorem (Recursion formula): *For $m \in I\!N$, $i \geq k \geq 0$ and $j_1, \ldots, j_{[k]} \in I\!N_0$:*

$$(\mu)_{k+1}^i = \sum_{j_1+\ldots+j_{[k]}=i-k} \frac{(i-k)!}{j_1! \cdot \ldots \cdot j_{[k]}!}(\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{[k]}}. \tag{16}$$

Especially for $k = 1$:

$$(\mu)_2^i = \sum_{j_1+\ldots+j_m=i-1} \frac{(i-1)!}{j_1! \cdot \ldots \cdot j_m!}(\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_m}. \tag{17}$$

Proof: *Case* $k = 0$: $[0] = 1$ for all $m \in I\!N$ and both sides of (16) are equal to $(\mu)^i$. *Case* $k = i$: both sides are equal to the identity on $T_{[l]}I\!R^n$.

In the linear case $m = 1$, i.e. $[k] \equiv 1$, the claim reduces to $(\mu)_{k+1}^i = (\mu)^{i-k} = \frac{(i-k)!}{(i-k)!}(\mu)^{i-k}$.

So it remains to show the recursion formula in the case $m \geq 2$ for $i \geq k+1 > 1$ by induction on i:

For $i = 2$ and $k = 1$ one computes $(\mu)_2^2 = d_{\mu,[2]}$ for the l.h.s. of (16) and for the r.h.s.:

$$\sum_{j_1+\ldots+j_m=1} \frac{1}{j_1! \cdot \ldots \cdot j_m!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_m} =$$

$$(\mu)^1 \otimes (\mu)^0 \otimes \ldots \otimes (\mu)^0 + (\mu)^0 \otimes (\mu)^1 \otimes \ldots \otimes (\mu)^0 + \ldots + (\mu)^0 \otimes \ldots \otimes (\mu)^0 \otimes (\mu)^1$$

$$\overset{(11)}{=} \mu \otimes \otimes^{m-1} id + id \otimes \mu \otimes \ldots \otimes id + \ldots + \otimes^{m-1} id \otimes \mu \overset{(8)}{=} d_{\mu,[2]}.$$

Suppose now validity of (16) for some i and all k with $0 < k < i$, then (16) is valid for $i+1$ and all k with $0 < k \leq i$ too:
Case $k = i$: $(\mu)_{i+1}^{i+1} = d_{\mu,[i+1]}$ and $d_{\mu,[i+1]}$ equals

$$\sum_{j_1+\ldots+j_{[i]}=1} \frac{1}{j_1! \cdot \ldots \cdot j_{[i]}!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{[i]}} = (\mu)^1 \otimes \otimes^{[i]-1} id + \ldots + \otimes^{[i]-1} id \otimes (\mu)^1.$$

Case $0 < k < i$:

$$(\mu)_{k+1}^{i+1} \overset{(15)}{=} (\mu)_{k+1}^{i} \circ d_{\mu,[i+1]} = \left(\sum_{j_1+\ldots+j_{[k]}=i-k} \frac{(i-k)!}{j_1! \cdot \ldots \cdot j_{[k]}!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{[k]}} \right) \circ d_{\mu,[i+1]}.$$

Fix now a $[k]$-tuple $(j_1, \ldots, j_{[k]})$ with the sum $j_1 + \ldots + j_{[k]} = i - k$, then $[j_1], \ldots, [j_{[k]}]$ is an ordered partition of $[i]$:

$$\sum_{\nu=1}^{[k]} [j_\nu] = \sum_{\nu=1}^{k(m-1)+1} \{j_\nu(m-1)+1\} = \{ \sum_{\nu=1}^{k(m-1)+1} j_\nu\}(m-1) + k(m-1) + 1$$

$$= \{i-k\}(m-1) + k(m-1) + 1 = i(m-1) + 1 = [i].$$

Therefore $d_{\mu,[i+1]} : T_{[i+1]}I\!R^n \to T_{[i]}I\!R^n$ can be written:

$$d_{\mu,[i+1]} \overset{(8)}{=} \sum_{\lambda=1}^{[i]} \otimes^{\lambda-1} id \otimes \mu \otimes \otimes^{[i]-\lambda} id = \sum_{\nu=1}^{[k]} \otimes^{[j_1]+\ldots+[j_{\nu-1}]} id \otimes d_{\mu,[j_\nu+1]} \otimes \otimes^{[j_{\nu+1}]+\ldots+[j_{[k]}]} id.$$

Hence:

$$((\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{[k]}}) \circ d_{\mu,[i+1]} = \tag{18}$$

$$\sum_{\nu=1}^{[k]} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{\nu-1}} \otimes \{(\mu)^{j_\nu} \circ d_{\mu,[j_\nu+1]}\} \otimes (\mu)^{j_{\nu+1}} \otimes \ldots \otimes (\mu)^{j_{[k]}}.$$

Finally using the substitution $j'_\nu = j_\nu + 1$ (note $j'_\nu > 0$):

$$(\mu)_{k+1}^{i+1} \overset{(18)}{=} \sum_{j_1+\ldots+j_{[k]}=i-k} \frac{(i-k)!}{j_1! \cdot \ldots \cdot j_{[k]}!} \left(\sum_{\nu=1}^{[k]} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_\nu+1} \otimes \ldots \otimes (\mu)^{j_{[k]}} \right)$$

$$= \sum_{\nu=1}^{[k]} \sum_{j_1+\ldots+j'_\nu+\ldots+j_{[k]}=i-k+1} \frac{(i-k)!}{j_1! \cdot \ldots \cdot (j'_\nu - 1)! \cdot \ldots \cdot j_{[k]}!} \times$$

$$\times (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j'_\nu} \otimes \ldots \otimes (\mu)^{j_{[k]}}$$

$$= \sum_{j_1+\ldots+j_{[k]}=i+1-k} \frac{(i+1-k)!}{j_1! \cdot \ldots \cdot j_{[k]}!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_{[k]}} ,$$

where the last step – omitting the primes at j'_ν – follows from the well known formula for multinomial coefficients, which generalizes the recursion relation for Pascal's triangle. □

Next we compute for vectors $v_1, \ldots, v_m \in I\!\!R^n$ with $v_\nu = (v_\nu^k)_{k=1,\ldots,n}$ $(\nu = 1, \ldots, m)$:

$$\mu(v_1 \otimes \ldots \otimes v_m) = \sum_{k_1,\ldots,k_m=1}^{n} v_1^{k_1} \cdot \ldots \cdot v_m^{k_m} \mu(e_{k_1} \otimes \ldots \otimes e_{k_m}) \tag{19}$$

$$\overset{(7)}{=} \left(\sum_{k_1,\ldots,k_m=1}^{n} a^i_{k_1\ldots k_m} v_1^{k_1} \cdot \ldots \cdot v_m^{k_m} \right)_{i=1,\ldots,n} .$$

For $v_1 = \ldots = v_m = v \equiv (v^k)_{k=1,\ldots,n}$ this specializes to

$$\mu^1(v) \overset{(13)}{=} (\mu \circ \delta_m)(v) = \mu(\otimes^m v) = \left(\sum_{k_1,\ldots,k_m=1}^{n} a^i_{k_1\ldots k_m} v^{k_1} \cdot \ldots \cdot v^{k_m} \right)_{i=1,\ldots,n} \overset{(5)}{=} D(v),$$

where in the last step clearly $D(v) \equiv (D_i(v_1, \ldots, v_n))_{i=1,\ldots,n}$. As an important consequence we have:

$$\mu^1 \equiv D \text{ as mappings of } I\!\!R^n \text{ into itself.} \tag{20}$$

Moreover, for all mappings A of $I\!\!R^n$ into itself and all $m \in I\!\!N$:

$$\mu^1 \circ A = \mu \circ \delta_m \circ A = \mu \circ (\otimes^m A) \circ \delta_m. \tag{21}$$

Finally we need the identity

$$\delta_{p_1+\ldots+p_m} = (\delta_{p_1} \otimes \ldots \otimes \delta_{p_m}) \circ \delta_m \text{ for } p_1, \ldots, p_m, m \in I\!\!N, \tag{22}$$

which we apply with $p_\nu = [j_\nu]$, $\sum j_\nu = i$ and $\sum [j_\nu] = [i+1]$.

Theorem (Solution property of the Exponential Formula): *For $m, n \in I\!\!N$ let μ correspond to the n-dimensional polynomial vector field* (4) *according to* (7). *Then $X(t) = \exp(t\mu)$ solves the equation $\frac{d}{dt}X = D \circ X$ and the IVP $\dot{x} = D(x)$, $x(0) = x_0 \in$*

$I\!R^n$ is solved by $x(t) = \exp(t\mu)(x_0) = \sum_{i=0}^{\infty} \frac{\mu^i(x_0)}{i!} t^i$. The latter is a vector of convergent power series.

Proof: Formal differentiation of the power series gives:

$$\frac{d}{dt}\exp(t\mu) \overset{(14)}{=} \frac{d}{dt}\Big(\sum_{i=0}^{\infty} \frac{t^i}{i!}\mu^i\Big) = \sum_{i=1}^{\infty} \frac{t^{i-1}}{(i-1)!}\mu^i = \sum_{i=0}^{\infty} \frac{t^i}{i!}\mu^{i+1}$$

$$\overset{(13)}{=} \sum_{i=0}^{\infty} \frac{t^i}{i!}(\mu)^{i+1} \circ \delta_{[i+1]} \overset{(15)}{=} \sum_{i=0}^{\infty} \frac{t^i}{i!}\mu \circ (\mu)_2^{i+1} \circ \delta_{[i+1]}$$

$$\overset{(17)}{=} \sum_{i=0}^{\infty} \frac{t^i}{i!}\mu \circ \left(\sum_{j_1+\ldots+j_m=i} \frac{i!}{j_1! \cdot \ldots \cdot j_m!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_m} \right) \circ \delta_{[i+1]}$$

$$\overset{(22)}{=} \sum_{i=0}^{\infty} t^i \mu \circ \left(\sum_{j_1+\ldots+j_m=i} \frac{1}{j_1! \cdot \ldots \cdot j_m!} (\mu)^{j_1} \otimes \ldots \otimes (\mu)^{j_m} \right) \circ (\delta_{[j_1]} \otimes \ldots \otimes \delta_{[j_m]}) \circ \delta_m$$

$$\overset{(13)}{=} \mu \circ \left\{ \sum_{i=0}^{\infty} t^i \left(\sum_{j_1+\ldots+j_m=i} \frac{\mu^{j_1}}{j_1!} \otimes \ldots \otimes \frac{\mu^{j_m}}{j_m!} \right) \right\} \circ \delta_m$$

$$= \mu \circ \left(\otimes^m \sum_{i=0}^{\infty} \frac{t^i}{i!}\mu^i \right) \circ \delta_m \overset{(21)}{=} \mu^1 \circ \exp(t\mu) \overset{(20)}{=} D \circ \exp(t\mu)\,.$$

The second part of the claim concerning $x(t)$ follows at once and the convergence will be discussed below. □

Remark 1.5: In the linear case $m = 1$ the Theorem specializes to the familiar form $\frac{d}{dt}\exp(tM) = M\exp(tM)$, where of course $\mu \in L(I\!R^n, I\!R^n)$ is represented by the matrix M.

1.3 The flow property

The solution property proven in the last section is of formal nature; but it is well known ([H, p.207 ff.]), that the local power series solution for a finite dimensional analytic vector field has a non vanishing radius of convergence. The formal solution represents therefore a vector of convergent power series.

On the other hand in the nonlinear case the radii of convergence will not be bounded away from zero uniformly for all starting points, whence one can expect “only” a *formal* or *local* flow. More precisely: the formal flow as an analytical mapping $I\!R^n \times I\!R \supset U \to$

$I\!R^n$ is defined on an open neighborhood $U \supset I\!R^n \times \{0\}$, but in general there does not exist $\varepsilon > 0$ such that $U \supset I\!R^n \times (-\varepsilon, \varepsilon)$.

Already for C^1-vector fields the solutions of all IVP's generate a formal flow ([A, Thm.10.3]), so that one can conclude indirectly by the solution property of $\exp(t\mu)(x_0)$ that $\exp(t\mu)$ is a formal flow and therefore a formal 1-parameter-group.

We will show this property of $\exp(t\mu)$ now directly by means of the recursion formula (16).

Theorem: $\exp(t\mu)$ *is a formal 1-parameter-group.*

Proof: $\exp(t\mu)|_{t=0} = \frac{1}{0!}\mu^0 0^0 \overset{(13)}{=} id$. It remains to show:

$$\exp((s+t)\mu) = \exp(s\mu) \circ \exp(t\mu)\ . \tag{23}$$

R.h.s.(23) $= \sum_{k=0}^{\infty} \frac{s^k}{k!}(\mu)^k(\otimes^{[k]} \exp(t\mu))$, and l.h.s.(23) gives:

$$\sum_{i=0}^{\infty} \frac{(s+t)^i}{i!}\mu^i = \sum_{i=0}^{\infty}(\sum_{k=0}^{i} \frac{s^k}{k!}\frac{t^{i-k}}{(i-k)!})\mu^i$$

$$= \sum_{i=0}^{\infty} \left\{ \sum_{k=0}^{i} \frac{s^k}{k!}(\mu)^k \circ \frac{t^{i-k}}{(i-k)!}(\mu)^i_{k+1} \right\} \circ \delta_{[i]}\ .$$

Now for $0 \le k \le i$ one has by the recursion formula (16):

$$\frac{t^{i-k}}{(i-k)!}(\mu)^i_{k+1} = \sum_{j_1+\ldots+j_{[k]}=i-k} \frac{t^{j_1}}{j_1!}(\mu)^{j_1} \otimes \ldots \otimes \frac{t^{j_{[k]}}}{j_{[k]}!}(\mu)^{j_{[k]}}$$

and by collecting terms with the same k the argument of $\frac{s^k}{k!}(\mu)^k$ becomes:

$$\sum_{i=k}^{\infty} \left(\sum_{j_1+\ldots+j_{[k]}=i-k} \frac{t^{j_1}}{j_1!}(\mu)^{j_1} \otimes \ldots \otimes \frac{t^{j_{[k]}}}{j_{[k]}!}(\mu)^{j_{[k]}} \right) = \otimes^{[k]} \exp(t\mu).$$

□

1.4 Some remarks on the 'history' of the Exponential Formula

The correspondence of homogeneous vector fields with multilinear mappings has been observed the first time by L.Marcus in 1960 [M] for the quadratic case $m = 2$ and has been used by him for the classification of plane homogeneous quadratic vector fields. C.Coleman [C] has generalized this correspondence 1970 for arbitrary homogeneous polynomial vector fields. Marcus and Coleman didn't isolate μ as a multilinear mapping, but interpreted it as an in general nonassociative nonunitary binary resp. m-nary algebra product on the underlying vector space $I\!R^n$. 1977 Röhrl has introduced and investigated in [R] categories of m-nary algebras with certain linear and nonlinear morphism.

Certain elementary dynamical phenomena correspond to structural properties of m-nary algebras ([R, Cor.(1.10)]), e.g. straight lines through the origin invariant under the flow correspond to the idempotent elements of the algebra. Therefore one may try to use homological methods as in algebraic topology to investigate the category of m-nary algebras and then to translate the results to dynamics. Unfortunately the first step turns out to be too complex: indeed the m-nary algebras with structure preserving morphism form an algebraic category, but finitely generated free objects are infinite dimensional in general, so that the cohomology theories which are canonically available for this type of categories are practically useless.

Back to the Exponential Formula: Röhrl [R], inspired by an unpublished manuscript of M. Köcher, came rather close to the Exponential Formula by investigating automorphism of m-nary algebras which map solutions to solutions. He didn't find the Formula, because of his predominantly algebraic-categorical viewpoint, unfavorable notation and not recognizing the 1-parameter-group property. We found the Exponential Formula and the fundamental recursion formula (16) by trying to establish the 1-parameter-group property for Röhrl's formalism. □

1.5 Estimating the radius of convergence

As already noted above (Sec.1.3), analytic vector fields are well known to possess locally analytic solutions. In this section we give a new proof of this result in the case of polynomial vector fields by establishing a lower bound for the radius of convergence of $\exp(t\mu)(x_0)$ depending on μ and the distance of x_0 to the origin.

Clearly $a_i = i!$ ($i \in \mathbb{N}_0$) solves the recursion $a_{i+1} = \sum_{\nu=0}^{i} \binom{i}{\nu} a_\nu a_{i-\nu}$ with $a_0 := 1$. To solve the more general recursion $a_{i+1} = \sum_{j_1+\ldots+j_m=i} \frac{i!}{j_1!\cdot\ldots\cdot j_m!} a_{j_1}\cdot\ldots\cdot a_{j_m}$ with $a_0 := 1$ for arbitrary $m \in \mathbb{N}$, we introduce *generalized factorials* for $n \in \mathbb{N}_0$ and all $r \in \mathbb{R}$:

$$(n+1)!_r := n!_r \cdot (1+nr) \quad \text{with } 0!_r := 1 \tag{24}$$

As special cases we have $n!_1 = n!$ and $n!_0 = 1$. Then $a_i := i!_{m-1}$ solves the above recursion:

Lemma: *For all $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$:*

$$(i+1)!_{m-1} = \sum_{j_1+\ldots+j_m=i} \frac{i!}{j_1!\cdot\ldots\cdot j_m!}\, j_1!_{m-1}\cdot\ldots\cdot j_m!_{m-1}\,. \tag{25}$$

Proof: The case $m=1$ is trivial and $m=2$ is already done. Using Appell's symbol $(x,n) := x(x+1)\cdot\ldots\cdot(x+n-1)$ and the binomial series ${}_1F_0(a;z) \equiv \sum_{i=0}^{\infty} \frac{(a,i)}{i!} z^i = (1-z)^{-a}$ for $|z|<1$ one can show

$$\sum_{j_1+\ldots+j_m=i} \frac{(b_1,j_1)}{j_1!}\cdot\ldots\cdot\frac{(b_m,j_m)}{j_m!} = \frac{(b_1+\ldots+b_m,i)}{i!}$$

([Ca, Thm.2.3-2]). Since $n!_r = 1(1+r)(1+2r)\cdot\ldots\cdot(1+(n-1)r) = r^n(\frac{1}{r},n)$ for all $r \in \mathbb{R}\setminus\{0\}$, an easy calculation with $b_1 = \ldots = b_m = \frac{1}{m-1}$ gives the result. □

Let $\| \ \|$ be an arbitrary norm on $I\!R^n$ with $|x^i| \leq \|x\|$ for each component x^i of the vector $x \in I\!R^n$. For concrete calculations the maximum norm is most easily handled and gives the sharpest results.

The following estimate is valid uniformly for all initial conditions in a ball of radius r (with respect to the chosen norm) around the origin.

Theorem: *Let for $X = (X^1, \ldots, X^n)^T$ $p \in I\!R^n[X]$ be homogeneous of degree $m \geq 1$ and μ the mapping associated by (7) to the ODE $\dot{X} = p(X)$. For $r > 0$ let $X(0) = x_0 \in I\!R^n$ be any initial condition with $\|x_0\| \leq r$, then there exists an explicit $P > 0$ depending on μ , such that*

$$\left\|\mu^i(x_0)\right\| \leq i!_{m-1} P^i \ r^{i(m-1)+1} \text{ for all } i \in I\!N_0. \tag{26}$$

Proof: In the case $m = 1$ choose P as the operator norm of the matrix of coefficients M and the estimate is valid. Therefore let $m \geq 2$ subsequently.

We need the notations $\nu = (\nu_1, \ldots, \nu_m) \in \{1, \ldots, n\}^m$ for a multi-index, $\sum_\nu \equiv \sum_{\nu_1,\ldots,\nu_m=1}^n$, $p(X) = \sum_\nu p_\nu X^{\nu_1} \cdot \ldots \cdot X^{\nu_m}$ with coefficient vector $p_\nu = (p_\nu^k)_{k=1,\ldots,n}$ and $x_i := \mu^i(x_0) \in I\!R^n$; define $P := \sum_\nu \|p_\nu\|$. Then:

$$\|\mu(x_{j_1} \otimes \ldots \otimes x_{j_m})\| \leq P \, \|x_{j_1}\| \cdot \ldots \cdot \|x_{j_m}\| \ , \tag{27}$$

because l.h.s.(27) $\overset{(19)}{=} \left\|\sum_\nu p_\nu x_{j_1}^{\nu_1} \cdot \ldots \cdot x_{j_m}^{\nu_m}\right\| \leq \sum_\nu \left\|p_\nu x_{j_1}^{\nu_1} \cdot \ldots \cdot x_{j_m}^{\nu_m}\right\| =$
$\sum_\nu (\|p_\nu\| \ |x_{j_1}^{\nu_1}| \cdot \ldots \cdot |x_{j_m}^{\nu_m}|) \leq \sum_\nu (\|p_\nu\| \ \|x_{j_1}\| \cdot \ldots \cdot \|x_{j_m}\|) =$ r.h.s.(27).

Using (27) and the recursion formula (17) we now show the estimate by induction on i:

For $i = 0$ clearly $\|x_0\| = \|\mu^0 x_0\| \leq r = 0!_{m-1} P^0 r^{0(m-1)+1}$. Assuming (26) for all numbers $0, \ldots, i \in I\!N_0$ yields validity for $i+1$: $\left(\sum \binom{i}{j} \equiv \sum_{j_1+\ldots+j_m=i} \frac{i!}{j_1! \cdot \ldots \cdot j_m!} \right)$

$$\left\|\mu^{i+1}(x_0)\right\| \overset{(13)}{=} \left\|\mu \circ (\mu)_2^{i+1} \circ \delta_{[i+1]}\right\| \overset{(17)}{=} \sum \binom{i}{j} \|\mu(x_{j_1} \otimes \ldots \otimes x_{j_m})\|$$

$$\leq \sum \binom{i}{j} \|\mu(x_{j_1} \otimes \ldots \otimes x_{j_m})\| \leq \sum \binom{i}{j} P \, \|x_{j_1}\| \cdot \ldots \cdot \|x_{j_m}\|$$

$$\leq \sum \binom{i}{j} j_1!_{m-1} \cdot \ldots \cdot j_m!_{m-1} P \cdot P^{j_1+\ldots+j_m} \times r^{(i+1)(m-1)+1},$$

because $\sum_{\nu=1}^m (j_\nu(m-1)+1) = (i+1)(m-1)+1$. Formula (25) now proves the claim for $\mu^{i+1}(x_0)$. The remaining statement in the theorem is obvious. □

Corollary (Estimate of the radius of convergence): *With the above notations the common radius of convergence of all power series in the components of* $\exp(t\mu)(x_0)$ *is at least* $\rho := (P(m-1)\, r^{m-1})^{-1}$.

Proof: Let $\exp(t\mu) = \sum_{i=0}^\infty \frac{\mu^i(x_0)}{i!} t^i \equiv \sum_{i=0}^\infty a_i t^i$. By $i!_{m-1} \leq i!(m-1)^i$ for $m \geq 2$:

$\|a_i\| \leq (m-1)^i P^i r^{i(m-1)+1} = r((m-1)\, Pr^{m-1})^i$ and consequently the common radius of convergence is bounded from below by $(\limsup_{i\to\infty} \|a_i\|^{1/i})^{-1} \geq (P(m-1)\, r^{m-1})^{-1}$. The case $m = 1$ is trivial. $\square$

Corollary (Estimate of the remainder term): *With the above notations let* $|t| \leq t_0 < \rho$ *and* $q := t_0/\rho = P(m-1)r^{m-1}t_0 < 1$. *Then the error of the Taylor approximation of order* k *for* $|t| \leq t_0$ *is at most* $rq^{k+1}/(1-q)$.
Proof: By assumption $|q| < 1$. Therefore:

$$\left\| \sum_{i=k+1}^{\infty} \frac{\mu^i(x_0)}{i!} t^i \right\| \leq \sum_{i=k+1}^{\infty} r \mid P(m-1)\, r^{m-1} t_0 \mid^i = r \sum_{i=k+1}^{\infty} q^i = \frac{r\, q^{k+1}}{1-q}.$$

$\square$

1.6 Homogenizing variable and radius of convergence

In Remark 1.4 we have seen that for polynomial systems homogenized with the variable z all n-dimensional hyperplanes $\{z \equiv \text{ const.}\}$ are invariant under the flow. Clearly $\{z \equiv 1\}$ contains the original system. Moreover all restrictions of the flow to the hyperplanes are modulo orientation topologically equivalent, with the exception of $\{z \equiv 0\}$, which contains only the homogeneous subsystem of highest degree. Because of this equivalence one might hope that for power series solutions with corresponding starting values, i.e. starting values on the same ray emanating from the origin, when choosing the "right" hyperplane, a "greater" part of the orbit is contained in the circle of convergence of the respective power series, thereby giving more dynamical information.

This turns out to be wrong: the union of all parts of orbits contained in the circles of convergence of power series solutions with corresponding starting values form a *homogeneous set* in $\mathbb{R}^{n+1}$, i.e. a set which is the union of rays emanating from the origin. We now briefly indicate the proof of this fact. A simple calculation shows:
Lemma: *Let* $\varphi(t, x_0)$ *be the solution of the IVP* $\dot{x} = f(x), x(0) = x_0$ *with* $x = (x_1, \ldots, x_n)^T$ *and* f *homogeneous of degree* m*; then for all* $\lambda > 0$*:* $\varphi(t, \lambda x_0) = (\lambda\varphi)(\lambda^{m-1}t, x_0)$ *is the solution of* $\dot{x} = f(x), x(0) = \lambda x_0$.
Corollary: *With the above notations let* $y := \varphi(T, x_0)$ *for some* $T > 0$*, then* $\lambda y = \varphi(T', \lambda x_0)$ *with* $T' = \lambda^{-(m-1)}T$.

Theorem: *Let* $\exp(t\mu)(x_0)$ *be the solution of the IVP from the Lemma above with one of its* n *components having radius of convergence* R*, then for all* $\lambda > 0$ $\exp(t\mu)(\lambda x_0)$ *has radius of convergence* $R' = \lambda^{-(m-1)}R$ *in the same component.*
Proof: $\mu^i(\lambda x) = (\mu)^i \circ \delta_{[i]}(\lambda x) = (\mu)^i \circ (\lambda^{[i]}\delta_{[i]}(x)) = \lambda^{[i]}(\mu)^i \circ \delta_{[i]}(x) = \lambda^{[i]}\mu^i(x)$ by (10-13) and $\mathbb{R}$-linearity of $(\mu)^i$. Let a_i be the i^{th} Taylor coefficient of the component in question of $\exp(t\mu)(x_0)$ and a'_i the coefficient in the coresponding component of $\exp(t\mu)(\lambda x_0)$, then: $R'^{-1} = \limsup_{i\to\infty} |a'_i|^{1/i} = \limsup_{i\to\infty} (\lambda^{i(m-1)+1}|a_i|)^{1/i} = \lambda^{m-1} \limsup_{i\to\infty} \lambda^{1/i}|a_i|^{1/i} = \lambda^{m-1}R^{-1}$. $\square$

1.7 Some remarks on homogeneous polynomial vector fields

In the introduction we have mentioned already that the global dynamics of all homogeneous vector fields are completely determined by the local dynamics in any neighborhood of the origin and that for linear systems these local dynamics are determined by the infinitesimal or algebraic information of the Jordan normal form. Therefore the question arises whether there is some substitute for the Jordan normal form in the case of finite dimensional homogeneous polynomial vector fields.

First of all we investigate dimension 2 (dimension 1 is trivial): Quadratic homogeneous plane vector fields have been investigated and classified by several authors. Marcus [M] proceeded by first classifying all nonassociative binary 2-dimensional algebras up to linear isomorphism and than grouping together the corresponding vector fields into topological equivalence classes, using his earlier results on plane foliations. Date [D] and Sibirskii [Si] used the theory of invariants of tensors and Newton [N] inspired by a paper of Lyagina has found a third approach, which appears to us as the most natural: one observes, that all lines through the origin for all homogeneous systems in all dimensions are isoclinals and that especially in two dimensions invariant lines (the "eigenspaces") form natural topological barriers separating the plane in hyperbolic, parabolic, and elliptic sectors. The analysis in [N] uses only elementary facts on polynomials and there is no principal obstacle for extending the analysis to higher degrees.

The situation in dimensions ≥ 3 is completely different:

- Invariant lines don't separate anything now, and invariant hyperplanes – when they exist at all – do not reveal much information.
- 3-dimensional systems include as special cases the homogenized plane polynomial vector fields, which shows that a local analysis at the origin in 3 dimensions (homogeneous case) is at least as difficult as a global analysis in 2 dimensions (non-homogeneous case).
- van Strien and dos Santos have shown [SS] that for 3-dimensional homogeneous quadratic systems moduli appear, i.e. one-parameter families of topologically non-equivalent vector fields. This clearly destroys all hopes for a neat classification of homogeneous polynomial systems in higher dimensions.

2 The Simplex and the Convolution Formula

Direct computation of $\mu^i(x_0)$ needs at least $(i-1)!$ evaluations of μ using (19) (with different arguments).

Assuming μ commutative there is a recursive calculation scheme, which needs an in i exponentially growing number of evaluations of μ . This calculation scheme is the same for all polynomial degrees m and seems therefore suited for numerical applications, namely a lower order Taylor method. This will be explained in detail in the forthcoming paper [W1].

In this section we discuss two further recursion formulas for the Taylor coefficients in the Exponential Formula. In both cases these are "total history recursions", i.e. for the calculation of the $(i+1)^{th}$ coefficient one needs all preceding coefficients. These formulas need to be programmed for each polynomial degree m separately, but then yield coefficients up to a rather high degree, e.g. for a plane quadratic IVP a 486-PC easily calculates the first 100 or 200 coefficients of the local power series solution. The two recursion formulas are:

1. The Simplex Formula: This formula provides insight into how the different homogeneous parts of the original polynomial vector field contribute to the $\mu^i(x_0)$; commutative form of μ is needed. The number of evaluations of μ grows polynomially of degree $m-1$ in i.

2. The Convolution Formula: This formula dispenses with multilinear algebra and gives a connection to the properties of polynomial algebras; no homogenization or commutative form is needed. The effort for calculating the Taylor coefficients grows linearly in i for all degrees m.

2.1 The Simplex Formula

Notations: A commutative $\mu \in L(T_m I\!R^n, I\!R^n)$ is written in this section as the sum of its homogeneous parts μ_ν of degree ν: $\mu = \mu_1 + \mu_2 + \ldots + \mu_m$ corresponding to the homogeneous parts of the original vector field p in (1), which in general is nonhomogeneous. Let $\xi_0 \equiv x_0 \in I\!R^n$ be the initial value and $\xi_i := \mu^i(\xi_0)$ for $i > 0$. Moreover for a multi-index $j \equiv (j_1, \ldots, j_m) \in I\!N_0^m$ with $j! := j_1! \cdot \ldots \cdot j_m!$ let $\otimes_j \xi := \xi_{j_1} \otimes \ldots \otimes \xi_{j_m}$. $\Delta_{m,i} := \{j \,| j_1 + \ldots + j_m = i,\ j_\nu \geq 0\}$ is the set of lattice points of $Z\!\!Z^m$ contained in the simplex of $I\!R^n$ spanned by $i \cdot e_1, \ldots, i \cdot e_m$. $\Delta_{m,i}$ is called the *summation simplex of degree m and size i.* A special case is $\Delta_{m,0} = \{(0, \ldots, 0)\}$. The *dimension* of $\Delta_{m,i}$ is zero for $i = 0$ and $m-1$ for $i > 0$. For $i > 0$ and $0 \leq l \leq m$ let $\Delta^l_{m,i} := \{j \in \Delta_{m,i}|$ at most l of the j_ν are $\neq 0\}$. A special case is $\Delta^0_{m,i} = \emptyset$. $\Delta^l_{m,i}$ is called the $(l-1)$-*dimensional boundary of* $\Delta_{m,i}$, because the sum $j_1 + \ldots + j_m = i$ has $(l-1)$ degrees of freedom in the j_ν. Especially, $\Delta^l_{m,i}$ is the set of vertices for $l = 1$ and the set of edges for $l = 2$; $\Delta^m_{m,i} = \Delta_{m,i}$. For $1 \leq l \leq m$ we call $\Delta^{l \backslash l-1}_{m,i} := \Delta^l_{m,i} \setminus \Delta^{l-1}_{m,i} = \{j \in \Delta_{m,i}|$ exactly l $j_\nu \neq 0\}$ *the inside of* $\Delta^l_{m,i}$. □

With this notation the recursion formula (17) reads:

$$\xi_{i+1} = \sum_{j\in\Delta_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi)\ . \tag{28}$$

Observe that $\Delta^l_{m,i}$ decomposes as:

$$\Delta^l_{m,i} = \Delta^{l\backslash l-1}_{m,i} \cup \ldots \cup \Delta^{2\backslash 1}_{m,i} \cup \Delta^{1\backslash 0}_{m,i} \tag{29}$$

In $\mu_\nu(y_1 \otimes \ldots \otimes y_m)$ the components of $y_1, \ldots, y_m$ belonging to the homogenizing variable appear exactly $m - \nu$ times as a factor in each summand (see (19)); hence μ_ν applied to

$$\underbrace{\begin{pmatrix} * \\ \vdots \\ * \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} * \\ \vdots \\ * \\ 0 \end{pmatrix}}_{l} \otimes \underbrace{\begin{pmatrix} * \\ \vdots \\ * \\ 1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} * \\ \vdots \\ * \\ 1 \end{pmatrix}}_{m-l}$$

gives zero, if $m - \nu > m - l$, i.e. if $\nu < l$.

In ξ_i the coefficient of the homogenising variable is zero for $i > 0$ (Rem.1.4), hence for $1 < l \leq m$

$$\sum_{j\in\Delta^{l\backslash l-1}_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi) = \sum_{j\in\Delta^{l\backslash l-1}_{m,i}} \frac{i!}{j!}(\mu_l + \ldots + \mu_m)(\otimes_j \xi)\ , \tag{30}$$

because every $j \in \Delta^{l\backslash l-1}_{m,i}$ has exactly l components $j_\nu \neq 0$, and consequently $\mu(\otimes_j \xi)$ has exactly l factors ξ_i with $i > 0$. Since for $i > 0$:

$$\xi_{i+1} = \sum_{j\in\Delta_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi) = \sum_{j\in\Delta^1_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi) + \sum_{j\in\Delta^{2\backslash 1}_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi) + \ldots + \sum_{j\in\Delta^{m\backslash m-1}_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi)$$

and using (30) one gets

Formula A:

$$\xi_{i+1} = \sum_{j\in\Delta^{1\backslash 0}_{m,i}} \frac{i!}{j!}\mu(\otimes_j \xi) + \sum_{j\in\Delta^{2\backslash 1}_{m,i}} \frac{i!}{j!}(\mu_2 + \ldots + \mu_m)(\otimes_j \xi) + \ldots + \sum_{j\in\Delta^{m\backslash m-1}_{m,i}} \frac{i!}{j!}\mu_m(\otimes_j \xi).$$

Collecting summands belonging to μ_l and using (29) gives for all $i > 0$:

Formula B:

$$\xi_{i+1} = \sum_{j\in\Delta^1_{m,i}} \frac{i!}{j!}\mu_1(\otimes_j \xi) + \sum_{j\in\Delta^2_{m,i}} \frac{i!}{j!}\mu_2(\otimes_j \xi) + \ldots + \sum_{j\in\Delta^m_{m,i}} \frac{i!}{j!}\mu_m(\otimes_j \xi)$$

Remark 2.1: Formulas A and B are valid also for $i = 0$, because by definition $\Delta^l_{m,0} = \{(0, \ldots, 0)\}$ for $1 \leq l \leq m$ and $\Delta^{l\backslash l-1}_{m,0} = \emptyset$ for $1 < l \leq m$. Clearly formulas A and B are extreme cases of a whole set of formulas using other partitions and collections of the μ_ν and $\Delta_{m,i}^{l\backslash l-1}$.

Remark 2.2: $\sum_{j\in\Delta_{m,i}^1}\frac{i!}{j!}\mu_1(\otimes_j\xi) = m\mu_1(\xi_i\otimes\otimes^{m-1}\xi_0) = M\xi_i$ for all m, where M is the matrix of coefficients of the linear part of the polynomial vector field. By $\xi_{i+1} = M\xi_i + \ldots = M^{i+1}\xi_0 + \ldots$ one sees $\exp(t\mu)(\xi_0) = e^{Mt}\xi_0 + remainder$, where "remainder" is of course the crucial part in the nonlinear case.

Remark 2.3: Let $D_{m,i}$, $D_{m,i}^l$ and $D_{m,i}^{l\backslash l-1}$ the respective cardinalities of the sets $\Delta_{m,i}$, $\Delta_{m,i}^l$ and $\Delta_{m,i}^{l\backslash l-1}$. Then by simple combinatorial considerations and different partitions of the summation simplex one can show the following formulas:

$D_{m,i} = c^*(m-1,i+1) = \begin{pmatrix} i+m-1 \\ m-1 \end{pmatrix}$, $D_{m+1,i} = \sum_{\nu=0}^i D_{m,\nu}$, $D_{m,i}^{l\backslash l-1} = \begin{pmatrix} m \\ l \end{pmatrix} D_{l,i}^{l\backslash l-1}$ for $1\le l\le m$, $D_{m,i}^{m\backslash m-1} = D_{m,i-m}$, $D_{m,i+1} = \sum_{\nu=1}^m D_{\nu,i}$, $D_{m,i} = \sum_{l=1}^m D_{m,i}^{l\backslash l-1}$, $D_{m,i} = D_{m,i-m} + D_{m,i}^{m-1}$ and for $i = sm+r$ with $0\le r<m$: $D_{m,i} = D_{m,r} + \sum_{\nu=1}^s D_{m,i-\nu m}^{m-1}$.

2.2 The Convolution Formula

In order to avoid the occurrence of multinomial coefficients, we choose in this section the notation $\exp(t\mu)(x_0) = \sum_{i=0}^\infty \frac{\mu^i(x_0)}{i!}t^i \equiv \sum_{i=0}^\infty \frac{\xi_i}{i!}t^i \equiv \sum_{i=0}^\infty x_i t^i$, i.e. $x_i = \mu^i(x_0)/i! = \xi_i/i!$ with $x_i \equiv (x_i^l)_{l=1,\ldots,n} \in I\!R^n$.

We depart from formula B ($i\ge 0$):

$$\xi_{i+1} = \sum_{\nu=1}^m \sum_{j\in\Delta_{m,i}^\nu}\frac{i!}{j!}\mu_\nu(\otimes_j\xi) \Leftrightarrow (i+1)x_{i+1} = \frac{\xi_{i+1}}{i!} = \sum_{\nu=1}^m \sum_{j\in\Delta_{m,i}^\nu}\mu_\nu(\otimes_j x),$$

because $\frac{1}{j!}\mu_\nu(\otimes_j\xi) = \frac{1}{j_1!\cdot\ldots\cdot j_m!}\mu_\nu(\xi_{j_1}\otimes\ldots\otimes\xi_{j_m}) = \mu_\nu(x_{j_1}\otimes\ldots\otimes x_{j_m}) = \mu_\nu(\otimes_j x)$. Hence:

$$x_{i+1} = \frac{1}{i+1}\left(\sum_{\nu=1}^m \sum_{j\in\Delta_{m,i}^\nu}\mu_\nu(\otimes_j x)\right) \tag{31}$$

Next we sum over j the individual monomials of μ_ν separately; this generates convolutions of the Taylor coefficients.

We demonstrate this in the case $m = n = 2$ for the system:

$$\begin{cases} \dot{x} = Mx + Ny + Ax^2 + Bxy + Cy^2 \\ \dot{y} = M'x + N'y + A'x^2 + B'xy + C'y^2 \end{cases}$$

By (31) and (19) (writing $x_i \equiv x_i^1$ and $y_i \equiv x_i^2$) one gets for the first equation: (The computation for the second equation is the same with primed capitals etc. .)

$$x_{i+1} = \frac{1}{i+1}\left(Mx_i + Ny_i + \sum_{j+k=i}\left(Ax_jx_k + \frac{B}{2}(x_jy_k + y_jx_k) + Cy_jy_k\right)\right)$$

$$= \frac{1}{i+1}\left(Mx_i + Ny_i + A(\sum_{j+k=i}x_jx_k) + B(\sum_{j+k=i}x_jy_k) + C(\sum_{j+k=i}y_jy_k)\right)$$

$$= \frac{1}{i+1}\,(Mx_i + Ny_i + A(x*x)_i + B(x*y)_i + C(y*y)_i\ ,)$$

Note that the commutative form of μ is no longer necessary. In the last step we used the following

Notation: For ν sequences $x^{(k)} = (x_0^{(k)}, x_1^{(k)}, x_2^{(k)}, \ldots)$ $(1 \le k \le \nu)$ of elements of a commutative ring R the *ν-fold convolution* is defined by:

$$*_\nu(x^{(1)}, \ldots, x^{(\nu)}) := \left(\sum_{j_1+\ldots+j_\nu=i} x_{j_1}^{(1)} \cdot \ldots \cdot x_{j_\nu}^{(\nu)} \right)_{i \in \mathbb{N}_0} .$$

$*_{\nu,i}(x^1, \ldots, x^\nu) := (*_\nu(x^1, \ldots, x^\nu))_i$ the i^{th} component of the ν-fold convolution.

Clearly $*_2(x, y) \equiv x*y$ (the usual binary convolution) and $*_1(x) = x$, hence $*_{1,i}(x) = x_i$. The ν-fold convolution is commutative, ν-linear and $(x*y)*z = x*(y*z) = *_3(x, y, z)$ implies, that the ν-fold convolution can be written as a product of (binary) convolutions with arbitrary bracketing.

Definition of R[x]: Let R be a commutative ring with unit and $F(R) \equiv R^{\mathbb{N}_0}$ the free R-module of all R-sequences on $\mathbb{N}_0$. The convolution $*$ defined above turns $F(R)$ into a R-algebra $\mathbf{R} \equiv (F(R), *)$. R as a R-algebra is embedded into $\mathbf{R}$ by the mapping $\Phi : R \to \mathbf{R}$ with $r \mapsto \mathbf{r} := (r, 0, 0, \ldots)$. Let $\mathbf{R}[\mathbf{x}] \equiv \mathbf{R}[x^{(1)}, \ldots, x^{(n)}]$ the ring of polynomials over $\mathbf{R}$ in the variables $x^{(1)}, \ldots, x^{(n)}$; of course $x^{(k)} = (x_0^{(k)}, x_1^{(k)}, x_2^{(k)}, \ldots)$ for $1 \le k \le n$ and $*$ is the multiplication.

Note that $\mathbf{R}[\mathbf{x}]$ is an R-algebra with componentwise multiplication and $r\,\mathbf{p}(\mathbf{x}) = \mathbf{r} * \mathbf{p}(\mathbf{x})$. Moreover $\Phi : R \to \mathbf{R}$ extends to a R-algebra homomorphism $\Phi : R[x] \to \mathbf{R}[\mathbf{x}]$ by $\Phi(x^l) := x^{(l)}$.

Now let p as in (2) a polynomial vector field in $\mathbb{R}^n$, i.e. a n-tuple of polynomials $p^l \in R[x]$ in the variables $x^1, \ldots, x^n$, then we denote by $\mathbf{p}$ the n-tuple with $\mathbf{p}^l := \Phi(p^l) \in \mathbf{R}[\mathbf{x}]$ $(l = 1, \ldots, n)$, i.e. $\mathbf{p}$ maps a n-tuple $\mathbf{x}$ of sequences $x^{(k)} \in \mathbf{R}$ to the n-tuple $\mathbf{p}(\mathbf{x})$ of sequences with the components $\mathbf{p}_{,i}(\mathbf{x})$. Let $\mathbf{p}_\nu$ denote the homogeneous part of degree ν and the $\mathbf{p}_{\nu,i}$ its components.

Completely as in the special case above one can deduce from (31) the:

Convolution Formula:

$$x_{i+1} = \frac{1}{i+1}\,\mathbf{p}_{,i}(\mathbf{x}) = \frac{1}{i+1}\left(\sum_{\nu=1}^{m} \mathbf{p}_{\nu,i}(\mathbf{x})\right) \ \forall i \in \mathbb{N}_0. \tag{32}$$

In addition to the deduction of the Convolution Formula from the Exponential Formula via formula B and (31) we give now a

Direct proof: Substitute a vector of power series to the system $\dot{x} = p(x)$ and equate coefficients for t^i. On the l.h.s. one has $(i + 1)x_{i+1}$ and on the r.h.s. for any ν-homogeneous monomial the i^{th} component of a ν-fold product of power series, i.e. the i^{th} component of the ν-fold convolution of their respective coefficients. □

Remark 2.4 Since Φ is a R-algebra homomorphism, the first equality in (32) is valid for any form of the polynomial vector field compatible with the rules of calculation for polynomials, for example a factorised p.

Remark 2.5 In the case $m = 1$ the Convolution Formula gives by $*_{1,i}(x) = x_i$: $x_{i+1} = \frac{1}{i+1}\, \mathbf{p}_{1,i}(\mathbf{x}) = \frac{1}{i+1}\, Mx_i = \frac{1}{(i+1)!}\, M^{i+1}x_0$, where M is the matrix of coefficients of $\mathbf{p}_1$.

Remark 2.6 For fixed m and n clearly the effort for calculating the Taylor coefficients grows linearly in i. It is not necessary to store completely the intermediary results of the ν-fold convolutions for $\nu = 1$ up to $\nu = m - 1$ (see the next example). For the Simplex Formula it is not clear how to store intermediary results efficiently, because for all $i, i', m \in I\!N$ we have $\Delta_{m,i} \cap \Delta_{m,i'} = \emptyset$, if $i \neq i'$.

Example 2.1: Let $n = 2$, $m = 3$. We have to store as intermediary results the coefficients x_i, y_i, and the numbers $(x{*}x)_i$ and $(y{*}y)_i$, but not the numbers $(x{*}y)_i$. Then all 3-fold convolutions can be calculated: e.g. $*_{3,i}(x, y, y) = (x{*}(y{*}y))_i = \sum_{\nu=0}^{i} x_\nu \cdot (y{*}y)_{i-\nu}$.

2.3 Faà di Bruno's Formula and the Convolution Formula

Let $\varphi(t) \equiv \sum_{i=0}^{\infty}(\xi_i/i!)t^i \equiv \sum_{i=0}^{\infty} x_i t^i$ be the power series solution of the polynomial IVP $\dot{x} = p(x)$, $x(0) = 0$, where $p(x) \equiv \sum_{k=0}^{m}(q_k/k!)x^k \equiv \sum_{k=0}^{m} Q_k x^k$. (Note that the initial value $x(0) = 0$ is uncomfortable but no restriction, because p can be translated.) Then clearly

$$\xi_{i+1} = (\frac{d}{dt})^i\ (p \circ \varphi(t))|_{t=0}. \tag{33}$$

For $i = 0, 1$ one has $\xi_0 = x_0 = 0$ and $\xi_1 = x_1 = p(0) = Q_0 = q_0$; so it remains to investigate $i \geq 1$. In dimension $n = 1$ the r.h.s. of (33) can be evaluated using Faà di Bruno's Formula and the formula for Bell polynomials (cf. [Co]):

$$\xi_{i+1} = \sum_{k=1}^{i}\ q_k\ B_{i,k}(\varphi^{(1)}, \varphi^{(2)}, \ldots, \varphi^{(i-k+1)})|_{t=0} = \sum_{k=1}^{i}\ q_k\ B_{i,k}(\xi_1, \xi_2, \ldots, \xi_{i-k+1}) =$$

$$\sum_{k=1}^{i}\ q_k\ \sum \frac{i!}{c_1!c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots}\ \xi_1^{c_1}\xi_2^{c_2}\ldots\ ,$$

where the summation takes place over all $c_1, c_2, \ldots \in I\!N_0$, such that $c_1 + c_2 + c_3 + \ldots = k$ and $c_1 + 2c_2 + 3c_3 + \ldots = i$. This can be transformed further to

$$\xi_{i+1} = \sum_{k=1}^{i} \frac{q_k}{k!} \sum_{j_1+\ldots+j_k=i} i!\ x_{j_1}\ldots x_{j_k},$$

where $j_1, j_2, \ldots \in I\!N$, and finally to:

$$(i+1)x_{i+1} = \xi_{i+1}/i! = \sum_{k=1}^{m}\ Q_k \sum_{j_1+\ldots+j_k=i} x_{j_1}\ldots x_{j_k}\ ,$$

which is our Convolution Formula. Therefore the use of Faà di Bruno's Formula in one dimension amounts to a desymmetrisation of the Convolution Formula, which obscures the structural simplicity of the convolution process, and is not well suited to higher dimensional generalization.

3 The Exponential formula and the iteration of polynomial mappings

Let k be a field and A a k-algebra; then every k-linear map from A to $A \otimes A$ is called a *comultiplication* on A. We are interested in this section especially in two important examples of comultiplications: $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta^*(x) = x \otimes x$. The following 'excursion' will put this two examples into perspective; we will use without further comment some elementary facts about coalgebras, Hopf algebras and affine algebraic groups, a full account of which the reader may find in the books of Sweedler [Sw] and Abe [Abe]. It is perfectly possible to understand the main constructions without reading the 'excursion', but skimming through it should give a feeling of the context.

Excursion: A Hopf algebra H over a field k is called *affine*, if it is finitely generated as an algebra and commutative. The set of all k-algebra mappings $G(k) := Alg(H, k)$ is then a group under the *convolution* $*$ defined by $f * g := m_k \circ (f \otimes g) \circ \Delta$, where m_k is the multiplication in k and Δ the comultiplication of H. The $G(k)$ constructed in this way is called an *affine algebraic group*. We are interested here in the following two special cases:

$G_a(k) := Alg(H, k)$ for $H = k[x]$ with comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$, i.e. x is *primitive*, counit $\varepsilon(x) = 0$ and antipode $S(x) = -x$. $G_a(k) \cong (k, +)$, i.e. $G_a(k)$ and the additive group $(k, +)$ of k are isomorphic.

$G_m(k) := Alg(H, k)$ for $H = k[x, x^{-1}]$ with comultiplication $\Delta^*(x) = x \otimes x$, i.e. x is *grouplike*, counit $\varepsilon^*(x) = 1$ and antipode $S^*(x) = x^{-1}$. $G_m(k) \cong (k^*, \cdot)$, i.e. $G_m(k)$ and the multiplicative group $(k^*, \cdot)$ of nonzero elements of k are isomorphic.

Moreover, if one uses the above two Hopf algebras as *measuring coalgebras*, i.e. let $Hom_k(A, B)$ be the space of k-linear mappings between the k-algebras A and B, and $\varphi : H \to Hom_k(A, B)$ a k-linear mapping, such that the comultiplication of H is "compatible" with the multiplications of A and B (cf. [Sw]), then the image of x under φ in the case $H = k[x, x^{-1}]$ will be an algebra homomorphism and in the case $H = k[x]$ a derivation. □

We will show now that *the sequence of μ^i's in the Exponential Formula is associated to the comultiplication $\Delta(x) = x \otimes 1 + 1 \otimes x$ in the same way as the sequence of iterates of real polynomial mappings is associated to the comultiplication $\Delta^*(x) = x \otimes x$.*

Define $\Delta_0 = id$, $\Delta_1 = \Delta$ and recursively: $\Delta_n = (\otimes^{n-1} 1 \otimes \Delta)\ \Delta_{n-1}$ for $n > 1$; by coassociativity, i.e. $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$, the definition does not depend on the place of Δ in the n-fold tensor product. In the same way one defines the 'higher order comultiplications' Δ_n^* for Δ^* and in fact this definition is possible in any coassociative coalgebra. More concretely one computes:

$$\Delta_n(x) = \sum_{\nu=1}^{n+1} \otimes^{\nu-1} 1 \otimes x \otimes^{n+1-\nu} 1 \quad \text{and} \quad \Delta_n^*(x) = \otimes^{n+1} x\ . \tag{34}$$

Let now $\mu \in L(T_m\mathbb{R}^n, \mathbb{R}^n)$ (as in Sec.1.1) correspond to a polynomial mapping or vector field $p : \mathbb{R}^n \to \mathbb{R}^n$, which is (made) homogeneous of degree m. Substitution of μ for x and id for 1 in the above formulas yields the $\mathbb{R}$-linear mappings:

$$\Delta_{p-m}(\mu) \overset{(8)}{=} d_{\mu,p}\ , \tag{35}$$

if we define $\Delta_{p-m}(\mu) := 0$ for $p < m$, and

$$d^*_{\mu,pm} := \Delta^*_{p-1}(\mu) = \otimes^p \mu : T_{pm}\mathbb{R}^n \to T_p\mathbb{R}^n \quad \text{for all } p \in \mathbb{N}\ . \tag{36}$$

In Section 1 we have seen that the mappings $d_{\mu,p}$ suitably composed (cf. (10-11)) and evaluated on the 'diagonal' (cf. (12-13)) yield the coefficients of power series solutions for polynomial IVP's. We investigate now the analogous construction for the $d^*_{\mu,p}$:

$$(\mu^*)^i := \begin{cases} id & , \text{if } i = 0 \\ d^*_{\mu,m} \circ d^*_{\mu,m^2} \ldots \circ d^*_{\mu,m^i} & , \text{if } i > 0, \end{cases} \quad \text{and} \tag{37}$$

$$\mu^{*\ i} := \begin{cases} id & , \text{if } i = 0 \\ (\mu^*)^i \circ \delta_{m^i} & , \text{if } i > 0. \end{cases}\ . \tag{38}$$

The 'evaluation formula' (19) then shows that $\mu^{*\ i}$ is nothing else than the i-fold composition of the polynomial mapping $p : \mathbb{R}^n \to \mathbb{R}^n$ with itself expressed in terms of multilinear algebra. Note that $\mu^{*\ i}$ corresponds to a homogeneous polynomial of degree m^i.

Remark 3.1: The above constructions are valid more generally, if we replace $\mathbb{R}$ by an arbitrary commutative ring R with unit, and also for arbitrary affine algebraic groups, respective, affine Hopf algebras. For example, let $GL_N(R)$ be the general linear group of order N over R, which is represented by a Hopf algebra with comultiplication $\Delta(T_{ij}) = \sum_{k=1}^N T_{ik} \otimes T_{kj}$ (cf. [Abe, Ex.4.3]); then following the lines of the above construction one substitutes for the N^2 variables $T_{ij},\ i,j = 1,\ldots,N$ the N^2 mappings $\mu_{ij} \in L(T_m\mathbb{R}^n, \mathbb{R}^n)$, such that for the corresponding polynomial mappings $p_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ one has $\det(p_{ij}) \neq 0$.

Remark 3.2: Suppose $\mu, \tilde{\mu} \in L(T_m \mathbb{R}^n, \mathbb{R}^n)$ and $B : \mathbb{R}^n \to \mathbb{R}^n$ a bijective and in general nonlinear 'transformation of coordinates', such that a given μ is transformed to a 'nice' $\tilde{\mu}$ according to $\tilde{\mu} := B \circ \mu \circ \otimes^m B^{-1}$. We choose the letter B, because B is called the "Böttcher function" in the iteration theory of 1-dimensional polynomials, where a given polynomial $p(x) = x^m + \textit{lower order terms}$ is transformed to $\tilde{p}(x) = x^m = B \circ p \circ B^{-1}(x)$. If in the above construction one uses $\tilde{\mu}$ instead of μ, then it is not hard to see (note: $\otimes^p B^{-1} \circ \Delta_p = \Delta_p \circ B^{-1}$), that $\tilde{\mu}^{*\ i} = B \circ \mu^{*\ i} \circ B^{-1}$ and $\tilde{\mu}^i = B \circ \mu^i \circ B^{-1}$.

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