DIAGRAM RULES FOR THE GENERATION OF SCHUBERT POLYNOMIALS

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Abstract: We prove an elegant combinatorial rule for the generation of Schubert polynomials based on box diagrams, which was conjectured by A. Kohnert. The main tools for the proof are (1) a recursive structure of Schubert polynomials and (2) a partial order on the set of box diagrams. As a byproduct we obtain (combinatorial) proofs for two other rules for the generation of Schubert polynomials based on box diagrams: (1) the more complicated rule of N. Bergeron, and (2) the rule of P. Magyar, which we show to be a simplified Bergeron rule.

The well known fact that the Schubert polynomials associated to Grassmannian permutations are in fact Schur polynomials is derived from Kohnert's rule.

To every finite permutation π of natural numbers contained in one of the symmetric groups S_n there is associated a Schubert polynomial $X_\pi \in \mathbb{Z}[x] = \mathbb{Z}[x_1, x_2, \dots]$, such that the collection of all X_π forms a $\mathbb{Z}$-basis of $\mathbb{Z}[x]$. The significance of Schubert polynomials rests mainly on two facts: (1) the ring of Schubert polynomials represents faithfully the ring of cohomology classes of flag manifolds under the cup product, and (2) Schur polynomials are special Schubert polynomials. The theory of Schubert polynomials has been established in a sequence of works by A. Borel (1953), I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand (1973), M. Demazure (1973-74), and finally A. Lascoux and M.-P. Schützenberger (1982-87). Since we discuss below only those properties of Schubert polynomials which are strictly necessary for our presentation, the interested reader may wish to consult as further reading on their basic geometric and algebro-combinatorial properties, e.g., [Hi] and [LS3, Mac1, Mac2, W1].

There are currently many different (and of course strongly interconnected) approaches for the understanding and computation of a Schubert polynomial X_π ($\pi \in S_n$):

(1) The original definition of Lascoux and Schützenberger: X_π can be computed by applying a certain π-dependent sequence of divided difference operators to the 'top'-monomial $x_1^{n-1} \dots x_n^0$. Instead of proceeding from a top-monomial there are also recursive methods, one of which will be of fundamental importance in the present paper.

(2) The recursive generation of X_π without divided differences based on Monk's rule and using the Bruhat order on permutations. There are two variants of this method: (1) the 'transition formula' variant of Lascoux and Schützenberger departing from the identity permutation [LS2] (see also [Bi]), and (2) the 'ascent-descent' variant departing from one of the top permutations $\omega_n := n-1\ n-2\ \dots\ 1$ [W1, Section 6].

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(3) The "generating function" or "generating product" approach using the nil-Coxeter relations [FS].
(4) The formula of S. Billey, W. Jockusch, and R.P. Stanley using the set of reduced words for π [BJS].
(5) The approach via sums of 'mixed shift and multiplication operators' applied to the sequence of variables $(x_1, x_2, x_3, \dots)$ [W3].
(6) The approach via 'balanced labeled tableaux' [FGRS].
(7) The approach via 'configurations of labeled pseudo-lines' [FK] or 'rc-graphs' [BB].
(8) The approach via 'flagged Schur modules' associated to a diagram [KP].
(9) And finally the combinatorial generation via sets of 'box diagrams', which is the theme of the present paper.

Since in the present paper we rely on the divided difference definition 1.), we recall some basic facts about divided differences:

Let f be an arbitrary function of $x_1, x_2, \dots$ and $\sigma_k = (k, k+1)$ an elementary transposition for $k \in \mathbb{N}$, then the divided difference operator ∂_k is defined as

$$\partial_k f = \frac{f - \sigma_k(f)}{x_k - x_{k+1}} \ , \quad \text{where } \sigma_k(f(\dots, x_k, x_{k+1}, \dots)) := f(\dots, x_{k+1}, x_k, \dots).$$

But with regard to the Schubert polynomials we are interested only in the case $f \in \mathbb{Z}[x]$, where it is not hard to see that $\partial_k f$ can be expressed more explicitly as a k-*symmetrisation* of f:

$$\partial_k(x_k^{m+1} x_{k+1}^0) = \sum_{\nu=0}^{m} x_k^{m-\nu} x_{k+1}^{\nu} \quad \text{for } m \in \mathbb{N}_0. \tag{0.1}$$

Note that ∂_k commutes with multipliction by all functions, which are symmetric in x_k and x_{k+1}, and that interchanging x_k and x_{k+1} in the preceding formula introduces a minus sign on the right side, because $\partial_k(\sigma_k(f)) = -\partial_k f$.

There are two main reasons to expect or to look for a combinatorial rule in terms of box diagrams for the generation of Schubert polynomials:

(1) The coefficients appearing in the X_π are non-negative integers and should therefore count some discrete objects.
(2) In the special case of *Grassmannian* permutations $\pi \equiv \pi(\lambda, m)$ associated to partitions of integers $\lambda \equiv \lambda_1 \dots \lambda_s$, $\lambda_1 \geq \dots \geq \lambda_s \geq 0$ $(s \leq m)$ one has

$$X_{\pi(\lambda,m)} = s_\lambda(x_1, \dots, x_m) \ , \tag{0.2}$$

i.e. the Schubert polynomial for $\pi(\lambda, m)$ is equal to the Schur polynomial in $x_1, \dots, x_m$ for λ. Therefore the natural question arises how the well known combinatorial definition of Schur polynomials using semistandard tableaux (cf. [Mac3,Sa] and Section 4) can be extended to Schubert polynomials.

Other combinatorial approaches to Schubert polynomials, especially 6. above, were also inspired by the combinatorics of Schur polynomials.

The very elegant and easily applicable combinatorial rule for the generation of Schubert polynomials, which will be proved in this paper, was first conjectured by A. Kohnert in his 1990 Ph.D. dissertation [Ko] at the University of Bayreuth and we learned about it from Macdonald's article [Mac1]. Its general idea is the following: begin with a *diagram of boxes* $B(\pi)$ associated to the permutation π, which in the case of a Grassmannian permutation $\pi(\lambda, m)$ is almost the Ferrer diagram of λ; than generate a set of box diagrams $\mathbb{K}(\pi)$ by certain *admissible moves of boxes*; finally associate to every box diagram $B \in \mathbb{K}(\pi)$ a monomial term, the sum of which gives X_π. Bergeron has given in [B] a combinatorial rule for the generation of Schubert polynomials, which is of the above type, but with other admissable moves, which are moreover controlled by certain labeling rules. In order to distinguish between the admissable moves as defined by Bergeron and Kohnert we will speak of B-moves and K-moves, respectively. A precise description of both rules will be given in Section 1.

In Section 2 we introduce the main tools for our proof of the K- and the B-rule, namely the recursive structure of Schubert polynomials and the partial order on the sets $\mathbb{K}(\pi)$. It will turn out that the proof of "Kohnert's conjecture" in Section 3 includes as an intermediary step a simplified proof of Bergeron's rule.

In Section 4 we derive equality (0.2) from Kohnert's rule by producing a natural bijection between the semistandard tableaux from the combinatorial definition of Schur polynomials and the set $\mathbb{K}(\pi(\lambda, m))$ of box diagrams.

A closer inspection of our simplified proof of Bergeron's rule shows that in fact the rule itself can be simplified (Theorem 5.1), then being very similar to a combinatorial rule for the generation of Schubert polynomials given recently by Magyar [M3]. Magyar's rule is formulated in terms of words in the 'alphabet' of natural numbers $\mathbb{N}$, but proven by algebro-geometric means. In Section 5 we deduce Magyar's rule from our 'simplified Bergeron rule', thereby making a connection, which can be understood in two directions: on one hand Magyar's results show that Bergeron's rule and our proof of this rule have a natural algebro-geometric underpinning, and on the other hand it makes apparent the possibility to give combinatorial proofs of Magyar's more general results, which are as follows:

To every diagram D (as finite subset of $\mathbb{N}\times\mathbb{N}$) it is possible to associate Specht, Schur, and flagged Schur modules (see e.g. [RS3]), and a result of Kraskiewicz and Pragacz [KP] says that a Schubert polynomial X_π is the character of the flagged Schur module associated to the diagram $D = B(\pi)$ (used in both Kohnert's and Bergeron's rule). Subsequently much of the algebra and combinatorics of Schur and Schubert polynomials has been generalized from diagrams associated to permutations to more general 'northwest' and even more general 'percent-avoiding' diagrams [M1,M2,M3,RS1,RS2,RS3], but there are not as yet fully combinatorial proofs.

1. The rules of Kohnert and Bergeron

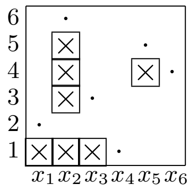
To every permutation $\pi \in S_n$ one can associate its *Lehmer code* $L(\pi) \equiv \overline{l_{n-1} \dots l_0}$ with $l_{n-i}(\pi) := \#\{\ j \mid j > i, \pi(j) < \pi(i)\ \}$, i.e. $l_{n-i}(\pi)$ is the number of 'letters' less than $\pi(i)$ right to the 'place' i. For example: $L(263154) = \overline{141010}$. L is for all n a bijection between the set of permutations S_n and and the set of Lehmer codes $\mathbb{L}_n := \{\ l \equiv \overline{l_{n-1} \dots l_0} \mid 0 \le l_\nu \le \nu,\ \nu = 0, \dots, n-1\ \}$; the inverse L^{-1} is given by the following procedure: $\pi(1)$ is the $(l_{n-1}+1)$-th element of the naturally ordered set $\{1,\ldots,n\}$, $\pi(2)$ the $(l_{n-2}+1)$-th element of $\{1,\ldots,n\}\setminus\{\pi(1)\}$ etc. .

A *box diagram* B is a subset of an $n\times n$-array of unit squares or *boxes* in the plane: $B\subset\{[i,j]\in\mathbb{Z}\times\mathbb{Z}\mid 1\le i,j\le n\}$ for some $n\in\mathbb{N}$. The *position* "row i, column j," will be denoted by (i,j), the *box at position* (i,j) by $[i,j]$. We use the notation $[i,j]\in B$ ($[i,j]\notin B$) as an abbreviation for: B contains (does not contain) the box $[i,j]$.

The *diagram* [1] or *Rothe diagram* $B(\pi)$ of a permutation $\pi\in S_n$ is the box diagram, which originates from $\{[i,j]\mid 1\le i,j\le n\}$ by cancellation of the 'hooks' of boxes

$$\{[\pi(j),j']\mid j'\ge j\}\cup\{[i',j]\mid i'\ge\pi(j)\}$$

for $j=1,\ldots,n$. For example, $\pi=263154$ has the diagram



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where we have added: dots in the positions $(\pi(j),j)$, row numbers $i=1,\ldots,6$ at the left, and variables x_j in columns $j=1,\ldots,6$ at the bottom of the diagram. For the columns we have taken variables instead of numbers in view of the following *evaluation rule*: to every box diagram B one associates a monomial $x^B := x_1^{\beta_1}x_2^{\beta_2}\ldots$, where $\beta_j := |B^{[j]}|$ is the number of boxes in $B^{[j]}$, the column j of B. The most important part of the K-rule is now a prescription, how to move a box $[i,j]$ of a given box diagram B:

Definition 1.1. (of K-moves) *Let $[i,j]\in B$ with $\{(i',j)\mid i'>i\}\cap B=\emptyset$, i.e. there is no box above $[i,j]$ in B, and assume that*

$$M_B(i,j) := \{(i,j')\mid j'<j,\ [i,j']\notin B\}\neq\emptyset\ .$$

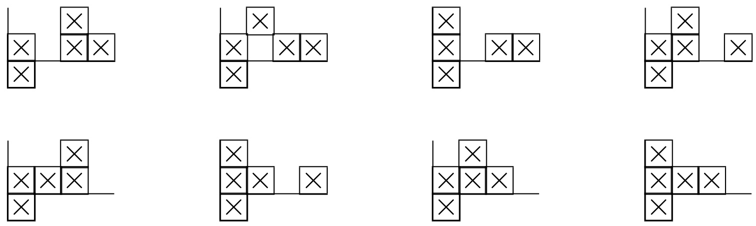
Then $[i,j]$ is allowed to move to the position in $M_B(i,j)$ with the greatest column number j', i.e. the closest empty position left to $[i,j]$ in row i of B.

Theorem 1.2. *Let $\mathbb{K}(\pi)$ denote the set of all box diagrams, which can be derived by (repeated) K-moves from $B(\pi)$; then*

$$(*)\qquad X_\pi=\sum_{B\in\mathbb{K}(\pi)}x^B\ .$$

Example 1.3. $\pi=31542$: $L(\pi)=\overline{20210}$ and $\mathbb{K}(\pi)$ contains the following box diagrams ($B(\pi)$ appears as the first box diagram in reading order; the empty third level has been omitted from all box diagrams):

[1] For the Rothe diagram we use the same convention as in [LLT], which depicts a permutation as a *mapping*.



And indeed: $X_\pi = x_1^2x_3^2x_4 + x_1^2x_2x_3x_4 + x_1^3x_3x_4 + x_1^2x_2^2x_4 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_3 + x_1^3x_2x_3$.

We collect some elementary notions concerning K-moves, which will be useful later:
If a box $[i, j]$ is K-moved to the next empty position (i, j') with $j' = j-1-m$ for some $m \in \mathbb{N}_0$, then we will speak of a $K^m - move$, i.e. m is the number of boxes between box $[i, j]$ and position (i, j'). If $m = 0$, a K^0-move will also be called a *free* K-move; if $m > 0$, we call this kind of move *tunneling* and if the special value of $m > 0$ is not of interest, we denote it by K^t; the m-tuple of traversed boxes $([i, j' + 1], \ldots, [i, j - 1])$ is called the *tunneling array*, and the boxes $[i, j' + 1]$ and $[i, j - 1]$ its *left* and *right boundaries*, respectively.
Sometimes it is convenient to view a K^t-move not as a traversal of the tunneling array, but as a *shunting* move, where all boxes in the *shunting array* ($\equiv$ tunneling array) are pushed one position to the left by the movement of box $[i, j]$ to position $(i, j - 1)$.
A K-move is called *irreducible* if it can not be represented as the composition of two non-identity K-moves. Note that a K^0-move is always irreducible and a K^t-move is irreducible iff no box in the tunneling array is the highest box in its column. Subsequently we will always use irreducible K-moves, if not explicitly stated otherwise.
Boxes of a box diagram B, which can be moved [freely] by a K-move, are called *[freely] movable*, otherwise *non-movable*. The set $\mathbb{K}(B)$ of all box diagrams, which can be derived from B by K-moves is called the K-*derived set of B*. For $B \equiv B(\pi)$ we have already introduced the notation $\mathbb{K}(\pi) := \mathbb{K}(B)$ called the K-*derived set of* π. A non-movable box in B, which remains non-movable in every box diagram $B' \in \mathbb{K}(B)$, is called *inertial*.

In order to describe concisely the B-moves introduced by N. Bergeron and also for later arguments, we introduce some further terms: let B be any box diagram, then for all $k \in \mathbb{N}$ let $L_k(B)$, $M_k(B)$, $R_k(B)$ and $B^{[k]}$ denote the subdiagrams of B consisting of the columns $\{ 1, \ldots, k-1 \}$, $\{ k, k+1 \}$, $\{ k+2, \ldots \}$ and $\{k\}$, respectively. $L_k(B)$, $M_k(B)$ and $R_k(B)$ are called the *k-left, middle, and right part of B*, respectively. Restricting now the attention to $M_k(B)$, we can observe: empty levels, levels with *paired boxes*, i.e. $[i, k] \in B \Longleftrightarrow [i, k + 1] \in B$, and levels with a *left* or *right unpaired box*, i.e. '$[i, k] \in B$ and $[i, k + 1] \notin B$' or '$[i, k] \notin B$ and $[i, k + 1] \in B$'.
In contrast to K-moves, the admissibility of a B-move is controlled not only by the "geometry" of a box diagram B but in addition by a labeling $e_B : B \longrightarrow \mathbb{N}$.

Definition 1.4. *The rule for* **B-moves** *is subdivided into a rule for* forward *(=right to left)-moves, the* B^f*-moves, and* backward *(=left to right)-moves, the* B^b*-moves. Let* $\mathbb{B}(\pi)$ *denote the B-*derived set of π. $B(\pi)$ *has the labeling* $e_{B(\pi)}(i, k) := k$.

Let $k \in \mathbb{N}$ and B be any box diagram with k-middle part $M_k(B)$.
A right unpaired box $[i, k+1]$ can be B^f-moved *to (i, k), if $[i, k+1]$ is the* highest *unpaired box in $M_k(B)$ or the next unpaired box* above *is left unpaired; the labeling of the unmoved boxes and the moved box remain the same.*
A left unpaired box $[i, k]$ can be B^b-moved *to $(i, k+1)$, if 1.) $[i, k]$ is the* lowest *unpaired box in $M_k(B)$ or the next unpaired box* below *is right unpaired, and 2.) $e := e_B(i, k) > k$. In this case all unmoved boxes and the moved box retain their labels except for the boxes in column $k+1$ below row i: if any of these boxes has a label greater than e, then it is changed to e.*

Therefore the labeling rules do not restrict the B^f-moves as allowed by the "geometry" of box diagrams, but they do restrict B^b-moves: they prevent boxes from moving 'behind' (i.e., to the right of) their starting position in $B(\pi)$, and they "trap" certain boxes in lower rows.

Note that for a given box diagram B the K-movabilty must be checked only for the highest box in every column in a very simple fashion, whereas the B-movability must be checked for all unpaired boxes relative to every two-column subdiagram $M_k(B)$. This clearly demonstrates the superiority of K-moves for the generation of Schubert polynomials.

Theorem 1.5. *Let $\mathbb{B}(\pi)$ denote the set of all box diagrams, which can be derived by (repeated) B-moves from $B(\pi)$; then*

$$(**) \qquad X_\pi = \sum_{B \in \mathbb{B}(\pi)} x^B \ .$$

Corollary 1.6. $\mathbb{K}(\pi) = \mathbb{B}(\pi)$.

Theorem 1.2 has been conjectured by Kohnert [Ko], who proved it for so called vexillary permutations (cf.[Mac1,Mac2]) – which make up a geometrically vanishing proportion of permutations of S_n for growing n; Theorem 1.5 has been proved by Bergeron [B], who also verified by computer to some extent that $\mathbb{K}(\pi) = \mathbb{B}(\pi)$. Subsequently we will show the validity of Kohnert's construction and as a byproduct give a simplified proof of Bergeron's result.

Remark 1.7. : The assertion "$\mathbb{K}(\pi) \subset \mathbb{B}(\pi)$ for all π", which is clearly true as a consequence of Corollary 1.6, cannot be shown *in general* as easly as suggested in [B, p.181], but under the *special* circumstances of Lemma 3.10 the arguments of [B] as described below apply:

The assertion is equivalent to "every K^m-move can be represented by a sequence of B-moves "; idea: induction over m. A K^0-move is clearly a special B^f-move. Now for $m > 0$ let $[i, k+1]$ be the left boundary of a tunneling array, i.e. $[i, k+1]$ is right unpaired in $M_k(B)$ for some B; assume further some more right unpaired boxes $[i_1, k+1], \ldots, [i_s, k+1]$ ($s > 0$ and $i < i_1 < \cdots < i_s < i'$) above $[i, k+1]$ and below the next left unpaired box $[i', k]$ (which need not exist), otherwise there is no difficulty to complete the argument. So one has first to B^f-move the boxes $[i_s, k+1], \ldots, [i_1, k+1]$, before moving $[i, k+1]$ to (i, k). Next by induction the tunneling of a box to $(i, k+1)$ can be simulated and it remains to B^b-move the boxes $[i_1, k], \ldots, [i_s, k]$. This can be achieved only if the next unpaired box below level i_1 – $[i,k]$ is paired now ! – is not left unpaired. But this condition is clearly irrelevant for the execution of the K^m-move. On the other hand under the conditions of Lemma 3.10 it can be guaranteed that the “next unpaired box below level i_1 is not left unpaired”.

2. Long induction and the poset structure on $\mathbb{K}(\pi)$

Every attempt to prove Theorem 1.2 faces the imminent danger “to get lost in boxes”. In this section we therefore introduce two devices, which cut down considerably the complexity of our task: *long induction* over all finite permutations of $\mathbb{N}$ using the recursive structure of Schubert polynomials (cf. [W1, Section 3]) and a *poset structure on* $\mathbb{K}(\pi)$. (For the basics on posets(=partially ordered sets) see e.g. [DP] or [St]).

We begin with *long induction.* In [W1] it has been established that many properties of Schubert polynomials (as objects indexed by permutations) can be proved by showing first that they hold in the case of $\pi = 1 \equiv id_1 \in S_1$, which is trivial in general, and then that these properties are preserved under steps of the following two kinds:

(1) (+)-*steps*: for arbitrary $\pi \in S_n$, $(n \in \mathbb{N})$ let $\pi' := 1_+(\pi)\ 1 \in S_{n+1}$, i.e. $\pi' \equiv (\pi(1)+1)\ \ldots(\pi(n)+1)\ 1$. For the Lehmer codes this means similarly $L(\pi') = 1_+(\ L(\pi)\ )\ \overline{0} \equiv \overline{(l_{n-1}(\pi)+1)\ldots(l_0(\pi)+1)\ \ 0}$; and for the corresponding Schubert polynomials: $X_{\pi'} = x_1x_2\ldots x_n \cdot X_\pi$.

(2) (∂)-*steps*: let $\pi \in S_n$, $(n \in \mathbb{N})$ be such that $k+1$ is the place of 1 in π for some $k \in \{1,\ldots,n-1\}$, i.e. $\pi(1) \neq 1$, and $s := \pi(k) > 1$; then $\pi' := \pi\sigma_k \in S_n$, i.e. $\pi'(\nu) = \pi(\nu)$ for all $\nu \neq k, k{+}1$, $\pi'(k) = 1$ and $\pi'(k{+}1) = s$. In terms of Lehmer codes $L(\pi) \equiv \overline{l_{n-1}\ldots l_0}$ this means: $L(\pi) = \overline{\ldots\ l_{n-k}(\pi)\ 0\ \ldots}$ and $L(\pi') = \overline{\ldots\ 0\ (l_{n-k}(\pi)-1)\ \ldots}$; and for the corresponding Schubert polynomials: $X_{\pi'} = \partial_k X_\pi$.

Lemma 2.1. *Properties* $(*)$ *and* $(**)$ *of Theorems 1.2 and 1.5 are preserved under (+)-steps.*

Proof. Assume properties $(*)$ and $(**)$ for some π. $B(\pi')$ is the same as $B(\pi)$ except for an additional lowest level of n boxes. This lowest level contains only inertial boxes, all other boxes move exactly as in the case of $B(\pi)$. Therefore $\mathbb{K}(\pi')$ is the same as $\mathbb{K}(\pi)$ [and $\mathbb{B}(\pi')$ is the same as $\mathbb{B}(\pi)$], where every box diagram of $\mathbb{K}(\pi)$ $[\mathbb{B}(\pi)]$ is supplemented by this lowest level of n boxes. But because $X_{\pi'} = x_1x_2\ldots x_n \cdot X_\pi$ this already establishes properties $(*)$ and $(**)$ for π'. □

To establish that properties $(*)$ and $(**)$ are preserved also under (∂)-steps is the hard part of the proof, which will be deferred to the next section. The *general idea of the proof* is the following: if the divided difference operator ∂_k is applied to the monomials of X_π (note: k is fixed by π), then $\partial_k x^d \equiv \partial_k(\ldots x_k^{d_k} x_{k+1}^{d_{k+1}} \ldots)$ is a sum with positive coefficients, zero, or a sum with negative coefficients, if $d_k > d_{k+1}$, $d_k = d_{k+1}$, or $d_k < d_{k+1}$, respectively. We will subdivide the set $\mathbb{K}(\pi) = \mathbb{B}(\pi)$ in such a way that a ‘combinatorial’ analog ∂_k of the ‘algebraic’ k-symmetrisation ∂_k (Definition 3.2) has to be applied only to certain box diagrams $B \in \mathbb{K}(\pi)$ with the property $|B^{[k]}| > |B^{[k+1]}|$, and that the remaining set of box diagrams (containing all the "critical" box diagrams with the property $|B^{[k]}| \le |B^{[k+1]}|$) can be neglected. This will be essentially a consequence of our second tool:

Definition 2.2. (Poset structure on $\mathbb{K}(\pi)$ *Let $B, B' \in \mathbb{K}(\pi)$ for some π be given. The partial order on $\mathbb{K}(\pi)$ is then completely determined by defining " B covers B' " if B' originates from B by an irreducible K-move, or if B' originates from B by moving with a forward B-move exactly one box $[i,j] \in B$, which is* not *the highest box in $B^{[j]}$, to the empty position $(i, j-1) \in B$. Movements of the last kind will be called* improper K^0-moves.

We emphasize that $\mathbb{K}(\pi)$ contains only box diagrams derived by (proper) K-moves according to Definition 1.1, and that the improper K^0-moves come into play only in defining the covering relation for the already existing set $\mathbb{K}(\pi)$. Subsequently we will usually tacitly assume the poset structure of Definition 2.2, when speaking of $\mathbb{K}(\pi)$ (or $\mathbb{B}(\pi)$).

Figures 1 and 2 and also Example 2.11 below show the Hasse diagrams of some posets $\mathbb{K}(\pi)$. (For typographic reasons we took circles instead of boxes. The different types of covering relations are indicated by different styles of lines.)

Next we show that $\mathbb{K}(\pi)$ is a *pseudoranked poset*; the different levels in the Hasse diagrams indicate the *pseudorank* $R(B)$ of the box diagrams B. We use the prefix 'pseudo' to indicate that the 'rank' of a box diagram B in $\mathbb{K}(\pi)$ is not determined solely by the partial order, but by the 'internal structure' of the box diagrams, too. Let

$$W(B) := |B^{[1]}| + 2 \cdot |B^{[2]}| + 3 \cdot |B^{[3]}| + \ \dots$$

denote the *column weight* of any box diagram B. Below we will see that every $\mathbb{K}(\pi)$ contains a unique minimal element: the *bottom element* $B_\perp \equiv B_\perp(\pi)$. Finishing the definition of the pseudorank for all $B \in \mathbb{K}(\pi)$ we set

$$R(B) := W(B) - W(B_\perp).$$

Remark 2.3. An immediate consequence of the definition of the covering relation is that $B' < B$ in $\mathbb{K}(\pi)$ implies $W(B') < W(B)$ and $x^{B'} > x^B$, where for monomials $m', m \in \mathbb{Z}[x]$ the relation "$m' > m$" is defined as: "the exponent tuple of m' is greater then the exponent tuple of m in the lexicographic order induced by the usual linear order on $\mathbb{N}_0$". The reverse implications are of course valid only if B' and B are comparable in $\mathbb{K}(\pi)$.

Remark 2.4. The pseudorank is a good tool to organize the computation of $\mathbb{K}(\pi)$: put all box diagrams derivable from an already constructed B into sets consisting of box diagrams with equal pseudorank; begin with $\{B(\pi)\}$ and work downwards through all levels.

Next we will investigate the poset structure of an $\mathbb{K}(\pi)$ in the light of the (+)- and (∂)-steps of long induction. Recall that a *principal order ideal* of a poset P is defined as a *down set* of P generated by one element $p \in P$, i.e. as the subposet $\downarrow p := \{ \ p' \in P \mid p' \le p \ \}$ of P.

Theorem 2.5. *Let π' originate from π by a (+)- or a (∂)-step. Then there exists $B \in \mathbb{K}(\pi')$ such that $\mathbb{K}(\pi)$ is isomorphic to the principal order ideal of $\mathbb{K}(\pi')$ generated by B: $\mathbb{K}(\pi) \cong\downarrow B$.*

Proof. First assume π and π' as in the description of the (+)-step above. Then from the definition of a (+)-step and the proof of Lemma 2.1 it is clear that $\mathbb{K}(\pi) \cong\downarrow B(\pi') = \mathbb{K}(\pi')$.

Now assume π and π' as in the description of the (∂)-step above. For $B(\pi)$ this means that column $k+1$ is empty, $[1,1],\dots,[1,k] \in B(\pi)$ and possibly $r > 0$ further boxes in column k above $[1,k]$; for $B(\pi')$ it means that column k is empty, $[1,1],\dots,[1,k-1] \in B(\pi')$, and the r boxes of $B(\pi)^{[k]}$ above level 1 are now on the same levels in $B(\pi')^{[k+1]}$. Symbolically we express this relationship between $B(\pi)$ and $B(\pi')$ as : $B(\pi') = \partial_k^0 B(\pi)$.

More generally for any $B \in \mathbb{K}(\pi)$ we have $[1,1],\dots,[1,k] \in B$, because these boxes are inertial. We denote by $\overline{B}$ the box diagram, which originates from B by removing the box $[1,k]$, and by $\overline{S}$ the set $\{ \ \overline{B} \mid B \in S \ \}$ for any subset S of $\mathbb{K}(\pi)$.

With $r = |B(\pi')^{[k+1]}| \geq 0$, a chain of r free K-moves leads in $\mathbb{K}(\pi')$ from $B(\pi')$ to $\overline{B(\pi)}$, which shows that $\mathbb{K}(\pi) \cong\downarrow \overline{B(\pi)}$. □

Definition 2.6. *Let S_∞ be the union of all permutations of all sets $\{1,\dots,n\}$, where every S_n is identified with the stabilizer of $(n+1)$ in S_{n+1}. We define a partial order on S_∞ through the covering relation: " $\pi' \succ \pi$" iff " π' originates from π by a (+)- or (∂)-step".*

From the definition of these steps for permutations it is clear that every $\pi \in S_\infty \setminus \{1\}$ covers exactly one other permutation, whence S_∞ is a infinite rooted tree with root $id = 1$; consequently there is for every π in the poset S_∞ a unique descending chain $C(\pi) : \pi_0 := \pi, \pi_1, \pi_2, \dots \ , id$. S_∞ is a (complete) meet-semi lattice; for example: $241653 \ \wedge \ 426513 = 31542$ and $12543 \ \wedge \ 15432 = 321$. Therefore:

Corollary 2.7. *Two arbitrary posets $\mathbb{K}(\pi_1)$ and $\mathbb{K}(\pi_2)$ ($\pi_1, \pi_2 \in S_\infty$) always contain principal order ideals, which are both isomorphic to $\mathbb{K}(\pi_1 \wedge \pi_2)$.*

Remark 2.8. Corollary 2.7 and the above example $12543 \ \wedge \ 15432 = 321$ explain why the posets $\mathbb{K}(\pi)$ in Figs.1 and 2 are so different: $\mathbb{K}(321)$ is the (trivial) one-element poset.

Theorem and Definition 2.9. *Every $\mathbb{K}(\pi)$ contains a unique minimal bottom element $B_\perp$. The numbering of the boxes in each row of $B(\pi)$ from left to right with the natural numbers $1, 2, \dots$ will be called the natural numbering of $B(\pi)$. Shifting all boxes of $B(\pi)$ to the columns indicated by the natural numbering yields $B_\perp$; moreover $B_\perp$ can be generated from $B(\pi)$ by a sequence of free moves and $W(B_\perp)$ is simply the sum of all entries of the natural numbering.*

Proof. Fix some $\pi \in S_\infty$ and recall the existence of the unique chain $C(\pi) : \pi_0 \equiv \pi, \pi_1, \pi_2, \dots, id$ in S_∞. Theorem 2.5 then guarantees the existence of a sequence of *different* box diagrams $B_0(\pi) \equiv B(\pi), B_1(\pi), B_2(\pi), \dots, B_s(\pi) := B_\perp$ contained in $\mathbb{K}(\pi)$, such that every $B_\nu(\pi)$ corresponds to some $B(\pi_i)$ with $\mathbb{K}(\pi_i) \cong\downarrow B_\nu(\pi)$ and $B_{\nu-1}(\pi)$

originates from $B_\nu(\pi)$ by a nonempty sequence of free moves. We call these box diagrams $B_\nu(\pi)$ the *principal box diagrams* and the unique chain of free moves connecting them the *principal chain.*

Now for $\pi \in S_{n+1}$ the box diagram $B(\pi)$ of π is contained in the $n \times n$-square: the $(n+1)^{th}$ column is empty, because $l_0(\pi) = 0$ for all π, and the $(n+1)^{th}$ level is empty, because for every column ν ($\nu \in \{1, \ldots, n\}$) at most $l_{n+1-\nu} \leq n+1-\nu$ boxes are piled up on $n+1-(\nu-1)$ possible levels.

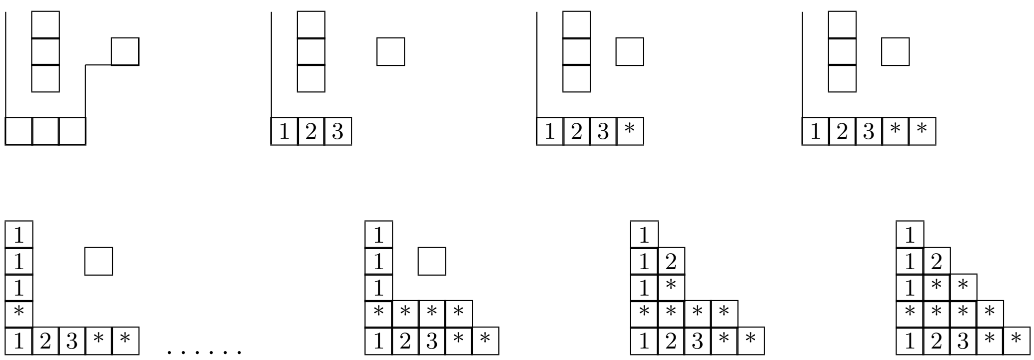
Let column k be the first empty column in $B(\pi)$. We can assume $k \geq 2$, because otherwise the whole box diagram can be shifted columnwise by free moves to the left without affecting the natural numbering. Then $B(\pi)$ contains the (inertial) boxes $[1,1], \ldots, [1, k-1]$, which we number with their column number. As long as $k < n$ we perform inverse ∂_k^0-steps (cf. proof of Theorem 2.5), i.e. shift the boxes of column $k+1$ (freely) to column k and add a *virtual box* $[1, k]$, thereby creating one of the principal box diagrams $B_\nu(\pi_i)$. In the case $k = n$ we only add a virtual box $[1, n]$.

If level 1 is filled with n (original boxes of $B(\pi)$ or virtual) boxes, continue the above procedure on the $(n-1) \times (n-1)$-square subdiagram of levels 2 to n and columns 1 to $n-1$.

Proceeding in this way eventually leads to a numbered box diagram, which allows no further movement, i.e. which is $B_\perp$ with all (non-virtual) boxes numbered by their column numbers. Reversing all free moves along the principal chain gives the natural numbering of $B(\pi)$ with the desired properties.

It remains to be shown that $B_\perp$ is indeed the bottom element of $\mathbb{K}(\pi)$. Assume that there is some $B \in \mathbb{K}(\pi)$ with $B \not\geq B_\perp$. Without loss of generality one can assume further that B is minimal, i.e. there is no further K-move possible. Since B is different from $B_\perp$, there must exist somewhere an empty position (i, j) in B such that $[i, j+1] \in B$ and such that there is no (irreducible) K-move possible, which closes this "gap" (in $B_\perp$ no such gap exists). This occurs when all possible boxes $[i, j'] \in B$ with $j < j'$ are fixed by boxes in higher rows and these boxes cannot be K-moved further. Hence (1) not all places (i', j) with $i < i'$ can be free and (2) this must have been so in $B(\pi)$ already, since otherwise the gap could have been closed. But this is a contradiction to the definition of a Rothe diagram and $B_\perp$ is the bottom element of $\mathbb{K}(\pi)$. □

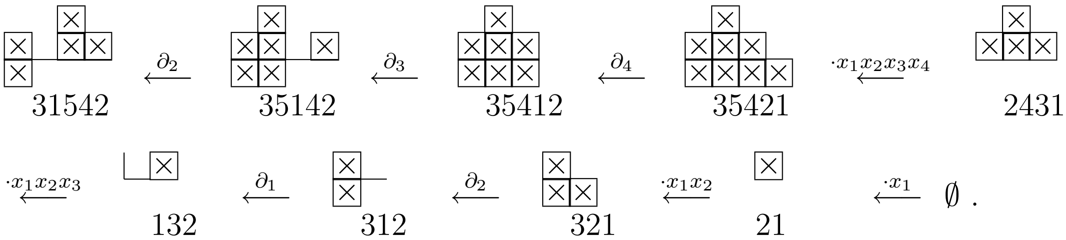
Example 2.10. $\pi = 263154$: We begin with $B(\pi)$ and illustrate the procedure of the proof of Theorem 2.9 by giving most intermediary diagrams. Virtual boxes are marked by an star.



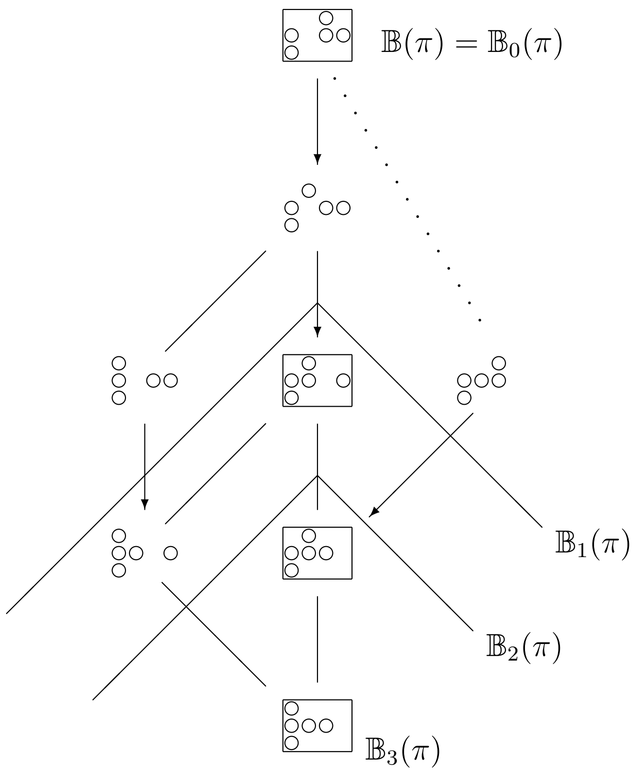
$W(B_\perp) = 11$, $W(B(\pi)) = 17$, and therefore $R(B(\pi)) = 6$. Note that $B_\perp$ is a subdiagram of $B(654321)$.

In order to illustrate the definitions and results of this section we continue the discussion of Example 1.3 with the following:

Example 2.11. Let again $\pi = 31542$. We first depict the diagrams of the sequence of permutations from π down to the identity in S_∞ with respect to the partial order of Definition 2.6. The arrows between the diagrams wear as superscripts the operations needed for the recursive computation of the respective Schubert polynomials.



In short: $X_\pi = \partial_2\partial_3\partial_4 x_1x_2x_3x_4x_1x_2x_3\partial_1\partial_2x_1x_2x_1(1)$, where the operations are applied successively from right to left, and indeed one computes in this way $X_\pi = x_1^2x_3^2x_4 + x_1^2x_2x_3x_4 + x_1^3x_3x_4 + x_1^2x_2^2x_4 + x_1^2x_2x_3^2 + x_1^3x_2x_4 + x_1^2x_2^2x_3 + x_1^3x_2x_3$. The poset $\mathbb{K}(\pi)$ of K-derived box diagrams is seen to be:



with principal box diagrams in frames. Recall from the proof of Theorem 2.9 that the principal box diagrams are the former top elements of the posets $\mathbb{B}(\pi')$ for π' prior to π w.r.t. the recursive structure, such that order-isomorphic copies of the $\mathbb{B}(\pi')$ are included now as principal order ideals in the posets $\mathbb{B}(\pi)$. In the example above there are 4 principal box diagrams and order ideals: the one element order ideal $\mathbb{B}_3(\pi)$ represents the permutations *id*, 21, 321, and 312; the two element order ideal $\mathbb{B}_2(\pi)$ represents the permutations 132, 2431, 35421, and 35412; the order ideal $\mathbb{B}_1(\pi)$ represents 35142; and the whole poset is of course $\mathbb{B}(\pi) = \mathbb{B}_0(\pi)$ for $\pi = 31542$.

Furthermore the 2-partition (see Proposition 3.6 below) of all box diagrams, which appear as new ones in the step from $\mathbb{B}(35142)$ to $\mathbb{B}(31542)$, is indicated by chains of arrows, which point to certain 'surface elements' of the principal order ideal $\downarrow(35142) = \mathbb{B}_1(\pi)$ in $\mathbb{B}(31542)$. Notice that all these new box diagrams are generated by backward Bergeron moves for the columns 2 and 3 from 'surface elements' B with $\Delta_2(B) = 1$.

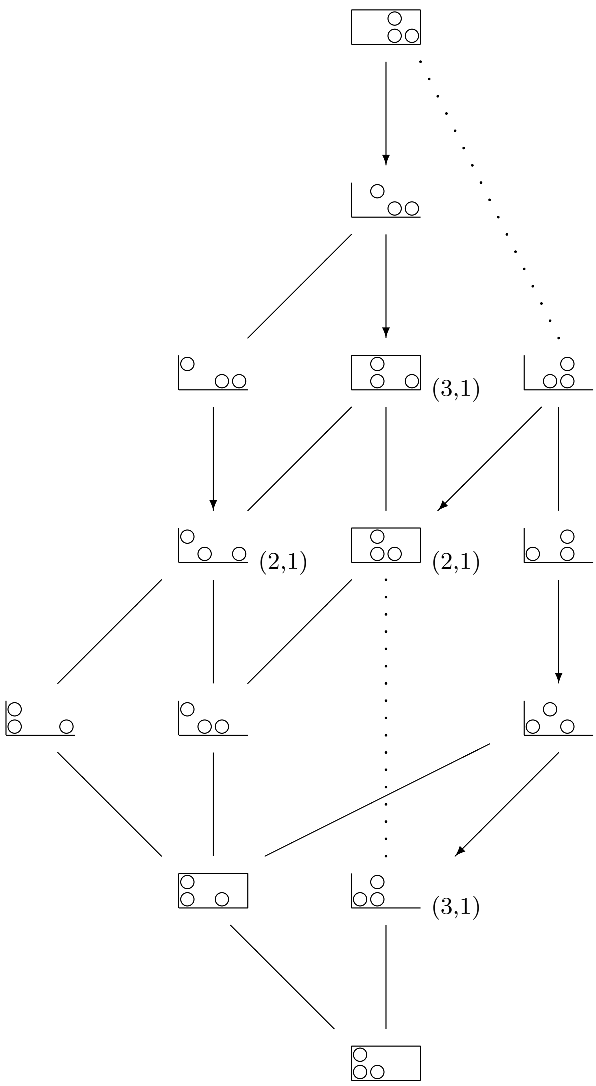


Figure 1: $\mathbb{B}(\pi)$ for $\pi = 12543$. $L(\pi) = \overline{00210}$ and $X_\pi = x_3^2x_4 + x_2x_3x_4 + x_1x_3x_4 +$

$x_2^2x_4 + x_2x_3^2 + x_1x_2x_4 + x_2^2x_3 + x_1x_3^2 + x_1^2x_4 + 2x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2$.
(K^0-move = normal line, K^t-move = punctured line; principal box diagrams = boxed; the 2-partition of $\tilde{V}(21543)$ is indicated by the arrows; the pairs of numbers show $(\Delta_2(\_), up(2, \_))$ for the respective box diagrams, which should be viewed as complemented with boxes $[1, 1]$, $[1, 2]$.)

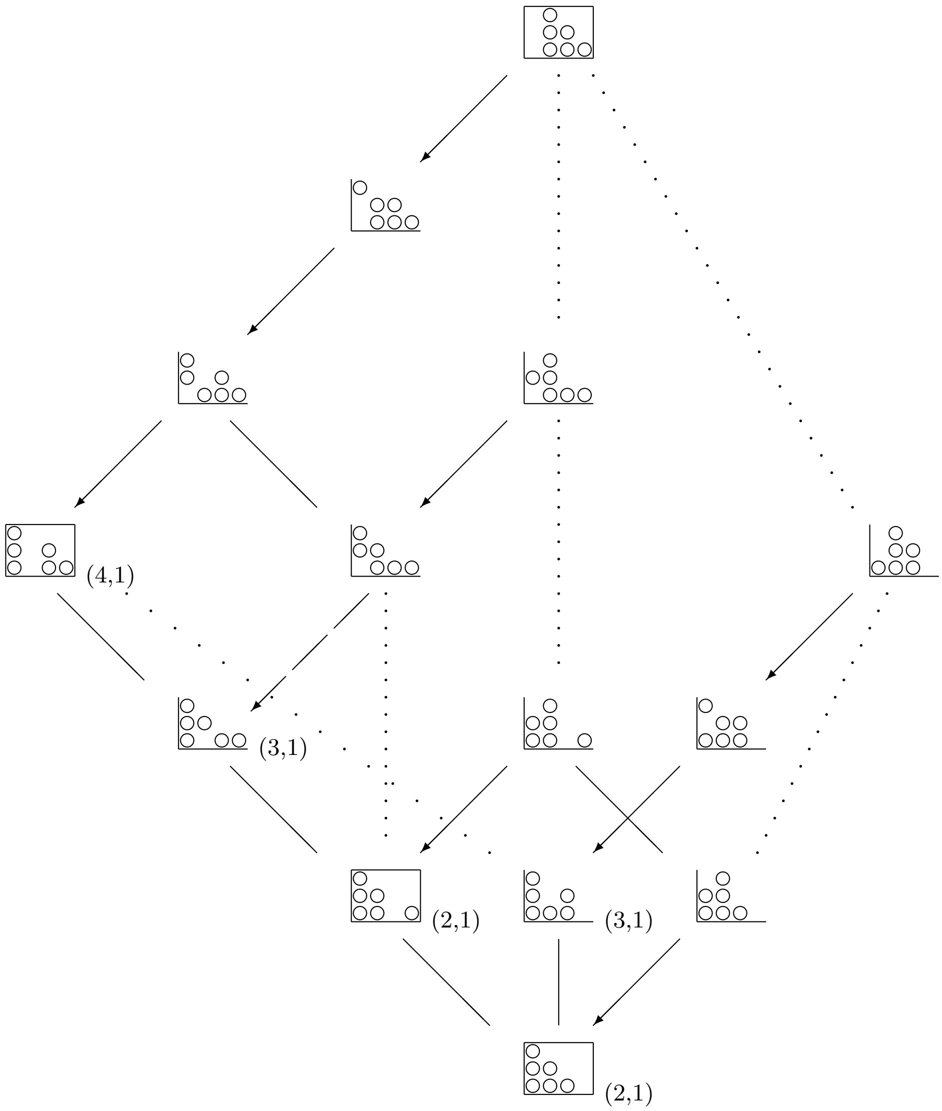


Figure 2: $\mathbb{B}(\pi)$ for $\pi = 15432$. $L(\pi) = \overline{03210}$ and $X_\pi = x_2^3x_3^2x_4 + x_1x_2^2x_3^2x_4 + x_1^2x_2x_3^2x_4 + x_1x_2^3x_3x_4 + x_1^3x_3^2x_4 + x_1^2x_2^2x_3x_4 + x_1x_2^3x_3^2 + x_1^3x_2x_3x_4 + x_1^2x_2^3x_4 + x_1^2x_2^2x_3^2 + x_1^3x_2^2x_4 + x_1^3x_2x_3^2 + x_1^2x_2^3x_3 + x_1^3x_2^2x_3$.
(K^0-move = normal line, K^t-move = punctured line, improper K^0-move = dashed line; principal box diagrams = boxed; the 1-partition of $\tilde{V}(51432) = \mathbb{B}(15432)$ is indicated by the arrows; the pairs of numbers show $(\Delta_1(\_), up(1,\_))$ for the respective box diagrams, which should be viewed as complemented with a box $[1,1]$.)

3. Proof of the Main Theorems

First we give an informal outline of the proof with the aid of the schematic Fig.3 (for "realistic" details see Figs.1 and 2):

Since the properties $(*)$ and $(**)$ are preserved under $(+)$-steps (Lemma 2.1), it remains to be shown that they are preserved also under (∂)-steps. The overall assumptions in this section are therefore: $n \in \mathbb{N}$, $\pi \in S_n$ with $1 \leq k \leq n-1$ such that $\pi(k+1) = 1$, i.e. k is fixed by the choice of π, and $\pi' := \pi\sigma_k$. We have to show that the 'algebraic' equation $X_{\pi'} = \partial_k X_\pi$ remains true in terms of our combinatorial rule, i.e., if we replace $X_{\pi'}$ by $\mathbb{K}(\pi')$, X_π by $\mathbb{K}(\pi)$, and the algebraic operator ∂_k by some yet to be defined combinatorial analog, which we denote by ∂_k, too.

In Section 2 we have seen that $\mathbb{K}(\pi)$ (the empty triangle of Fig.3) is a pseudoranked poset, which is contained (with boxes $[1,k]$ removed from all box diagrams) as an principal order ideal in $\mathbb{K}(\pi')$ (the whole shape of Fig.3). Assume now that this containment is proper – otherwise the necessary (∂)-step will turn out to be very easy –, then we will see (1) that we must consider only the newly generated box diagrams, and (2) that these new box diagrams can be organized (Definition 3.2 and Proposition 3.6 below) into chains (the vertical lines of Fig.3) with bottom elements (= big dots in Fig.3) in $\overline{\mathbb{K}(\pi)}$ (recall from the proof of Theorem 2.4 that $\overline{B} \in \mathbb{K}(\pi')$ is B from $\mathbb{K}(\pi)$ with the box $[1,k]$ removed). *Every such chain stands for the sum of terms of r.h.s.(0.1), its bottom element for the first term of this sum, and the corresponding box diagram in* $\mathbb{K}(\pi)$, *i.e. the same box diagram with an additional box* $[1,k]$, *to such a bottom element stands for the term of l.h.s.(0.1).*

The set of all these chains forms a partition of $\mathbb{K}(\pi) \setminus \overline{\mathbb{K}(\pi)}$ "parallel" to the principal chain (= double vertical line between $B(\pi)$ and $B(\pi')$ in Fig.3). We call the box diagrams in $\mathbb{K}(\pi)$ corresponding to the bottom elements of all these chains the *surface elements* (= big dots in Fig.3 and the elements of $V_0(\pi)$ in the notation below) of $\mathbb{K}(\pi)$, from which the chains of new box diagrams "sprout" by application of our combinatorial k-symmetrisation ∂_k. Since backward B-moves are perfectly suited to this sprouting, Bergeron's rule follows. For Kohnert's rule the situation is more complicated: in Lemma 3.9 we show that given a chain with bottom point B, the irreducible K-move from B to say C in $\mathbb{K}(\pi)$ "lifts" to all other points in the respective chains; or in other words, if the chain $\partial_k B$ can be generated from the principal chain by K-moves, the same follows for the "neighborly" chain $\partial_k C$.

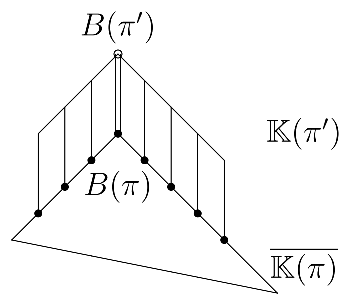


Figure 3

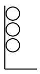
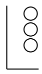
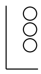
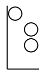
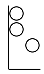
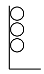
Our first task is now to find a combinatorial analog for the k-symmetrisation operator ∂_k. The set of box diagrams along the principal chain provides a good example of how to proceed. Let $B \in \mathbb{K}(\pi)$ with $\Delta_k(B) := |B^{[k]}| - |B^{[k+1]}| \geq 1$, then $\partial_k(B)$ is computed by the following (still incomplete) prescriptions:

(1) Remove the box $[1, k]$; this gives $\overline{B}$.

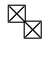
(2) Shift exactly $\Delta_k(B) - 1$ boxes (in their respective rows) from column k of $\overline{B}$ to column $k + 1$. The result of the operations done so far is denoted by $\partial_k^0 B$ (compare the proof of Theorem 2.4).

(3) Move the boxes, which have been shifted in step 2 to column $k + 1$, one by one back to column k proceeding from the top level downwards, thereby getting a set $\partial_k(B)$ of box diagrams of cardinality $\Delta_k(B)$, which obviously includes $\overline{B}$ and $\partial_k^0 B$.

Example 3.1. Let B be a box diagram with $|B^{[k]}| = s \geq 1$ and $|B^{[k+1]}| = 0$. This is the case for example, when B is a principal box diagram. To avoid trivialities let $s \geq 2$, e.g. $s = 4$. Then one has (depicting only the k-middle parts)

$B =$  , $\overline{B} =$  , $\partial_k^0 B =$  , and $\partial_k B = \{$  ,  ,  ,  $\}$.

Clearly this procedure gives for $B = B(\pi)$ the box diagrams of the principal chain and for general $B \in \mathbb{K}(\pi)$ it properly reflects the algebraic definition of k-symmetrisation — as long as $\Delta_k(B) \geq 1$! It will turn out that we need not consider the case $\Delta_k(B) \leq 0$ (thereby circumventing the problems of an appropriate definition of ∂_k in this case and of proving something like [B, Proposition 1.4]). But with respect to step 2 above of course the problem arises, *which $\Delta_k(B) - 1$ boxes are to be selected in general* ? This difficult problem has been solved by Bergeron ([B, p.172 f.]) through the following definition:

Definition 3.2. *Let $B \in \mathbb{K}(\pi)$ with $k := \pi^{-1}(1) - 1 \geq 1$ and $\Delta_k(B) \geq 1$. (This means that every $B \in \mathbb{K}(\pi)$ contains the inertial boxes $[1, 1], \ldots, [1, k]$ and no further boxes on level 1.) Let $i_1, i_2, \ldots$ with $1 < i_1 < i_2 < \ldots$ be the list of levels greater 1 of $M_k(B)$, on which unpaired boxes occur. One then repeatedly removes all* 'fixed pairs', *i.e. consecutive numbers $i_\nu, i_{\nu+1}$ with $[i_\nu, k], [i_{\nu+1}, k+1] \in M_k(B)$, from this list until there is no more fixed pair. The resulting list is denoted by $f(k, B) \equiv (f_1, \ldots, f_r)$ with $1 < f_1 < \cdots < f_r$. We have chosen the name 'fixed pair', because the "fixed configuration" of boxes*  *is exactly opposed to the configuration*  *, which allows B-moves of both boxes. Note that by the process of construction:*

$$[f_\nu, k] \in B \quad \Longrightarrow \quad [f_{\nu+1}, k] \in B \ .$$

We can therefore define

$$up(k, B) := \min\{ \ \nu \mid [f_\nu, k] \in B \ \} \ ,$$

if the r.h.s. set is nonempty; if $\{ \ \nu \mid [f_\nu, k] \in B \ \} = \emptyset$, then there are two cases:
$up(k, B) := r + 1$, if all unpaired non-fixed boxes occur in column k+1, and
$up(k, B) := 0$, if there does not occur any unpaired non-fixed box.
If $up(k, B) = 0$, then set $\partial_k B := \{\overline{B}\}$.

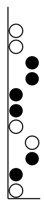
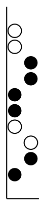
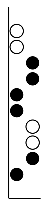
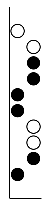
If $up(k,B) \geq 1$, *then one observes again that the unpaired non-fixed boxes in* $M_k(B)$ *are right unpaired on levels* $(f_1, \ldots, f_{up(k,B)-1})$ *and left unpaired on levels* $(f_{up(k,B)}, f_{up(k,B)+1}, \ldots)$. *Then* $\partial_k B$ *is the set containing the following* $\Delta_k(B)$ *box diagrams:*
$B_{up(k,B)-1} := \overline{B} = B \setminus \{ \ [1,k] \ \}$ *and recursively*

$$B_{up(k,B)+i} := B_{up(k,B)+i-1} \cup \{ \ [f_{up(k,B)+i}, k+1] \ \} \setminus \{ \ [f_{up(k,B)+i}, k] \ \} \tag{3.1}$$

for i *running from 0 to* $\Delta_k(B) - 2$.

Note, that $\partial_k^0 B = B_{up(k,B)+\Delta_k(B)-2}$ and $up(k, B_m \cup \{[1,k]\}) = m+1$ for all $B_m \in \partial_k B$. Note further, that the 'combinatorial' definition of the k-symmetrisation ∂_k is done without reference to the allowed moves, but that B-moves are clearly defined in such a way that $\partial_k B$ as a whole can be B-derived easily from every one of its elements; this makes the proof of $(**)$ much easier than that of $(*)$.

Example 3.3. We investigate the meaning of the above definition for a box diagram B, whose k-middle part without the empty and paired levels we have depicted below. The fixed boxes are printed as filled circles. $\Delta_k(B) = 3$, $up(k,B) = 2$, and $\partial_k B = \{B_1, B_2, B_3\}$.

$B =$  , $\overline{B} = B_1 =$  , $B_2 =$  , and $B_3 = \partial_k^0 B =$  .

Remark 3.4. It is very likely that the above (partially) defined combinatorial operators ∂_k are the only ones, which satisfy the nil-Coxeter relations

(i) $\partial_k \partial_k = 0$, (ii) $\partial_k \partial_{k'} = \partial_{k'} \partial_k$, if $|k - k'| \geq 2$, (iii) $\partial_k \partial_{k+1} \partial_k = \partial_{k+1} \partial_k \partial_{k+1}$,

and which are thereby the "correct" ones. Relation (i) cannot be checked, because we did not say what $\partial_k B$ means in case of $\Delta_k(B) \leq 0$ (but see [B]). Relation (ii) is trivially true, and for relation (iii) one observes that the transition from $\overline{B}$ to $\partial_k^0 B$ is actually a variant of the 'plactic action on words' of Lascoux and Schützenberger [LS1] in disguise (see Definition 5.2 below), which has been proved in [LS1] to obey relation (iii). The connection between the 'plactic action on words' and the operators ∂_k has been first pointed out in [W3, Section 4], where the 'plactic action' on semistandard Young tableaux was investigated. We can understand the combinatorial operators ∂_k as an 'interpolated plactic action' on box diagrams (Definition 5.3).

Remark 3.5. Bergeron has overlooked the possibility: $up(k,B) = 0$. As examples show (e.g. ∂_2 on $\mathbb{K}(L^{-1}(\overline{13010}))$, c.f. Fig.1), box diagrams D with $up(k,D) = 0$ must be added to set $\Omega_0(v)$ defined on p.175 of [B] in order to retain the validity of assertion [B, Proposition 1.3], but this does not affect the correctness of the proof in principal.

By the induction hypothesis we have that $(*)$ and $(**)$ are true for π, and that $\mathbb{B}(\pi) = \mathbb{K}(\pi)$. We show first that $(**)$ and later that $(*)$ are true for π', too.

For $\Delta_k(B) = 1$ one has $\partial_k(B) = \overline{B}$; hence in this case the (∂)-step is rather simple: $B(\pi') = \overline{B(\pi)}$ and consequently $\mathbb{K}(\pi') = \overline{\mathbb{K}(\pi)}$ (resp. $\mathbb{B}(\pi') = \overline{\mathbb{B}(\pi)}$). In fact in this case one can deduce algebraically from Monk's rule (see [Mac1,Mac2] or [W, Cor.6.8]) that $X_{\pi'} = \partial_k X_\pi = X_\pi / x_k$. We assume therefore subsequently $\Delta_k(B(\pi)) \geq 2$, whence

$$V(\pi) := \mathbb{B}(\pi') \setminus \overline{\mathbb{B}(\pi)} \neq \emptyset \ ,$$

and: $\{ \ \nu \mid [f_\nu, k] \in B \ \} \neq \emptyset$ and $up(k, B) \geq 1$.

Proposition 3.6. *Let $\overset{k}{\sim}$ be the relation defined on $\mathbb{B}(\pi')$ by*

$$B \overset{k}{\sim} B' :\Longleftrightarrow \quad f(k,B) = f(k,B'),\ L_k(B) = L_k(B'),\ R_k(B) = R_k(B') \ .$$

Then $\overset{k}{\sim}$ has the following properties:
(1) It is an equivalence relation and we call the partition induced by it on $\mathbb{B}(\pi')$ the k-partition.
(2) The equivalence classes $[B]_{\overset{k}{\sim}}$ form chains w.r.t. the poset structure, i.e., consecutive elements of these chains have a difference in pseudorank of ± 1.
(3) If $B \overset{k}{\sim} B'$ and $B < B'$ in $\mathbb{B}(\pi')$, then all 'intermediary' B'' (in the sense of (3.1)) are also in $[B]_{\overset{k}{\sim}}$.
(4) $[B]_{\overset{k}{\sim}}$ is of the form $\partial_k(B_0 \cup \{[1,k]\})$ for some $B_0 \overset{k}{\sim} B$.

Proof. (1) is trivial. By definition different box diagrams in a class $[B]_{\overset{k}{\sim}}$ are distinguished only by the positions of their unpaired non-fixed boxes, which are exactly the ones that are B-movable back and forth (difference in pseudorank is ± 1) or movable by (improper) K^0-moves (difference in pseudorank is -1). This shows (2), and (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). □

Let $V_0(\pi)$ denote the set of all $B \in \mathbb{B}(\pi)$, such that $\overline{B}$ is covered by some box diagram of $V(\pi)$ in $\mathbb{B}(\pi')$, and set $\widetilde{V}(\pi) := V(\pi) \cup \overline{V_0(\pi)}$. In addition let

$$V_1(\pi) := \{ \ B \in V(\pi) \mid B \ \text{ is maximal in } [B]_{\overset{k}{\sim}} \ \} \ ,$$

so that $V_1(\pi) = \partial_k^0 \ V_0(\pi)$, and set

$$U(\pi) := \{ \ B \in \mathbb{B}(\pi) \mid \partial_k^0(B) \notin \overline{\mathbb{B}(\pi)},\ \Delta_k(B) \geq 2,\ up(k,B) = 1 \ \} \ .$$

In other words: $U(\pi)$ is the set of 'surface elements' of $\mathbb{B}(\pi)$ resp. $\mathbb{K}(\pi)$ (see our informal overview in the beginning of this section).

The following Lemma shows that the set $V_0(\pi)$ of minimal elements of all the chains in $\mathbb{B}(\pi')$, which partition the set $\widetilde{V}(\pi)$, can be characterized intrinsically in $\mathbb{B}(\pi)$ as the set $U(\pi)$.

Lemma 3.7. *For π with $\Delta_k(B(\pi)) \geq 2$ and $V_0(\pi)$ and $U(\pi)$ as defined above one has: $V_0(\pi) = U(\pi)$.*

Proof. Let $B \in V(\pi)$ and denote by B' the smallest element of $[B]_{\overset{k}{\sim}}$ (according to Proposition 3.6) supplemented with a box $[1,k]$. Then B' is an element of $\mathbb{B}(\pi)$: take the sequence of B-moves leading from $B(\pi')$ to B or to the maximal element in $[B]_{\overset{k}{\sim}}$ and apply it accordingly to $B(\pi)$. Moreover $up(k,B') = 1$ and $B \in \partial_k B' = \{ \ \overline{B'} =$

$B_0, B_1, \ldots, B_{\Delta_k(B')-1}$ }, whence $B' \in U(\pi)$. But B' is also an element of $V_0(\pi)$, i.e., $B_1 \in V(\pi)$:

Assume to the contrary that $B_1 \in \overline{\mathbb{B}(\pi)}$. Then there must be something to prevent B-moves from $B_1 \cup \{[1,k]\}$ to (the non-existing) $B \cup \{[1,k]\}$ inside $\mathbb{B}(\pi)$. Since the labeling clearly does not prevent the B^b-move in question, this can be due only to a box $[f_i, k]$ with $i > 1$, which is unmoved from its position in $B(\pi)$ and therefore not allowed to move backwards. But from the definition of the box diagram of a permutation it then follows that there was an original box at position $[f_1, k] \in B(\pi)$, which cannot move as long as $[f_i, k]$ is not moved. Hence B_1 cannot be obtained by K-moves from $B(\pi)$, a contradiction! □

Figs. 1 and 2 illustrate Lemma 3.7 and the k-partition introduced in Proposition 3.6. Let us recapitulate what we have accomplished so far:

Since $(**)$ is valid for π by the induction hypothesis and $\overline{B(\pi)} \in \mathbb{B}(\pi')$ can be reached through free K-moves from $B(\pi')$ in $\mathbb{B}(\pi')$ (cf. proof of 2.9), we conclude that the set $\overline{\mathbb{B}(\pi)} \subset \mathbb{B}(\pi')$ can be generated by B-moves and that

$$\sum_{B \in \overline{\mathbb{B}(\pi)}} x^B = \frac{1}{x_k} X_\pi \ . \tag{3.2}$$

Again one can deduce algebraically from Monk's rule (cf. [W, Cor.6.8]) that

$$X_{\pi'} = \partial_k X_\pi = \frac{1}{x_k} X_\pi + p_\pi(x) \ , \tag{3.3}$$

where $p_\pi(x)$ is a polynomial in $x = (x_1, \ldots, x_n)$ with non-negative integer coefficients.

Proposition 3.6 and Lemma 3.7 show that all box diagrams $B \in V(\pi) = \mathbb{B}(\pi') \setminus \overline{\mathbb{B}(\pi)}$ can be generated by B-moves: every such B can be reached by backward B-moves from some $B_0 \in \overline{U(\pi)}$ (– the labeling rules prevent them from moving further backwards –), and $\widetilde{V}(\pi)$ is the disjoint union of the sets $\partial_k(B')$ for all $B' \in V_0(\pi) = U(\pi)$, i.e., in terms of 'combinatorial' divided differences one has

$$\partial_k U(\pi) = \widetilde{V}(\pi)$$

and in terms of 'algebraic' divided differences

$$\partial_k \left(\sum_{B \in U(\pi)} x^B \right) = \sum_{B \in \widetilde{V}(\pi)} x^B = \frac{1}{x_k} \left(\sum_{B \in U(\pi)} x^B \right) + p'_\pi(x) \ . \tag{3.4}$$

Note that $p_\pi(x) = p'_\pi(x)$: let $U'(\pi)$ denote the subset of $\mathbb{B}(\pi)$, which generates the terms in $p_\pi(x)$, i.e.,

$$\partial_k \left(\sum_{B \in U'(\pi)} x^B \right) = \sum_{B \in \overline{U'(\pi)}} x^B + p_\pi(x) \ .$$

But since the algebraic and combinatorial ∂_k operate in 1-1 correspondence and by the definition of $U(\pi)$ one must have $U'(\pi) = U(\pi)$, whence $p_\pi(x) = p'_\pi(x)$. It then follows from (3.2–4) that

$$\partial_k \left(\sum_{B\in\mathbb{B}(\pi)\setminus U(\pi)} x^B \right) = \frac{1}{x_k} \left(\sum_{B\in\mathbb{B}(\pi)\setminus U(\pi)} x^B \right) . \tag{3.5}$$

Now the calculation

$$X_{\pi'} = \partial_k X_\pi = \partial_k \left(\sum_{B\in\mathbb{B}(\pi)\setminus U(\pi)} x^B + \sum_{B\in U(\pi)} x^B \right) = \frac{1}{x_k} \left(\sum_{B\in\mathbb{B}(\pi)\setminus U(\pi)} x^B \right) + \sum_{B'\in\widetilde{V}(\pi)} x^{B'}$$
$$= \sum_{B\in\overline{\mathbb{B}(\pi)\setminus U(\pi)}} x^B + \sum_{B\in\overline{U(\pi)}} x^B + \sum_{B\in V(\pi)} x^B = \sum_{B\in\mathbb{B}(\pi')} x^B$$

completes the proof that property $(**)$ is preserved under (∂)-steps, and thereby the proof of Theorem 1.5 (Bergeron's rule).

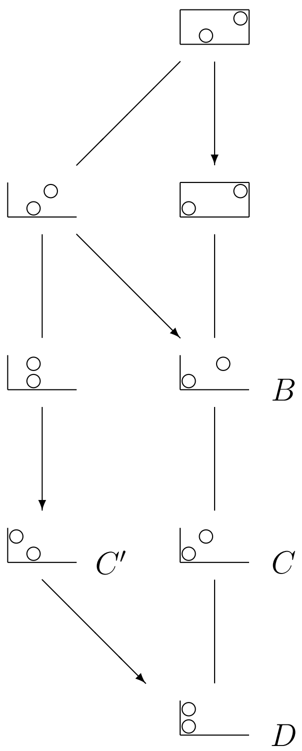
The next goal is to show that $\mathbb{B}(\pi') \subset \mathbb{K}(\pi')$. By induction hypothesis and Theorem 1.5 this amounts to showing $V(\pi) \subset \mathbb{K}(\pi')$. The reader may recall the last paragraph of our informal introduction to this section for the general idea how we proceed.

Every box diagram in $V(\pi)$ contains at least one box in column $k+1$, which is unmoved or 'original' relative to $B(\pi')$ and unpaired, and every $B \in U(\pi)$ contains at least one unpaired box in column k, which is the original of $B(\pi)$. Assume that $[i,k]$ is the only original unpaired box in B, and that it is possible for some box in row $j > i$ to K-move to position $(j, k+1)$, such that $[i,k]$ gets paired with $[j, k+1]$ in the resulting box diagram C. Since $C^{[k]}$ does not contain any unpaired original box, one has $\partial_k^0 C \notin V(\pi)$. It may be possible now to K-move $[j, k+1]$ to the (necessarily empty) position (j,k), which sets $[i,k]$ free again in the resulting box diagram D and implies that D is in $U(\pi)$. If this is the case we call C *transitory*. Observe that $\Delta_k(D) = \Delta_k(B) + 1$, whence the chain $\partial_k D$ contains one more element than $\partial_k B$.

We set $U_0(\pi) := \{\ B \in \mathbb{B}(\pi) \mid B \text{ is transitory}\ \}$ and $\widetilde{U}(\pi) := U(\pi) \cup U_0(\pi)$. Now $\widetilde{U}(\pi)$ is connected in $\mathbb{K}(\pi')$, where "connected" means that the subgraph of the Hasse diagram $\mathbb{K}(\pi')$ induced by the set of vertices $V_1(\pi)$ is connected. The question here is "If one leaves (by K-moves) $\widetilde{U}(\pi)$, when and how is it possible to return (by K-moves) to $\widetilde{U}(\pi)$?" If the last original box of some $B \in \widetilde{U}(\pi)$ is moved, then there is clearly no return. But it is also possible to leave $\widetilde{U}(\pi)$ by increasing the number of boxes in $B^{[k+1]}$, such that $\Delta_k(B) \le 0$, and still retaining some original (paired) boxes in $B^{[k]}$. In this case one is in a chain $\partial_k C \subset \overline{\mathbb{B}(\pi)}$ for some $C \in \widetilde{U}(\pi)$, and going down the chain to C can be accompanied by a number of anlogous steps inside $\widetilde{U}(\pi)$.

If B and $D \in U(\pi)$ are connected in $\widetilde{U}(\pi)$ by a transitory box diagram C, the transition from the chain $\partial_k B$ to $\partial_k D$ is analogous to Example 3.8 below, and otherwise Lemma 3.9 completes the proof of the inclusion $\mathbb{B}(\pi') \subset \mathbb{K}(\pi')$.

Example 3.8. Let $\pi = 31254$. Then $k = 1$, $\pi' = 13254$, and $\mathbb{K}(\pi')$ is given by



Here C is transitory between B and D, which results in an additional box diagram C' in the chain $\partial_k(D \cup \{[1,k]\})$, which is not covered by an element of $\partial_k(B \cup \{[1,k]\})$.

Lemma 3.9. *Assume for $B, C \in U(\pi)$ that $\partial_k B$ has been already generated by K-moves from $B(\pi')$ and that C originates from B by an irreducible K-move; then $\partial_k C$ can be generated by K-moves from $B(\pi')$.*

Proof. The principal difficulty with $\mathbb{K}(\pi)$ is the occurrence of *improper* K-moves. Figure 2 gives an idea of both the problem and its solution: the first box diagram with pseudorank two (and tag "$(3,1)$") is in its 1-equivalence class only reachable by an improper K^0-move, but "going back" to a higher (w.r.t. pseudorank) 1-equivalence class it is generated easily.

The proof proceeds by an exhaustive analysis of cases, which are specified (with the exception of (a)) by the type of K-move generating C from B. (Recall Definition 3.2 especially for the numbering of boxes in $\partial_k B$.)

(*a*) $M_k(C) = M_k(B)$: Apply the K-move from B to C to every $B_\nu \in \partial_k B$.

(*b*) *K^0-move of* $[i,k] \in B$: Immediate consequences are: $[i,k]$ highest box in $B^{[k]}$, $[i,k-1] \in C$, but $[i,k-1] \notin B$, no paired box above level i, $\Delta_k(C) = \Delta_k(B) - 1$, and $f(k,B) = (f_1, \ldots, f_s)$ for some $s > 0$.
If $[i,k]$ is paired or fixed in B, then $[f_s,k]$ will be fixed in C and C_ν will be the same as B_ν with $[i,k]$ freely moved for $\nu = 0, \ldots, s-1$. If $[i,k]$ is unpaired and non-fixed in B, then B_s will have no partner in $\partial_k C$.

(b') K^t*-move of* $[i,k] \in B$: Similar to (b).

(b'') $[i,k] \in B$ *is the left boundary of a* K^t*-move*: The same as (a), if the moving box comes from $R_k(B)$. Analogous to (b), if the moving box is $[i,k+1]$, but then the arguments are restricted to the case: $[i,k]$ paired in B.

(c) K^0*-move of* $[i,k+1] \in B$: Immediate consequences are: $[i,k+1]$ highest box in $B^{[k+1]}$, $[i,k] \in C$, but $[i,k] \notin B$, no paired box above level i, $\Delta_k(C) = \Delta_k(B) + 2$. Note that $i \in f(k,B)$ implies $\Delta_k B > 1$: a contradiction. Hence assume $i \notin f(k,B)$; then the next lower unpaired non-fixed box must be right unpaired, say $[i',k+1]$. Clearly, if it is possible to move the boxes on levels $f(k,B)$ to obtain $\partial_k B$, then it will be possible to move the same boxes and in addition $[i,k+1]$ and $[i',k+1]$.

(c') K^t*-move of* $[i,k+1] \in B$: Immediate consequences are: $\Delta_k(C) = \Delta_k(B) + 1$; $[i,k+1]$ is paired in B or $f(k,C)$ is the same as $f(k,B)$ plus i with $f_1 < \dots f_r < i < f_{r+1} < \dots < f_s$.
Then C_ν originates for $\nu = 0,\dots,r$ from B_ν by the K^t-move of $[i,k+1]$ and w.r.t. the higher boxes one has to proceed as follows: first move $[i,k] \in \partial_k^0 B$ (— $[i,k+1]$ the highest box in $B \Longrightarrow [i,k]$ the highest box in $\partial_k^0 B$ —) to the place to which $[i,k+1]$ tunnels, then move the higher boxes freely.

(c'') $[i,k+1] \in B$ *left boundary of a* K^t*-move*: Similar to (c): (i,k) empty in B, but $\Delta_k(C) = \Delta_k(B) + 1$; $[i,k+1]$ does not move.

(d) K^0*-move of* $[i,k+2] \in B$: $\Delta_k(C) = \Delta_k(B) - 1$ (; $up(k,C) = up(k,B) - 1 = 0$ possible). We distinguish three sub cases:
(d_1) $[i,k] \in B$ *and* $i \in f(k,B)$: $f(k,C)$ is the same as '$f(k,B)$ minus i' with $f_1 < \dots f_r < i = f_{r+1} < \dots < f_s$. Then C_ν originates for $\nu = 0,\dots,r$ from B_ν by: the K^0-move of $[i,k+1]$, then moving the boxes on levels $f_1,\dots,f_r$ appropriately and finally the K^0-move of $[i,k+2]$. The moves of the higher boxes are not affected by the K^0-move of $[i,k+2]$, but note that B_{r-1} and B_r are now identified.
(d_2) $[i,k] \in B$ *and* $i \notin f(k,B)$: i.e., $[i,k] \in B$ is fixed with a box $[i',k+1]$, $i' > i$. This box will be set free by the K^0-move of $[i,k+2]$ and it becomes fixed again by a box $[i'',k]$, $i'' < i$. If now $i'' \notin f(k,B)$, then the preceding step will be repeated with i'' playing the role of i; if $i'' \in f(k,B)$, then the situation is essentially that of case (d_1) and the arguments there apply.
(d_3) (i,k) *empty in* B: analogous to (d_2): $[i,k+1] \in C$ fixes a box $[i'',k]$, $i'' < i$.

(d') K^t*-move through* $[i,k+1] \in B$: The same as (a), if the moving box goes to $L_k(B)$, and the same as (c''), if it moves to (i,k).

(d'') $[i,k+2] \in B$ *is the left boundary of a* K^t*-move*: Analogous to (d). □

In order to complete the proofs of Theorem 1.2 (Kohnert's rule) and Corollary 1.6 it remains to be shown that $\mathbb{K}(\pi') \subset \mathbb{B}(\pi')$, which by induction hypothesis amounts to proving:

Lemma 3.10. *Under the general assumtions on π and π' of this section one has*

$$W(\pi) := K(\pi') \setminus K(\pi) \subset V(\pi) \ .$$

Proof. The assertion says that every box digramm $D \in W(\pi)$, which is reachable by K-moves from $B(\pi')$ resp. from the principal chain between $B(\pi)$ and $B(\pi')$ (see proof of Theorem 2.9) is also reachable by B-moves. In other words, we have to show that every tunneling K-move in $W(\pi)$ can by simulated by B-moves.

We first discuss, how the k-middle part $M_k(D')$ of a $D \in W(\pi)$ looks like: assume that column $k+1$ of $B(\pi')$ contains boxes on levels $1 < i_1 < i_2 < \cdots < i_s$. By the definition of $B(\pi')$ every level below i_s not occuring as an i_ν is empty right to column k – and therefore empty for every $D \in W(\pi)$. For a $D \in W(\pi)$ it is characterizing that at least $[i_1, k+1] \in D$ is unmoved (view all non-free K-moves as tunneling moves), but some other boxes $[i_\nu, k+1]$ of $B(\pi')$ might have been moved to column k (or further to the left) and some boxes above level i_s might have been moved from the right to the k-middle part. Some of the remaining boxes on levels i_ν in $M_k(D)$ may be paired now, but – most important – all left unpaired boxes must be above the right unpaired boxes.

If now $D' \in W(\pi)$ originates from some other $D \in W(\pi)$ by an irreducible tunneling K-move, then there are two possibilities: (1) The tunneling box moves to a position outside $M_k(D')$ or to a position above level i_s inside $M_k(D)$ or (2) the tunneling box moves to some position in $M_k(D')$ with level $\leq i_s$. In the first case one can find box diagrams $\bar{D}, \bar{D}' \in \mathbb{K}(\pi)$, which differ from D and D' only by having all unpaired boxes on levels i_ν, which did not leave $M_k(D)$, in column k. By induction hypotheses the K^t-move from $\bar{D}$ to $\bar{D}'$ is known to be substitutable by a sequence of B-moves, whence the same sequence of B-moves can simulate the K^t-move from D to D' inside $W(\pi)$. In the second case the argumentation of Remark 1.7 applies due to the properties of the unpaired boxes in $M_k(D)$ as discussed in the preceding paragraph. □

4. Schur polynomials and semistandard tableaux

The following well-known properties of Schubert polynomials (cf.[Mac1,Mac2,W1]) are immediate consequences of Theorem 1.2 (Kohnert's rule):

(P1) $X_{id} = 1$ and $X_{n\ (n-1)\ \dots\ 1} = x_1^{n-1} \dots x_n^0$;

(P2) X_π is homogeneous of degree $l(\pi) = |L(\pi)|$ and for every monomial x^d occurring in X_π, the exponent tuple d is lexicographically greater than or equal to $L(\pi)$;

(P3) $X_{\sigma_k} = x_1 + \cdots + x_k$;

(P4) $X_{\pi(1)\dots\pi(n)\ n+1\dots m} = X_\pi$ for $\pi \in S_n$ and all $m > n$;

(P5) for $\pi \in S_n$ and $\mu \in S_m$ let $\pi \times \mu := \pi(1) \dots \pi(n)\ (\mu(1)+n) \dots (\mu(m)+n) \in S_{n+m}$, then $X_{\pi\times\mu} = X_\pi \cdot X_{1\dots n\ (\mu(1)+n)\dots(\mu(m)+n)}$;

(P6) for dominant π, i.e., $L(\pi)$ weakly decreasing, is $X_\pi = x^{L(\pi)}$.

Note: (P2) implies, that the X_π form a basis of the polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ (cf. property (B) in [W1]).

This section is concerned mainly with a combinatorial proof of the following property of Schubert polynomials (see [Mac2,W1] for algebraic proofs):
(S) Let $\pi \equiv \pi(\lambda)$ be *Grassmannian*, i.e., π has a unique descent at place m ($\pi(m) > \pi(m+1)$) $\iff$ $L(\pi) \equiv \overline{\lambda_m \dots \lambda_1\ 0 \dots 0}$, where $\lambda \equiv \lambda_1 \dots \lambda_m$ with $\lambda_1 \geq \dots \geq \lambda_m \geq 0$ is a partition; then

$$X_{\pi(\lambda)} = s_\lambda(x_1, \dots, x_m) = \sum_{T \in SSYT_m(\lambda)} x^T \ .$$

Here $s_\lambda(x_1, \dots, x_m)$ denotes the Schur polynomial in the variables $x_1, \dots, x_m$ associated to the partition λ and the second equality is the well known (cf.[Mac3,Sa]) combinatorial characterization of $s_\lambda(x_1, \dots, x_m)$, which we shall briefly explain: $SSYT(\lambda)$ denotes the set of *semistandard* or *column strict Young tableaux of shape λ with entries in the set $\{1, \dots, m\}$* ; for example $T = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&6\\\cline{1-2}\end{array}$ is a numbering of the Ferrer diagram of shape $\lambda = 3\,2$, which weakly increases in rows from left to right and strictly increases in columns from top to bottom, and hence is an element of $SSYT_m(3\,2)$ for all $m \geq 6$; x^T is defined as the product of all x_ν, where ν runs through all numbers contained in T, e.g. for the above T we have $x^T = x_1^2 x_2 x_3 x_6$.

Kohnert [Ko] has proved (S) using K-moves and Bergeron [B] using B-moves. Bergeron even defined the notion of a 'retract' (cf. Definition 4.1). But we have included a proof of (S) here for the sake of completeness and with the hope to give a more transparent exposition.

Definition 4.1. *For every permutation π let $\mathbb{K}_0(\pi)$ denote the set of* freely generated box diagrams*, where 'freely generated' means: all box diagrams, which can be derived from $B(\pi)$ by free K-moves alone. Defining the covering relation as given by (improper) K^0-moves, i.e., without K^t-moves, turns $\mathbb{K}_0(\pi)$ into a ranked poset.*

For every $B \in \mathbb{K}(\pi)$ the retract $r(B)$ *of B is a numbering of $B(\pi)$, which originates from B by first attaching to each box $[i, j] \in B$ its column number j and second retract all boxes to their original position in $B(\pi)$ without changing their row order, i.e., all K^t-moves are viewed as shunting moves.*

The set $SST(\pi)$ of semistandard *or* row strict tableaux of shape $B(\pi)$ *is the set of all numberings of $B(\pi)$ with natural numbers, such that (1) the numbers strictly increase in the rows from left to right and weakly decrease in the columns from bottom to top and (2) the entries in each column j are $\leq j$.*

Proposition 4.2. *For all π the ranked poset $\mathbb{K}_0(\pi)$ is a lattice and $SST(\pi) = r(\mathbb{K}_0(\pi))$.*

Proof. $\mathbb{K}_0(\pi)$ has a unique top and bottom element, because it contains the unique principal chain from $B(\pi)$ down to $B_\perp$ (cf. proof of Theorem 2.9). For all $B, C \in \mathbb{K}_0(\pi)$ the meet $B \wedge C$ [join $B \vee C$] is given by taking in every row for two corresponding boxes in B and C the leftmost [rightmost] position as the position for this box in $B \wedge C$

$[B \vee C\ ]$. Correspondingly one can introduce a lattice structure on $SST(\pi)$: meet and join are defined by taking minimal and maximal values of corresponding box numbers or, alternatively, the covering relation in $SST(\pi)$ is defined by decreasing exactly one number by 1. Clearly the retraction then gives an isomorphism of the lattices $\mathbb{K}_0(\pi)$ and $SST(\pi)$. $\square$

Proposition 4.3. $\mathbb{K}(\pi(\lambda)) = \mathbb{K}_0(\pi(\lambda))$ *for any Grassmannian permutation* $\pi(\lambda)$.

Proof. Let $\lambda \equiv \lambda_1 \dots \lambda_m$ with $\lambda_1 \geq \dots \geq \lambda_m \geq 0$. Then $L(\pi(\lambda)) = \overline{\lambda_m \dots \lambda_1\, 0 \dots 0}$ and $B(\pi(\lambda))$ equals the suitably positioned Ferrer diagram of λ reflected on its main diagonal. All K^t-moves in this proof are to be understood as shunting moves.

For every permutation π and $B \in \mathbb{K}(\pi)$ we define the *retract distance* of some box $[i', k'] \in B$ as follows: let $[i, k] \in B(\pi)$ be the box corresponding to $[i', k']$ under retraction; then the retract distance of $[i', k']$ is the label of $[i, k]$ in $r(B)$ minus the label of $[i+1, k]$, if $[i+1, k] \in B(\pi)$, and zero otherwise. With this notion Proposition 4.2 can be rephrased as: $B \in \mathbb{K}(\pi)$ can be generated freely $\iff r(B) \in SST(\pi) \iff$ all retract distances of the boxes in B are non-negative.

Since trivially $\mathbb{K}_0(\pi(\lambda)) \subset \mathbb{K}(\pi(\lambda))$, we use this equivalence and induction over λ_1 – the number of non-empty rows of $B(\pi(\lambda))$ – to show that every shunting move in $\mathbb{K}(\pi(\lambda))$ can be reduced to a sequence of free K-moves:

For $\lambda_1 = 1$ one has $L(\pi(\lambda)) = \overline{0 \dots 0\, 1 \dots 1\, 0 \dots 0}$ and the assertion is immediate; assume therefore that the assertion is true for all λ with $\lambda_1 \leq r$. Let λ' be a partition with $\lambda'_1 = r+1$ and assume that a shunting move can be carried out with a box $[1, k]$ in the bottom row of some $B \in \mathbb{K}_0(\pi(\lambda'))$: evacuate first all positions in $B^{[k]}$ above $[1, k]$ – if necessary – and denote the resulting box diagram by B', and the box diagram B' without the bottom row by B''. Since the removal of the bottom row of $B(\pi(\lambda'))$ gives the Rothe diagram $B(\pi(\lambda))$ of some Grassmannian permutation $\pi(\lambda)$ with $\lambda_1 = r$, one knows by induction hypothesis that B'' can be generated using only K^0-moves. Therefore all boxes in B'' have non-negative retract distances and the retract distances of all boxes in the bottom row of B' are clearly ≥ 1. Since a shunting move of $[1, k]$ can decrease these latter numbers at most by 1, no negative relative distance can be produced by our shunting move, i.e., it is reducible to a sequence of free K-moves. $\square$

Proof. of **(S)**. Using Theorem 1.2, Proposition 4.3, Proposition 4.2, Definition 4.1 and the combinatorial definition of Schur polynomials (in this order) one has for all Grassmannian permutations $\pi \equiv \pi(\lambda)$ with $\lambda \equiv \lambda_1 \dots \lambda_m$, $\lambda_1 \geq \dots \geq \lambda_m \geq 0$:

$$X_{\pi(\lambda)} = \sum_{B \in \mathbb{K}(\pi(\lambda))} x^B = \sum_{B \in \mathbb{K}_0(\pi(\lambda))} x^B$$

$$= \sum_{T \in SST(\pi(\lambda))} x^T = \sum_{T \in SSYT_m(\lambda)} x^T = s_\lambda(x_1, \dots, x_m).$$

Only the fourth equality needs an explanation: take any T from $SST(\pi(\lambda))$, reflect it along the main diagonal and substitute a label i by a label $\omega_m(i) := m + 1 - i$. This clearly sets up a bijection between the sets $SST(\pi(\lambda))$ and $SSYT_m(\lambda)$. But the application of ω_m to the labels of boxes respectively indices of x-variables does not change the sum, because $s_\lambda(x_1, \dots, x_m)$ is symmetric in $x_1, \dots, x_m$. $\square$

The results in this section have shown that the sets of semistandard numberings $SST(\pi)$ of a box diagram $B(\pi)$ are the most direct generalization of the sets $SSYT_m(\lambda)$ of semistandard Young tableaux of shape λ, which are so important and ubiquitous. It is therefore tempting to define *modified Schubert polynomials* Y_π for all permutations $\pi \in S_\infty$ as

$$Y_\pi := \sum_{B \in \mathbb{B}_0(\pi)} x^B = \sum_{T \in SST(\pi)} x^T \ ,$$

where m is the number of the rightmost nonempty column in $B(\pi)$.

Proposition 4.4. *For modified Schubert polynomials the properties (P1) to (P6) and (S) are valid again, and in addition:* $X_\pi = Y_\pi + \text{nonnegative terms}$ *for all* π.

Proof. Immediate from Theorem 1.2 and Proposition 4.4. □

An interesting research problem is to find equivalent algebraic, combinatorial and geometric (in terms of flag manifolds) characterizations of the Y_π (the approach of [BJS] seems especially promising).

5. Magyar's rule

Our simplified proof of Bergeron rule in Section 3 shows that we have actually proven the following recursive rule for the generation of Schubert polynomials:

Theorem 5.1. (Simplified Bergeron rule) *Let $\pi' \in S_{n+1}$ and $k := (\pi')^{-1}(1)$.*

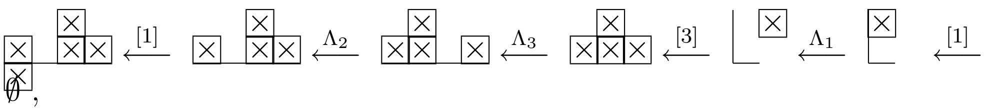
If $k = n+1$, then $\mathbb{B}(\pi')$ is generated from $\mathbb{B}(\pi)$ by adding a lowest level of boxes $[1,1],\ldots,[1,n]$ to all $B \in \mathbb{B}(\pi)$.

If $k \in \{1,\ldots,n\}$, then $\mathbb{B}(\pi')$ is generated from $\mathbb{B}(\pi)$ by (1) removing the box $[1,k]$ from all $B \in \mathbb{B}(\pi)$, and (2) applying backward B-moves in accordance with Definition 3.2 (3.1) to all $\overline{B} \in \overline{\mathbb{B}(\pi)}$ with $\Delta_k(B) \geq 2$ and $up(k,B) = 1$.

Instead of building up much terminology and notation we state and illustrate the correspondence between Magyar's rule and the simplified Bergeron rule with the help of our running example $\pi = 31542$ (see Example 1.3 and Example 2.11 above).

We review the rule of Magyar for the generation of Schubert polynomials in terms of words as described in [M3], [RS3, Theorem 23], and [Sh]. This rule has been proven with the help of algebro-geometric means and works also for the generation of the characters of flagged Schur modules in the case of 'northwest' and 'percent-avoiding' diagrams, which are more general than Rothe diagrams.

One starts with the diagram $D(\pi)$ of a permutation π and eliminates by successive transpositions σ_k of columns k and $k+1$ (in any order) all gaps between boxes in the first row until all boxes in row 1 are contained in columns 1 up to say r_1, where r_1 must be the number of boxes in row 1 of $D(\pi)$. Then row 1 is removed and the algorithm proceeds with row 2 etc., until the last box in the highest row of $D(\pi)$ is removed leaving the empty diagram. The sequence of transpositions and row removals is recorded as a sequence of operators Λ_k and $[r_i]$, respectively, applied to the empty diagram. In our example $\pi = 31542$ the procedure works as follows:



i.e., $D(\pi) = [1]\Lambda_2\Lambda_3[3]\Lambda_1[1](\emptyset)$. Note that in general the sequence of operators is not unique, because the transposition of columns can be done in different ways. If one requires that the leftmost possible transposition comes always first, then the resulting sequence is essentially the one given by the long induction. Note moreover that Magyar's way of setting up the sequence of operators Λ_k and $[r_i]$ is essentially the chain of principal box diagrams of Theorem 2.9 and Example 2.10.

Now **Magyar's rule for monomials** says that *the substitution of Λ_k and $[r_i]$ by $\pi_k := \partial_k x_k$ and $\cdot x_1 \ldots x_{r_i}$, respectively, applied to the constant 1 gives the Schubert polynomial X_π.* For example,

$$X_{31542} = x_1\pi_2\pi_3 x_1x_2x_3\pi_1 x_1(1) \ ,$$

but by the product rule for divided differences one has from long induction the expression

$$\begin{aligned} X_\pi =& \partial_2\partial_3\partial_4 x_1x_2x_3x_4x_1x_2x_3\partial_1\partial_2x_1x_2x_1(1) \\ =& x_1\partial_2x_2\partial_3x_3\partial_4x_4x_1x_2x_3\partial_1x_1\partial_2x_2x_1(1) \\ =& x_1\partial_2x_2\partial_3x_3x_1x_2x_3\partial_1x_1x_1(1) = x_1\pi_2\pi_3x_1x_2x_3\pi_1x_1(1) \ . \end{aligned}$$

Magyar's rule has therefore the advantage over long induction that operators π_k, which do not alter the expression computed so far, but only recompute it, are not introduced right from the outset. But on the other hand it involves manipulation of diagrams, which is certainly more complicated than the computation using permutations (or even simpler the code of a given permutation π as described in [W2, Section 4]).

Magyar's rule for *words* instead of *monomials* is also seen to be even more closely related to the simplified Bergeron rule (Theorem 5.1):

> Interpret the operator Λ_k as an 'interpolated plactic operator on words', and $[r_i]$ as the prefixing of the word $1\dots r_i$. Then **Magyar's rule for words** says that *this sequence of operators applied to the empty word gives the monomial content of the Schubert polynomial X_π in terms of words, which code in fact in a simple manner the box diagrams of $\mathbb{K}(\pi)$.*

The rest of this section will be used to explain the statements of the last paragraph, and we begin with the phrase 'interpolated plactic operator'. The 'plactic action' on words was first described by Lascoux and Schützenberger in [LS1] (see also Rem.3.4 above).

Definition 5.2. (Plactic action on words) *Let w be a word in the alphabet $\mathbb{N}$ and $r \in \mathbb{N}$. Then the word $\sigma_r(w)$ is given as follows: mark every occurrence of $(r+1)$ in w by a left parenthesis " (" and every occurrence of r by a right parenthesis ") ". The letters r and $(r+1)$, which correspond to paired parentheses in the usual sense, are called r-paired, and the remaining letters r and $(r+1)$ are called r-unpaired. The r-unpaired subword of w is necessarily of the form $r^s(r+1)^t$, and substituting it by the sequence $r^t(r+1)^s$ gives $\sigma_r(w)$.*

For example: $\sigma_2(243312423113231432) = 243312423113221432$.

Definition 5.3. (Interpolated plactic action on words) *Let w be a word in the alphabet $\mathbb{N}$ and $r \in \mathbb{N}$. w is called an r-*head word *if it contains no r-unpaired letter $(r+1)$.*
The interpolated plactic operator $\overline{\sigma}_r$ applied to a r-head word w is defined as the set of words containing w itself with the r-unpaired subword $r^s(r+1)^t$, and then
w with r-unpaired subword changed to $r^{s+1}(r+1)^{t-1}$, to $r^{s+2}(r+1)^{t-2}$, etc. up to $\sigma_r(w)$, if $s<t$, and
w with r-unpaired subword changed to $r^{s-1}(r+1)^{t+1}$, to $r^{s-2}(r+1)^{t+2}$, etc. down to $\sigma_r(w)$, if $s>t$.
If w is not a r-head word, then $\overline{\sigma}_r w := \emptyset$.

For example: the word 243312423113231432 is not a 1- or 2-head word, but a 3-head word, and $\overline{\sigma}_1(11211) = \{11211, 21211, 22211, 22212\}$. For our running example

$\pi = 31542$ one computes:

$$[1]\Lambda_2\Lambda_3[3]\Lambda_1[1](\emptyset) \;=\; [1]\Lambda_2\Lambda_3[3]\Lambda_1(1) \;=\; [1]\Lambda_2\Lambda_3[3](1+2) \;=$$
$$[1]\Lambda_2\Lambda_3(1231+1232) \;=\; [1]\Lambda_2(1231+1241+1232+1242) \;=$$
$$[1](1231+1241+1341+1232+1233+1242+1342+1343) =$$
$$11231+11241+11341+11232+11233+11242+11342+11343\ ,$$

which indeed gives the indices of the monomials contained in X_{31542}. (We did not use the clause on head words here, but other examples like $\pi = 12543$ show that it is indispensable.)

Let $w(B)$ be the reading word of a box diagram B, i.e., the word consisting of the column numbers of boxes, where one reads row-wise from left to right and the rows from bottom to top. Then it is easy to show the following remarkable facts:

(1) The interpolated plactic operator $\overline{\sigma}_k$ applied to a reading word $w(B)$ of a box diagram $B \in \mathbb{B}(\pi)$ yields the reading words of the set of box diagrams $\partial_k(B)$, where ∂_k is the combinatorial divided difference.
(2) The condition $up(k, B) = 1$ for $B \in \mathbb{B}(\pi)$ means that $w(B)$ is a k-head word. Especially important examples are the surface elements of $V_0(\pi)$ (Section 3) and their reading words. Therefore:
(3) The set of words derived by Magyar's rule for a given permutation π is the set of reading words of $\mathbb{B}(\pi)$ (or $\mathbb{K}(\pi)$).
(4) The main difference between Magyar's rule and the simplified Bergeron rule is that by virtue of the poset structure on the set $\mathbb{B}(\pi)$, we consider only the newly generated box diagrams, whereas Magyar's rule considers in each step all words. (There should be a natural grading of flagged Schur modules reflecting the structure of order ideals generated by the principal box diagrams in $\mathbb{B}(\pi)$.)

1 and 2 follow by comparison of Definitions 5.3 and 3.2 ; 3 and 4 by comparison of the simplified Bergeron's rule (Theorem 5.1) and Magyar's rule for words.

Note finally that Kohnert's rule for the generation of Schubert polynomials has been shown as an improvement of the simplified Bergeron rule through Lemma 3.9. The latter does not work for northwest and percent-avoiding diagrams, so that Kohnert's rule is not generalizable to these shapes. Does the simplified Bergeron rule generalize? And finally:

What are the special features of flagged Schur modules for Rothe diagrams? Is there an algebraic version and proof of Kohnert's rule?

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