SEQUENCES OF SYMMETRIC POLYNOMIALS AND COMBINATORIAL PROPERTIES OF TABLEAUX

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Abstract. In 1977 G.P. Thomas has shown that the sequence of Schur polynomials associated to a partition λ can be comfortably generated from the sequence of variables $x = (x_1, x_2, x_3, \dots)$ by the application of mixed Baxter/multiplication operators, which in turn can be easily computed from the set $SYT(\lambda)$ of standard Young tableaux of shape λ.

We generalize this construction, thereby making possible the explicit and effective computation of the Hall-Littlewood, Jack, and Macdonald polynomials used in representation theory, combinatorics, multivariate statistics, and quantum algebra. These generalized formulas have a pleasing recursive structure with respect to the Young lattice and they can easily be specialized to yield 'skew' forms in all cases and 'super' forms in the Schur case.

We introduce and investigate: (1) the 'descent polynomial of a partition λ ', which arises naturally in the enumeration of semistandard Young tableaux of shape λ; (2) the Boolean lattice $G(\zeta)$ associated to any $\zeta \in SYT(\lambda)$, which is fundamental for the 'weighted' generalization of Thomas' approach to Schur polynomials; and (3) an action of the symmetric groups on semistandard Young tableaux, which is connected with Knuth's combinatorial proof of the symmetry of Schur functions. Moreover we argue that a generalization of Thomas' approach is a natural starting point in search of 'universal weighted symmetric functions'.

Let $s_\lambda^{(m)}(x) \in \mathbb{Z}[x_1, \dots, x_m]$ denote the *Schur polynomial* in the variables $x_1, \dots, x_m$ associated to a partition $\lambda \vdash N$ of the natural number N. It is well known (cf. [M1, S, K]) that these Schur polynomials can be defined algebraically by various determinantal formulas or combinatorially by the formula

$$s_\lambda^{(m)} \equiv s_\lambda^{(m)}(x) := \sum_{\eta \in SSYT_{(m)}(\lambda)} x^\eta ,$$

where $x^\eta \equiv x^{\rho(\eta)} := x_1^{\rho_1(\eta)} x_2^{\rho_2(\eta)} x_3^{\rho_3(\eta)} \dots$ is the monomial associated to the content $\rho(\eta)$ of η and $SSYT_{(m)}(\lambda)$ is the set of semistandard Young tableaux of shape λ with entries in $\{1, \dots, m\}$. (The exact definition of these and other no(ta)tions appearing subsequently has been collected in an Appendix.)

The determinantal formulas are very compact and appropriate for many theoretical purposes, but difficult to evaluate: it is even hard to decide, which monomials in a given $s_\lambda^{(m)}(x)$ occur. To the contrary this is an easy task from the combinatorial definition, but the latter is clearly not very compact due to the large number of possible SSYT

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(semistandard Young tableaux). Therefore one seeks for a 'systematization' of all the SSYT of a given shape. It is well known that the partition $\mu \vdash N$ formed from the content vector ρ of any $\eta \in SSYT(\lambda)$ precedes λ in the dominance order '$\leq$':

$$\mu \not\leq \lambda \Longrightarrow [SSYT(\lambda, \rho) = \emptyset \quad \forall \rho \in F_N(\mu)] \ , \tag{0.1}$$

and that the cardinality of the set $SSYT(\lambda, \rho)$ of all $\eta \in SSYT(\lambda)$ with content ρ is invariant under arbitrary permutations of the components of ρ:

$$\forall \rho \in F_N(\mu) \ : \quad |SSYT(\lambda, \rho)| = |P(\lambda, \mu)| =: K_{\lambda\mu} \ , \tag{0.2}$$

where the non-negative numbers $K_{\lambda\mu}$ are called *Kostka numbers*. These two facts together with the obvious equivalence:

$$x^\eta = x^{\eta'} \Longleftrightarrow \rho(\eta) = \rho(\eta') \ ,$$

and the definition of *monomial symmetric functions* $m_\mu(x)$ (cf. [M1,S]) yield the expansion of *Schur functions* into monomial symmetric functions:

$$s_\lambda(x) := \sum_{\eta \in SSYT(\lambda)} x^\eta = \sum_{\mu \leq \lambda} K_{\lambda\mu} \ m_\mu(x) \ . \tag{0.3}$$

An alternative way of collecting the x^η's has been described by G.P. Thomas in [T1,T2]: fix $\lambda \vdash N$, then the sequence of Schur polynomials

$$(\ s_\lambda^{(1)}, s_\lambda^{(2)}, s_\lambda^{(3)}, \dots \ )$$

can be generated comfortably by applying a λ-dependent sum of certain mixed shift/multiplication operators to the sequence $x = (x_1, x_2, x_3, \dots)$. Every single operator in such a sum is easily computed from exactly one standard Young tableaux $\zeta \in SYT(\lambda)$, i.e. the problem of handling the infinite set $SSYT(\lambda)$ of semistandard Young tableaux is reduced to that of finding the finite set $SYT(\lambda)$ of standard Young tableaux. We will speak of *τPx-formulas*, because the operators appearing are build up from the shift operator 'τ', the Baxter operator 'P' and the multiplication operator 'x'.

Clearly, $SSYT_{(m)}(\lambda) = \emptyset$, if $m < s = l(\lambda)$ ($l(\lambda)$ being the length of λ), and $SSYT_{(m)}(\lambda) \subset SSYT_{(m')}(\lambda)$, if $m < m'$. Consequently $s_\lambda^{(m)} = 0$, if $m < s$, and $s_\lambda^{(m')} = s_\lambda^{(m)} +$ '*positive terms*', if $m < m'$, which can be concisely expressed by saying that the Schur polynomials are *cumulative*. Therefore it is enough to consider the sequence of difference polynomials

$$s_{[\lambda]}(x) := (s_\lambda^{[1]}(x), s_\lambda^{[2]}(x), s_\lambda^{[3]}(x), \dots \ ) \qquad \text{with} \ \ s_\lambda^{[m]}(x) := \sum_{\eta \in SSYT_{[m]}(\lambda)} x^\eta = s_\lambda^{[m]}(x) - s_\lambda^{[m-1]}(x) \ . \tag{0.4}$$

We call $s_{[\lambda]}$ the *graded Schur function for* λ, where the *part of 'degree'* m or m-*part* $s_\lambda^{[m]}$ is the sum of all monomials x^η with $\eta \in SSYT(\lambda)$, which contain x_m, but no x_ν with $\nu > m$.

The above definitions can now be extended from the Schur case to more general *weighted* symmetric functions, polynomials, and graded functions:

(0.5)

$$\{\lambda\}_w(x) := \sum_{\eta \in SSYT(\lambda)} w(\eta)\ x^\eta\ , \quad \{\lambda\}_w^{(m)}(x) := \sum_{\eta \in SSYT_{(m)}(\lambda)} w(\eta)\ x^\eta\ , \text{ and}$$

$$[\lambda]_w(x) := (\{\lambda\}_w^{[1]}(x), \{\lambda\}_w^{[2]}(x), \{\lambda\}_w^{[3]}(x), \dots) \text{ with } \{\lambda\}_w^{[m]}(x) := \sum_{\eta \in SSYT_{[m]}(\lambda)} w(\eta) x^\eta\ ,$$

where w is a *weight*, which associates to every $\eta \in SSYT(\lambda)$ an element of a ring. Obviously, in the Schur case one has the 'trivial' weight $w^S(\eta) \equiv 1$; the well known 'non-trivial' weighted generalizations investigated in the present paper are Hall-Littlewood (HL) functions ($w^{HL}(\eta) \in \mathbb{Z}[t]$), the Jack functions ($w^J(\eta) \in \mathbb{Q}(\alpha)$), and the Macdonald functions ($w^M(\eta) \in \mathbb{Q}(q,t)$). HL and Jack functions are generalizations of Schur functions and Macdonald functions generalize HL as well as Jack functions. For more details on the relationship between the different families of symmetric functions as well as their applications in representation theory, both in the classical and quantum case, in combinatorics, and in statistics the reader may consult the first paragraphs of sections 5,6, and 7, respectively.

It is not hard to see, that $\{\lambda\}_w$ is symmetric exactly when

$$\left[\lambda(\rho_1) = \lambda(\rho_2) \Longrightarrow \sum_{\eta_1 \in SSYT(\lambda,\rho_1)} w(\eta_1) = \sum_{\eta_2 \in SSYT(\lambda,\rho_2)} w(\eta_2)\right] , \tag{0.6}$$

i.e. the sum of weights $\sum w(\eta)$ over all $\eta \in SSYT(\lambda)$ with content vector ρ is invariant under arbitrary permutations of the components of ρ. Therefore in the case of a 'symmetric weight w' it makes sense to define the *w-Kostka factors*:

$$w_\lambda(\mu) := \sum_{\eta \in P(\lambda,\mu)} w(\eta) \ \text{ for all } \mu \vdash N = |\lambda|\ , \tag{0.7}$$

where $P(\lambda,\mu)$ is the set of all $\eta \in SSYT(\lambda)$ with partition like content μ. Then (0.3) generalizes to

$$\{\lambda\}_w(x) = \sum_{\mu \leq \lambda} w_\lambda(\mu)\ m_\mu(x)\ . \tag{0.8}$$

The aim of the present paper is to present 'weighted τPx-formulas' in the HL case (Section 5), the Jack case (Section 6) and the Macdonald case (Section 7). These weighted τPx-formulas enable the effective generation of the respective graded functions and polynomials, for which there are explicit formulas only in very special cases. Moreover they allow recursive computations with respect to the Young lattice $\mathcal{Y}$ and easy evaluations of the 'skew' forms in all cases and the 'super' forms in the Schur case (Section 2).

In fact the $w-Kostkafactors$ appearing in the expansion (0.8) are computed most effectively from weighted τPx-formulas: using (0.7) one has to find the sets $P(\lambda,\mu)$ for all $\mu \leq \lambda$, compute the weights in all cases, and sum up; using the weighted τPx-formulas one has to find only the the set $SYT(\lambda) = P(\lambda, 1^N)$, set up the weighted τPx-formula, which is roughly equivalent to finding the weights for the $\zeta \in SYT(\lambda)$, and finally to compute $\{\lambda\}_w^{(N)}(x)$, which is not very expansive and clearly sufficient for finding the coefficients $w_\lambda(\mu)$.

The approach of G.P. Thomas to the τPx-expansions of graded Schur functions, of which we give a simplified and concise account in Section 1, is to coarse to deal with the weighted case; therefore we give a new refined derivation in Section 5 (Thm.5.4). Central to this new approach is the Boolean lattice $G(\zeta)$ of gapless SSYT associated to every SYT ζ (Def.3.3). Moreover we introduce the *descent polynomial* $D_\lambda(\tau)$ *of a partition* λ in Section 3, which appears naturally in the counting of the sets $SSYT_{[m]}(\lambda)$, and describe an action of permutations on SSYT in Section 4, which 'improves' the bijection of Knuth appearing in the combinatorial proof of the symmetry of Schur polynomials.

It will turn out that a necessary condition for the *existence* of the τPx-representation is:

(0.9) $$w(\eta) = w(pr_G\, \eta) \text{ for all } \eta \in SSYT(\lambda)\ .$$

This means that the weight $w(\eta)$ does not depend on the absolute values of the numbers appearing in η, but only on its structure of horizontal stripes as represented by the gapless elements $G(\lambda)$ of $SSYT(\lambda)$. In order to get a *reasonably compact* τPx-representation of a weighted symmetric function one needs moreover that the weight w is *S-insected* as explained in Section 6 (Def.6.2) — indeed Schur, HL, Jack, and Macdonald functions have S-insected weights.

To our understanding the central point in the *combinatorial* approach to symmetric functions is that *weights encode combinatorial properties of semistandard Young tableaux*: the Schur case expresses the mere fact that a multiset of numbers (or indices of variables in a monomial) occurs as a semistandard numbering of a shape λ, whereas the more general HL, Jack, and Macdonald weights encode additional facts about the distribution of horizontal stripes placed in this shape. A question which therefore naturally arises, but doesn't seem to have been addressed before, is the existence of *combinatorially meaningful universal symmetric functions* U_λ, where the term 'combinatorially meaningful universal' in accordance with the above 'philosophy' means:

For $\eta \in SSYT(\lambda)$ the gapless representant $pr_G\, \eta \in G(\lambda)$ or at least the set of horizontal stripes $H(\eta)$ should be reconstructable from the given weight $w(\eta)$, and Schur, HL, Jack, and Macdonald symmetric functions should be contained as special cases. In Section 7 we briefly describe an important result of Kerov ([Ke]), which says that every essential step beyond the above mentioned special cases has to avoid the 'superorthogonality' of the weight.

Notice that the existence of an expansion (0.8) of a symmetric function into a weighted sum of monomial symmetric functions does not pose any restriction on the w-Kostka factors $w_\lambda(\mu)$ and can therefore not be regarded as 'combinatorially meaningfull'. To the contrary the τPx-approach achieves this goal in a natural way (compare the above discussion of (0.9) and S-insected weights); therefore the existence of weighted τPx-formulas appears to be a basic step in the construction of the universal symmetric functions U_λ.

Clearly, with some labor one can figure out weights w, which contain more or less complete information about the elements $\eta \in G(\lambda)$, but at present it seems very difficult to do this in a way, which makes $\{\lambda\}_w$ symmetric; the action of permutations on SSYT described in Section 4 may be helpfull in this respect.

The construction and investigation the universal symmetric functions U_λ is intimately connected with a *unified* treatment of the following problems:

(1) Combinatorial proofs of the symmetry of the weighted symmetric functions (HL, Jack, and Macdonald).
(2) Combinatorial proofs of 'Cauchy identities': the Robinson-Schenstedt-Knuth (RSK) correspondence 'solves' the Schur case (cf. [S, Sec.4.8]); and for super Schur functions a generalization has been given by Remmel ([Re]).
(3) The combinatorial treatment of Kostka-Foulkes polynomials (see Remarks 3.6 and 5.9).
(4) The possibility of choosing the symmetric weight w in such a way that the w-Kostka factors $w_\lambda(\mu)$ are *polynomials with integer coefficients*: for $w = w^S, w^{HL}$ this is trivially true, because all $w(\eta)$ are already such polynomials; in the Jack case $w = w^J$ the corresponding conjecture by R.P. Stanley and I.G. Macdonald [St2,M3] has been proven recently by F. Knop and S. Sahi [KS], who gave in addition a combinatorial formula for the computation of the $w^J_\lambda(\mu)$, and by L. Lapointe and L. Vinet [LV1, LV2]; in the Macdonald case $w = w^M$ proofs have been given independantely by L. Lapointe and L. Vinet [LV3], F. Knop [Kn], A.M. Garsia and J. Remmel [GR], and A.N. Kirillov and M. Noumi [KN].

An essential tool in proving the latter facts is the generation of symmetric functions associated to a partition $\lambda \equiv \lambda_1 \dots \lambda_s$ by a sequence of 'creation operators', say

$$(K_s)^{\lambda_s}(K_{s-1})^{\lambda_{s-1}-\lambda_s} \dots (K_1)^{\lambda_1-\lambda_2}(1) \ .$$

This approach has been pionered by J.N. Bernstein for Schur functions (cf.[Z, p.69]), and subsequently extended for Hall-Littlewood functions by N. Jing [J], for Jack functions by L. Lapointe and L. Vinet [LV1], and for Macdonald functions by L. Lapointe and L. Vinet [LV4] and A.N. Kirillov and M. Noumi [KN]. With regard to explicit computations of the respective polynomials however, one sees that setting up the creation operator formulas is easier as in the τPx-case, but the evaluation usually will involve the massive occurence of cancelations due to the appearence of divided difference operators.

We finally remark that the existence of τPx-formulas is not restricted to symmetric functions, but can be extended to sequences of (in general nonsymmetric) Schubert polynomials, which contain the Schur polynomials as special cases. This is the subject of the paper [W2] about 'graded Schubert functions'.

1. Schur polynomials

In this section we mainly present Thomas' results on τPx-formulas for Schur polynomials, but we carefully seperate the combinatorial basis of his construction in terms of SSYT from its algebraic translation. The result of Prop.1.10 is new.

In [T1] Thomas introduced a partitioning of $SSYT(\lambda)$ into equivalence classes $\overline{\zeta} \equiv SSYT(\zeta)$, which have as canonical representatives exactly the standard Young tableaux $\zeta \in SYT(\lambda)$. Every such $\overline{\zeta} \subset SSYT(\lambda)$ is $\mathbb{N}$-graded and cumulative:

$$SSYT_{[m]}(\zeta) := \overline{\zeta} \cap SSYT_{[m]}(\lambda) \text{ and } SSYT_{(m)}(\zeta) := \overline{\zeta} \cap SSYT_{(m)}(\lambda) ,$$

whence $SSYT(\zeta) = \biguplus_{m=1}^{\infty} SSYT_{[m]}(\zeta) = \bigcup_{m=1}^{\infty} SSYT_{(m)}(\zeta)$. The following definition, which seems to have been considered first by Schensted [Sch], introduces the basic concept for the whole paper:

Definition 1.1. *To every $\eta \in SSYT(\lambda)$ associate a numbering $\zeta(\eta)$ of the Ferrer diagram of shape λ by attaching the numbers $1, \dots, N$ to the boxes of the Ferrer diagram λ in the following linear order w.r.t. η:*
1. the box with smaller η-label precedes the box with greater η-label,
2. in case of the equal η-labels the lower box precedes, and
3. in case of equal η-labels and equal rows the box more to the left precedes.
The above linear order will be called the **standard order** *on η.*

Clearly $\zeta(\eta) \in SYT(\lambda)$, such that sets

$$\overline{\zeta} \equiv SSYT(\zeta) := \{\eta \in SSYT(\lambda) \mid \zeta(\eta) = \zeta\}$$

give the desired partition of $SSYT(\lambda)$.

If on the other hand $\zeta \in SYT(\lambda)$ is given and ρ is a 'suitable' content vector, then one can construct uniquely an $\eta \in SSYT(\lambda, \rho)$, such that $\zeta(\eta) = \zeta$: just number the boxes of λ in the order given by ζ with ρ_1 one's, ρ_2 two's, etc. . Hence for arbitrary $\rho \in F_N^{[m]}$ the set $\overline{\zeta} \cap SSYT(\lambda, \rho)$ has cardinality 0 or 1.

Example 1.2. For $\eta = \begin{array}{|c|c|c|c|c|}\hline 2&3&5&5&8\\ \hline 3&6&6&8\\ \cline{1-4} 5&8\\ \cline{1-2}\end{array} \in SSYT_{[8]}(\ 5\ 4\ 2\ )$ one gets $\zeta(\eta) = \begin{array}{|c|c|c|c|c|}\hline 1&3&5&6&11\\ \hline 2&7&8&10\\ \cline{1-4} 4&9\\ \cline{1-2}\end{array}$; η can be recovered from ζ and the content vector $\rho = (0, 1, 2, 0, 3, 2, 0, 3)$.

For fixed $\zeta \in SYT(\lambda)$ we speak of the box with ζ-label $\nu \in \{1, \dots, N\}$ in λ as the 'ζ-box ν'. If the row number of some ζ-box ν' is greater than the row number of another ζ-box ν, we simply say 'ν' is ζ-below ν' or 'ν is ζ-above ν''; similarly: 'ν' is ζ-left to ν', 'ν is ζ-right to ν' '.

For later use we introduce a related notion applicable to an arbitrary $\eta \in SSYT(\lambda)$: running through the boxes of η in standard order, the next box is reached by an *S-step*, if it is 'below', and by a *P-step* otherwise.

Lemmma 1.3. *Fix $\zeta \in SYT(\lambda)$ and let (– only for this Lemma –) η be an* arbitrary *numbering of λ. Let furthermore '$\eta(\nu)$' be a shorthand for 'the number of the $\zeta(\eta)$-box ν*

in η' and $M(\zeta)$ be the set of arbitrary numberings η of λ subjected to the two conditions:

$$1.)\ \eta(1) \leq \cdots \leq \eta(N),\ \textit{and}\ 2.)\ [\ \nu+1\ \textit{below}\ \nu \Longrightarrow \eta(\nu) < \eta(\nu+1)\ ]\ .$$

Then $\overline{\zeta} = M(\zeta)$.

Proof. First let $\eta \in \overline{\zeta}$, then the procedure given above for finding the representative ζ for a given $\eta \in SSYT(\lambda)$ immediately implies the validity of conditions 1 and 2. Let on the other hand $\eta \in M(\zeta)$ be given, then also immediately $\eta \in \overline{\zeta}$, *provided* $\eta \in SSYT(\lambda)$; but conditions 1 and 2 already imply $\eta \in SSYT(\lambda)$: let ν and ν' be two ζ-boxes and ν and ν' in the same row with ν' right to ν, then $\nu < \nu'$ and by condition 1: $\eta(\nu) \leq \eta(\nu')$; or ν and ν' in the same column with ν' below ν, then $\nu < \nu'$ and by condition 2: $\eta(\nu) < \eta(\nu')$. □

Now Lemma 1.3 and the definitions of s_λ, $s_\lambda^{(m)}$ and $s_\lambda^{[m]}$ immediately imply:

Proposition 1.4. *Let $\lambda \vdash N$, then*

$$s_\lambda = \sum_{\zeta \in SYT(\lambda)} \sum_{\eta \in M(\zeta)} x^\eta\ , \quad s_\lambda^{(m)} = \sum_{\zeta \in SYT(\lambda)} \sum_{\substack{\eta \in M(\zeta) \\ \eta(N) \leq m}} x^\eta\ ,$$

$$\textit{and} \quad s_\lambda^{[m]} = \sum_{\zeta \in SYT(\lambda)} \sum_{\substack{\eta \in M(\zeta) \\ \eta(N) = m}} x^\eta\ .$$

The definition of $M(\zeta)$ suggests another one: that of the *descent set* $D(\zeta)$ for any $\zeta \in SYT(\lambda)$:

$$D(\zeta) := \{\ i \mid i+1 \text{ below } i\ \}.$$

Then Prop 1.4 says, that the contribution of a $\zeta \in SYT(\lambda)$ to s_λ resp. $s_\lambda^{[m]}$ is

$$\sum_{\substack{1 \leq i_1 \leq \cdots \leq i_N \\ i_\nu \in D(\zeta) \Longrightarrow i_\nu < i_{\nu+1}}} x_{i_1} \ldots x_{i_N} \quad \text{resp.} \quad \sum_{\substack{1 \leq i_1 \leq \cdots \leq i_N = m \\ i_\nu \in D(\zeta) \Longrightarrow i_\nu < i_{\nu+1}}} x_{i_1} \ldots x_{i_N}\ .$$

Let R be a commutative ring with unit and $x = (x_1, x_2, x_3, \ldots)$ a sequence of variables; then

$$A \equiv A_R(x) := (\ R[x_1],\ R[x_1, x_2],\ R[x_1, x_2, x_3],\ \ldots)$$

is a R-algebra under componentwise operations. For every $a = (a_1, a_2, \ldots) \in A$ we denote the n^{th}-component a_n by $[a]_n$. The *shift operator*

$$\tau\ :\ A \longrightarrow A,\ (a_1, a_2, a_3, \ldots) \mapsto (0, a_1, a_2, \ldots)\ ,$$

i.e. $[\tau a]_n := a_{n-1}$, where $a_n = 0$ for $n \leq 0$, and all its powers τ^ν for $\nu \in \mathbb{N}_0$ ($\tau^0 := id$) are algebra endomorphism of A; consequently the same is true for all operators $f(\tau) \in R[\tau]$ or $f(\tau) \in R[[\tau]]$, since $[A]_n$ is not affected by the τ^ν with $\nu > n$. One can calculate as usual in the rings $R[\tau]$ and $R[[\tau]]$. Note that for $x = (x_1, x_2, \ldots) \in A$ and all $n \in \mathbb{N}$

the sets $\{ \ [\tau^\nu x]_n \mid \nu \in \mathbb{N}_0 \ \}$ generate the R-algebras $[A]_n = R[x_1, \dots, x_n]$. Especially important is the 'geometric shift operator'

$$P := \sum_{\nu=0}^{\infty} \tau^\nu \ , \quad P\ (a_1, a_2, a_3, \dots) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots)\ .$$

and its companion $S := \tau P$.

A *Baxter operator* on an arbitrary commutative R-algebra A is an R-linear mapping $B : A \longrightarrow A$ such that for some fixed $\theta \in R$:

$$B(aB(b)) + B(bB(a)) = B(a)B(b) + B(\theta ab) \quad \text{for all } a, b \in A.$$

Indeed the above defined operators P and S are Baxter for $\theta = 1$ and $\theta = -1$, respectively. Baxter operators have been introduced by G. Baxter ([B]), and investigated to some extend by G.-C. Rota [R1,R2,RS] and P. Cartier [C]. The basic result is the isomorphism between the *standard Baxter algebra*, containing the symmetric functions and the above Baxter operators P and S, and the *free Baxter algebra* (in the sense of universal algbra), which makes it possible to prove results for the free Baxter algebra with the help of symmetric functions. One example in this direction is the remarkebly short proof of the Bohnenblust-Spitzer formula of fluctuation theory by Rota.

It is not hard to see (e.g. by induction) that a sequence of the form

$$(\dots, a_n, \dots) \equiv (\ \dots\ ,\ \sum_{\substack{1 \le i_1 \le \dots \le i_N = n \\ i_\nu \in D \Longrightarrow i_\nu < i_{\nu+1}}} x_{i_1} \dots x_{i_N}\ ,\ \dots\ )\ \in\ A_{\mathbb{Z}}(x)$$

can be written as the *Baxter sequence*

$$B_1 \dots B_{N-1}(x) := x B_{N-1} \dots x B_1 x\ ,$$

where for $\nu = 1, \dots, N-1$: $B_\nu \in \{P, S\}$ and $B_\nu = S$ iff $\nu \in D$. Together with Prop.1.4 this shows

Theorem 1.5. (G.P. Thomas [T1]) *For* $\lambda \vdash N$ *and* $\zeta \in SYT(\lambda)$ *let* $B(\zeta) := B_1 \dots B_{N-1}$ *denote a sequence of operators* $B_\nu \in \{P, S\}$, *where* $B_\nu = S$ *iff* $\nu \in D(\zeta)$; *then*

(S)
$$s_{[\lambda]} = \sum_{\zeta \in SYT(\lambda)} B(\zeta)(x)\ .$$

Example 1.6. Let $\lambda = 2^2 1$. The columns below show $\zeta \in SYT(\lambda)$, $D(\zeta)$, $B(\zeta)$, and $\rho(\widehat{\zeta})$ (cf. Def.3.3).

1 2	1 2	1 3	1 4	1 3
3 4	3 5	2 5	2 5	2 4
5	4	4	3	5
$\{2,4\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,3,4\}$
$PSPS$	$PSSP$	$SPSP$	$SSPS$	$SPSS$
(2,2,1)	(2,1,2)	(1,2,2)	(1,1,2,1)	(1,2,1,1)

Therefore $s_{[2^2 1]}(x) = PSPS(x) + PSSP(x) + SPSP(x) + SSPS(x) + SPSS(x)$.

Proposition 1.7. *Let λ' be the conjugate partition to $\lambda \vdash N$ and $B'(\zeta)$ the same as $B(\zeta)$ in Thm.1.5, but with $P \leftrightarrow S$ (P exchanged by S and vice versa); then*

$$a)\ D(\zeta') = D'(\zeta) := \{1, \ldots, N-1\} \setminus D(\zeta) \ ;$$

$$b)\ B(\zeta') = B'(\zeta) \ \ \text{for all } \zeta \in SYT(\lambda) \ \text{and} \quad s_{[\lambda']}(x) = \sum_{\zeta \in SYT(\lambda)} B'(\zeta)(x) \ .$$

Proof. The conjugation of the Ferrer diagram λ induces a bijection $\ ' : SYT(\lambda) \longrightarrow SYT(\lambda')$. Fix $\zeta \in SYT(\lambda)$ and two consecutive ζ-boxes ν and $\nu+1$; then one has two possibilities: *either* $\nu+1$ is below ν in ζ, then $\nu+1$ not right to ν and therefore in ζ' is $(\nu+1)'$ not below ν', *or* $\nu+1$ not below ν, then $\nu+1$ right to ν and in ζ' is $(\nu+1)'$ below ν'. This shows (a) . Part (b) is then immediate by the definition of $B(\zeta)$ and $B'(\zeta)$. □

The elementary and complete symmetric polynomials have an especially simple representation in terms of Baxter sequences:

$$s_{[1^N]}(x) = (xS)^{N-1}(x) \quad \text{and} \quad s_{[N]}(x) = (xP)^{N-1}(x) \ .$$

Remark 1.8. Thomas developed his arguments more generally for 'numbered frames' = 'columnstrictly numbered finite subsets of unit squares in the $\mathbb{Z} \times \mathbb{Z}$-plane'. In fact the set D used for the definition of the (graded) functions of the type $B(\zeta)$ can originate in many different ways, for example as a set of ranks of a poset in [St1]. Functions of the type $B(\zeta)$, now called 'fundamental quasisymmetric functions', have been investigated more systematically by I. Gessel [G] and subsequently many others.

Remark 1.9. A natural question that comes to mind is: what happens, if P and S in Thm.1.5 are substituted more generally by shift operators $f(\tau) = \sum_{i=0}^{\infty} f_i \tau^i$, $(f_0 \neq 0)$ and $g(\tau) = \sum_{i=1}^{\infty} g_i \tau^i$, $(g_1 \neq 0)$? Without loss of generality we can assume $f_0 = g_1 = 1$. The requirement that the generalized sequences "$s_{[\lambda]}(x; f, g)$" are sequences of symmetric polynomials forces $f(\tau) = P$ and $g(\tau) = S$, as can be seen easily by investigating the cases $\lambda = 2$ and $\lambda = 1^2$. Therefore the only possibility to generate more general symmetric functions by τPx-formulas is to introduce appropriate 'weights', i.e. extend $A_{\mathbb{Z}}(x)$ to a ring $A_R(x)$, where $R \supsetneq \mathbb{Z}$.

If on the other hand one dispenses with symmetry and allows arbitrary scalars f_i, g_i in the style of "umbral calculus", then the resulting polynomials should share many properties of Schur polynomials.

Proposition 1.10. *i) Let $D(\lambda) := \{D(\zeta) \mid \zeta \in SYT(\lambda)\}$ and $D^*(\lambda) := \{D^*(\zeta) \mid \zeta \in SYT(\lambda)\}$ for $D^*(\zeta) := \{|\lambda| - i \mid i \in D(\zeta)\}$; then $D^*(\zeta) = D(\zeta)$.*
ii) For $\lambda \vdash N$ and $\zeta \in SYT(\lambda)$ associate to every $B(\zeta) = B_1 \ldots B_{N-1}$ as defined in Thm.1.5 the 'reversed Baxter sequence' $B^*(\zeta) := B_{N-1} \ldots B_1$; then

$$s_{[\lambda]}(x) = \sum_{\zeta \in SYT(\lambda)} B^*(\zeta)(x) \ .$$

Proof. Clearly i) and ii) are equivalent; we show i): every equivalence class $\overline{\zeta}$ contains a unique element $\widehat{\zeta}$ called the *maximal identification of* ζ, which originates from ζ by numbering the ζ-boxes $1,\dots,N$ consecutively with $\nu = 1,2,\dots$, where ν is increased by 1 iff the corresponding ζ-step is a S-step (cf. Def.3.3). Let $\rho = \rho(\widehat{\zeta})$ the content vector of $\widehat{\zeta}$ and consider the monomial x^ρ contained in some component of $s_{[\lambda]}$; by symmetry this component contains also a monomial x^{ρ^*}, where ρ^* is ρ (without end zeroes) in reversed order. This is possible only if there is some $\xi \in SYT(\lambda)$ with $\rho^* = \rho(\widehat{\xi})$, whence for this ξ one has $D(\xi) = D^*(\zeta)$. (As an illustration see Ex.1.6 .) □

2. Skew Schur and hook Schur polynomials

Let $\lambda \vdash N,\ \mu \vdash M, \mu \subset \lambda\ (M \leq N)$,and $m \in \mathbb{N}$, then for the skew shape λ/μ one defines the **skew Schur functions** ([M1, I (5.12)]) and *polynomials* as

$$s_{\lambda/\mu} := \sum_{\eta \in SSYT(\lambda/\mu)} x^\eta \quad \text{and} \quad s^{(m)}_{\lambda/\mu} := \sum_{\eta \in SSYT_{(m)}(\lambda/\mu)} x^\eta \ .$$

It is not hard to see that Thm.1.5 and Prop.1.7 apply similarly in the 'skew case', i.e.

$$\text{(sS)} \qquad s_{[\lambda/\mu]} := (s^{[m]}_{\lambda/\mu})_{m \in \mathbb{N}} = \sum_{\zeta \in SYT(\lambda/\mu)} B_{\lambda/\mu}(\zeta)(x)$$

and $s_{[\lambda'/\mu']} = \sum_{\zeta \in SYT(\lambda'/\mu')} B'_{\lambda/\mu}(\zeta)(x)$, where $SYT(\lambda/\mu)$, $D(\zeta)$, $B_{\lambda/\mu}(\zeta)$ and $B'_{\lambda/\mu}(\zeta)$ are defined completely analogous to the case $\mu = \emptyset$. Now

$$\text{(2.1)} \qquad SYT(\lambda/\mu) = M_{-}\{\zeta \in SYT(\lambda) \mid \lambda(\zeta^{(M)}) = \mu\} \ ,$$

which should be understood as follows: the set $SYT(\lambda/\mu)$ can be obtained from the set $SYT(\lambda)$ by first selecting all $\zeta \in SYT(\lambda)$ for which the numbers $1,\dots,M$ fill exactly a sub-SYT of shape μ and second subtracting M from all entries and cancel all boxes with entries ≤ 0.

This procedure extends to an easy method for obtaining the 'skew Baxter sequences' $B_{\lambda/\mu}(\zeta)$ from the 'full Baxter sequences' $B_\lambda(\zeta) \equiv B(\zeta)$: take the Baxter sequences $B_\lambda(\zeta)$ of all elements in $\{\zeta \in SYT(\lambda) \mid \lambda(\zeta^{(M)}) = \mu\}$ and cancel the first M symbols; this gives the 'skew Baxter sequences' $B_{\lambda/\mu}(\zeta)$ for $\zeta \in SYT(\lambda/\mu)$.

Example 2.1. Let $\lambda = 2^2 1$ and $\mu = 1^2$. Then in Ex.1.6 exactly the last three SYT have the property that the numbers 1 and 2 fill a sub-SYT of shape μ. Deletion of the first two entries of their Baxter sequences gives SP, PS and SS, whence: $s_{[2^2 1/1^2]}(x) = SP(x) + PS(x) + SS(x)$.

The above described procedure also yields an economic way to compute 'extended τPx-formulas' for the **hook Schur** or **super Schur polynomials**, the definition of which we recall next (for details see [BR,Re]):

For $k,l \in \mathbb{N}_0$ and $\lambda \vdash N$ one defines the set $SST_{(k,l)}(\lambda)$ of (k,l)-*semistandard tableaux of shape* λ as the set of all numberings of the Ferrer diagram λ with numbers from $\{1,\dots,k\} \cup \{\overline{1},\dots,\overline{l}\ \}$ (– the second set is the set of 'overlined' numbers $1,\dots,l$ –), such that there is some β, $\emptyset \subset \beta \subset \lambda$ with 'β columnstrict with entries in $\{1,\dots,k\}$' and 'λ/β

rowstrict with entries in $\{\bar{1},\ldots,\bar{l}\}$'. Hence the set $SST_{(k,l)}(\lambda)$ can be more conveniently described as a set of pairs of SSYT:

$$SST_{(k,l)}(\lambda) \cong \biguplus_{\beta:\ \emptyset\subset\beta\subset\lambda} SSYT_{(k)}(\beta) \times SSYT_{(l)}(\lambda'/\beta') \ . \tag{2.2}$$

The *hook Schur polynomials* $HS_\lambda^{(k,l)}(x;y)$ are defined combinatorially as

$$HS_\lambda^{(k,l)}(x;y) \equiv HS_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_l) := \sum_{T\in SST_{(k,l)}(\lambda)} w(T) \ ,$$

where for $T \equiv (\eta,\eta') \in SSYT_{(k)}(\beta)\times SSYT_{(l)}(\lambda'/\beta')$ we forget the bars and set $w(T) := x^\eta y^{\eta'}$. The definition and the above bijection (2.2) immediately imply:

$$HS_\lambda^{(k,l)}(x;y) = \sum_{\beta:\ \emptyset\subset\beta\subset\lambda} s_\beta^{(k)}(x)\ s_{\lambda'/\beta'}^{(l)}(y) \ . \tag{2.3}$$

Moreover ([BR, Thm.6.13]):

$$HS_\lambda^{(k,l)} = HS_{\lambda'}^{(l,k)} \ . \tag{2.4}$$

Setting

$$SST_{[k,l]}(\lambda) := \biguplus_{\beta:\ \emptyset\subset\beta\subset\lambda} SSYT_{[k]}(\beta) \times SSYT_{[l]}(\lambda'/\beta')$$

gives us the possibility to express the $HS_\lambda^{(k,l)}(x;y)$ for all pairs $(k,l)\in\mathbb{N}_0\times\mathbb{N}_0$ simultaneously as

$$HS_\lambda^{(k,l)}(x;y) = \sum_{i=0}^{k}\sum_{j=0}^{l} HS_\lambda^{[i,j]}(x;y) \ ,$$

where

$$HS_\lambda^{[i,j]}(x;y) := \sum_{T\in SST_{[i,j]}(\lambda)} w(T) = \sum_{\beta:\ \emptyset\subset\beta\subset\lambda} s_\beta^{[i]}(x)\ s_{\lambda'/\beta'}^{[j]}(y) \equiv \sum_{\beta:\ \emptyset\subset\beta\subset\lambda} HS_{\lambda,\beta}^{[i,j]}(x;y) \ . \tag{2.5}$$

and

(hS)

$$HS_{[\lambda]}(x;y) := \left(HS_\lambda^{[i,j]}(x;y)\right)_{i,j\geq 0} = \sum_{\beta:\ \emptyset\subset\beta\subset\lambda} \left(HS_{\lambda,\beta}^{[i,j]}(x;y)\right)_{i,j\geq 0} \equiv \sum_{\beta:\ \emptyset\subset\beta\subset\lambda} HS_{[\lambda],\beta}(x;y)$$

is an $\mathbb{N}_0\times\mathbb{N}_0$-grading of the product $s_\beta(x)\ s_{\lambda'/\beta'}(y)$. Note that $SST_{[0,0]}(\lambda) = \emptyset \implies HS_\lambda^{[0,0]}(x;y) = 0$, $HS_\lambda^{[k,0]}(x;y) = s_\lambda^{[k]}(x)$ and $HS_\lambda^{[0,l]}(x;y) = s_{\lambda'}^{[l]}(y)$.

In analogy to Section 1 we define for any commutative unitary ring R and sequences of variables $x = (x_1,x_2,x_3,\ldots)$, $y = (y_1,y_2,y_3,\ldots)$ the $\mathbb{N}_0\times\mathbb{N}_0$-array of polynomial rings

$$A_R(x;y) := (R[x_1,\ldots,x_i;y_1,\ldots,y_j])_{i,j\geq 0} \ ,$$

where $[A_R(x;y)]_{0,0} := R$, $[A_R(x;y)]_{i,0} := R[x_1,\ldots,x_i]$ and $[A_R(x;y)]_{0,j} := R[y_1,\ldots,y_j]$. $A_R(x;y)$ is a commutative R-algebra under componentwise operations. Furthermore we define $x, y \in A_R(x;y)$ by $[x]_{ij} := \delta_{0,j}\ x_i$ and $[y]_{ij} := \delta_{i,0}\ y_j$ and the two commuting shift operators $\tau, \overline{\tau}$ by

$$[\tau^k \overline{\tau}^l a]_{ij} := a_{i-k,j-l} \quad (\text{ with } a_{i-k,j-l} := 0, \text{ if } k > i \text{ or } l > j, \text{ for all } a = (a_{ij})_{i,j\geq 0}).$$

Let P and S as in Section 1 and in addition $\overline{P} := \sum_{\nu=0}^{\infty} \overline{\tau}^{\nu}$, $\overline{S} := \overline{\tau}\,\overline{P}$.

We observe next that one has the isomorphism of algebras $A_R(x;y) \cong A_R(x) \otimes A_R(y)$, where for $f(x) = (f_0, f_1(x_1), \ldots, f_i(x_1, \ldots, x_i), \ldots) \in A_R(x)$ and $g(y) = (g_0, g_1(y_1), \ldots, g_j(y_1, \ldots, y_j), \ldots) \in A_R(y)$ we define:

$$f(x) \otimes g(y) := (f_i(x_1, \ldots, x_i) \otimes g_j(y_1, \ldots, y_j))_{i,j\geq 0} \equiv (f_i(x_1, \ldots, x_i) g_j(y_1, \ldots, y_j))_{i,j\geq 0} \ .$$

Under the above isomorphism it follows from (2.5) and (hS) that

$$HS_{[\lambda],\beta}(x;y) = s_{[\beta]}(x) \otimes s_{[\lambda'/\beta']}(y) \ . \qquad (2.6)$$

Using the results of Section 1 and formula (2.6) it is easy now to compute the 'extended τPx-expansions' of the summands $HS_{[\lambda],\beta}(x;y)$ from the set $SYT(\lambda)$:

(1) Compute the (regular) 'τPx-expansion' of $s_\beta(x)$ according to Thm.1.5 (cf. Rem.2.3 below);
(2) Compute the (regular) '$\overline{\tau}\overline{P}y$-expansion' of the skew Schur functions as described in the beginning of this section;
(3) Take the $\otimes$-product over the two sums obtained in steps 1 and 2, where in the '$\overline{\tau}\overline{P}y$-factor' $\overline{P} \leftrightarrow \overline{S}$ ($\overline{P}$ are substituted by $\overline{S}$ and vice versa) by Prop.1.7.

Example 2.2. Let $\lambda = 2^2 1$. According to (hS) we list all β, $\emptyset \subset \beta \subset \lambda$ together with the 'extended τPx-expansion' of the $HS_{[\lambda],\beta}(x;y)$:

β	$HS_{[\lambda],\beta}(x;y)$
221	$(PSPS + PSSP + SPSP + SSPS + SPSS)(x)$
22	$(PSP + SPS)(x) \otimes y$
211	$(PSS + SPS + SSP)(x) \otimes y$
21	$(PS + SP)(x) \otimes (\overline{S} + \overline{P})(y)$
2	$P(x) \otimes (\overline{PS} + \overline{SP})(y)$
111	$SS(x) \otimes \overline{P}(y)$
11	$S(x) \otimes (\overline{PS} + \overline{SP} + \overline{PP})(y)$
1	$x \otimes (\overline{PSP} + \overline{PPS} + \overline{SPS} + \overline{PSP} + \overline{SPP})(y)$
$\emptyset$	$(\overline{SPSP} + \overline{SPPS} + \overline{PSPS} + \overline{PPSP} + \overline{PSPP})(y)$

Note that in accordance with formula (2.4) the sum $\sum_{\beta:\ \emptyset\subset\beta\subset\lambda} HS_{[\lambda],\beta}(x;y)$ of the above 'extended τPx-expressions' is invariant in a nontrivial way under the simultaneous exchanges $x \leftrightarrow y$, $P \leftrightarrow \overline{S}$ and $S \leftrightarrow \overline{P}$.

Remark 2.3. A very pleasant property of the τPx-expansions of Schur functions is its **recursive structure** with respect to the Young lattice $\mathcal{Y}$:

(1) Suppose the τPx-expansion of some $s_{[\lambda]}(x)$ is available; then the τPx-expansions of all $s_{[\overline{\lambda}]}(x)$ with $\overline{\lambda} \subset \lambda$ can be computed in a fashion similar to the skew case:
let $\lambda \vdash N$, $\overline{\lambda} \vdash M$ and $M < N$; for every $\overline{\zeta} \in SYT(\overline{\lambda})$ single out one $\zeta \in SYT(\lambda)$, which contains $\overline{\zeta}$ as sub-SYT; in the corresponding $B(\zeta)(x)$ delete the last $N - M$ symbols P or S, e.g. for $\lambda = 2^2 1$ and $\mu = 2^2$ we (must) choose the first and the last SYT in Ex.1.6 to compute: $s_{[2^2]}(x) = PSP(x) + SPS(x)$.

(2) Suppose that for all $\lambda \vdash N$ the τPx-expansions of the $s_{[\lambda]}(x)$ have been computed; let $\overline{\lambda} \vdash N + 1$ and

$$C(\overline{\lambda}) := \{ \lambda \vdash N \mid \overline{\lambda} \text{ covers } \lambda \text{ in } \mathcal{Y} \}$$

be the $\overline{\lambda}$-*covered set in* $\mathcal{Y}$; then $s_{[\overline{\lambda}]}(x)$ can be determined as the sum of all τPx-expansions of the $s_{[\lambda]}(x)$ with $\lambda \in C(\overline{\lambda})$, where a symbol P [or S] is added to the right of a $B(\zeta)$ iff the single box in $\overline{\lambda}/\lambda$ is right [or below] the ζ-box N. As an example study again Ex.1.6 and observe that $C(2^2 1) = \{21^2, 2^2\}$.

Remark 2.4. The **noncommutative Schur functions** as pioneered by Lascaux and Schützenberger and developed further e.g. by Fomin and Greene in [FG] (— not to be confused with the noncommutative symmetric functions of [GKLLRT] —) are defined as sums over SSYT, where the entries of some $\eta \in SSYT(\lambda)$ are to be read columnwise bottom-up and the columns from left to right in order to yield the sequence of non-commuting variables in the monomial x^η. Since for every $\zeta \in SYT(\lambda)$ the sequence of multiplications by x in $B(\zeta)(x)$ is done in the ζ-standard order, it is not hard to accustom the evaluation of $B(\zeta)(x)$ in such a way that one gets the noncommutative graded Schur functions.

3. The descent polynomial of a partition and the lattices $G(\zeta)$

In this section we investigate for all partitions λ the sequences $\lambda^\sharp := (\lambda_1^\sharp, \lambda_2^\sharp, \dots)$ of numbers $\lambda_m^\sharp := |SSYT_{[m]}(\lambda)|$. Obviously one has

$$\lambda^\sharp := \sum_{\zeta \in SYT(\lambda)} B(\zeta)(\mathbf{1}) \quad \text{with} \quad \mathbf{1} = (1, 1, \dots)\ . \tag{3.1}$$

Since all factors $x = \mathbf{1}$ except the rightmost can be neglected and shift operators in $R[\tau]$ commute, one has

$$B(\zeta)(\mathbf{1}) = \tau^{|D(\zeta)|}\ P^{N-1}(\mathbf{1}) = \tau^{|D(\zeta)|}\ N^\sharp \quad \text{for all } \zeta, \tag{3.2}$$

where we used the special τPx-formula (just in front of Rem.1.8) for the graded complete symmetric functions $s_{[N]}(x)$.

Definition 3.1. *For any partition* λ *the* descent polynomial $D_\lambda \in R[\tau]$ of λ *is defined as*

$$D_\lambda(\tau) := \sum_{\zeta \in SYT(\lambda)} \tau^{|D(\zeta)|}\ .$$

An immediate consequence from the above discussion is:

$$(\sharp) \qquad \lambda^\sharp := D_\lambda(\tau)\; N^\sharp \quad \text{for all } \lambda \vdash N \ .$$

Example 1.6 shows: $D_{2^21}(\tau) = 3\tau^2 + 2\tau^3$ and $D_{32}(\tau) = 2\tau + 3\tau^2$. In the special case where $\lambda = (n-k+1)1^k$, i.e. λ is a (n,k)-*hook* with 'arm length' $n-k$ and 'leg length' k, one has $D_\lambda(\tau) = \binom{n}{k}\tau^k$: each of the k boxes in the leg can be reached only by descent steps, which can be done in $\binom{n}{k}$ ways. In general a simple formula for the descent polynomial D_λ doesn't seem to exist, so that one has explicitly to compute the set of descent sets

$$D(\lambda) := \{D(\zeta) \mid \zeta \in SYT(\lambda)\} \ .$$

For the numbers $[P^N(\mathbf{1})]_m$ we have the recursion

$$[P^N(\mathbf{1})]_m = \sum_{\nu=1}^{m} [P^{N-1}(\mathbf{1})]_\nu = [P^{N-1}(\mathbf{1})]_m + [P^N(\mathbf{1})]_{m-1} \ ,$$

and the 'initial conditions' $[P^N(\mathbf{1})]_1 = 1 = [P^0(\mathbf{1})]_m$ for all $m, N \in \mathbb{N}$, whence we conclude

$$N^\sharp_m = [P^{N-1}(\mathbf{1})]_m = \binom{m+N-2}{N-1} \quad \text{for all } m, N \in \mathbb{N} \ .$$

We have therefore proven

Proposition 3.2. *For $\lambda \vdash N$ one has:*

$$(3.3) \qquad \lambda^\sharp_m = D_\lambda(\tau) N^\sharp_m = \sum_{\zeta \in SYT(\lambda)} \binom{m - |D(\zeta)| + N - 2}{N-1} \ .$$

As a special case we note

$$(1^N)^\sharp_m = \tau^{N-1}\binom{m+N-2}{N-1} = \binom{m-(N-1)+N-2}{N-1} = \binom{m-1}{N-1} \ .$$

Definition 3.3. *For any partition λ and every $\zeta \in SYT(\lambda)$ define the sets of gapless elements for ζ as*

$$G(\zeta) := SSYT(\zeta) \cap G(\lambda) \ .$$

We introduce now further no(ta)tions yielding as a byproduct that the sets $G(\zeta)$ can be equipped with a partial order turning them into boolean lattices:

Label the step from the ζ-box ν to the ζ-box $\nu+1$ with ν, and let $I(\zeta) \subset \{1, \dots, N-1\}$ be the subset of (ζ-relative) P-steps, i.e. $I(\zeta) = D(\zeta')$ (cf. Prop.1.7 a)) as sets of numbers. For later use we subdivide the set of P-steps in ζ further into the set $I_0(\zeta)$ of P_0-steps*, where the ζ-boxes ν and $\nu+1$ are in consecutive columns, and the set $I_1(\zeta)$ of* P_1-steps*, where the is at least one column between ζ-boxes ν and $\nu+1$.*

Given any subset $I \subset I(\zeta)$ let $\widehat{\zeta}^{\,I} \in G(\zeta)$ denote the unique element, which originates from ζ by the following procedure: label the upper left box with '1' and run through the boxes $1, \dots, N$ of ζ, where the label remains the same exactly in the steps contained in I and otherwise is increased by 1. Special cases are $\zeta = \widehat{\zeta}^{\,\emptyset}$ and the maximal identification *$\widehat{\zeta} := \widehat{\zeta}^{\,I(\zeta)}$. If on the other hand one begins with some $\eta \in G(\zeta)$, one can compute a set*

$I(\eta)$ by running through the η-boxes in standard order, such that $\eta = \widehat{\zeta}^{\,I(\eta)}$. We call the set $I \equiv I(\eta)$ characterizing one element of $G(\zeta)$ the (ζ-relative) identification set *of η. (Moreover, set $I_0(\eta) := I(\eta) \cap I_0(\zeta)$ and $I_1(\eta) := I(\eta) \cap I_1(\zeta)$.)*

Therefore $G(\lambda)$ can be equipped with the order structure induced by the boolean lattice $\mathcal{B}(I(\zeta))$ of all subsets of $I(\zeta)$ ordered by inclusion; the top element is $\widehat{\zeta}$ and the bottom element is ζ.

Example: Let $\eta = \begin{array}{|c|c|c|c|c|}\hline 1&1&2&3&5\\ \hline 2&2&4&5\\ \cline{1-4} 4&6\\ \cline{1-2}\end{array} \in G(\zeta)$, where $\zeta = \begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&10\\ \hline 3&4&8&9\\ \cline{1-4} 7&11\\ \cline{1-2}\end{array}$; then $D(\zeta) = \{2, 6, 10\}$, $I(\zeta) = \{1, 3, 4, 5, 7, 8, 9\}$, $I(\eta) = \{1, 3, 4, 7, 9\}$, $I_0(\zeta) = \{1, 3, 4, 5, 8, 9\}$, $I_0(\eta) = \{1, 3, 4, 9\}$, and $I_1(\zeta) = I_1(\eta) = \{7\}$.

Proposition 3.4. *For $\lambda \vdash N$ one has:*

(a) $$D_{\lambda'}(\tau) = \tau^{N-1} D_\lambda(\tau^{-1}) \ ;$$

(b) $$|G(\lambda)| = D_{\lambda'}(2) = 2^{N-1} D_\lambda(\frac{1}{2}) \ .$$

Proof. a) Since $|D(\zeta')| = N - 1 - |D(\zeta)|$ by Prop.1.7 a), the assertion follows from the definition of the descent polynomial. For b) observe $G(\lambda) = \biguplus_{\zeta \in SYT(\lambda)} G(\zeta)$ and $|\mathcal{B}(I(\zeta))| = 2^{|I(\zeta)|} = 2^{|D(\zeta')|}$; hence:

$$|G(\lambda)| = \sum_{\zeta \in SYT(\lambda)} |G(\zeta)| = \sum_{\zeta \in SYT(\lambda)} 2^{|D(\zeta')|} = D_{\lambda'}(2) \overset{a)}{=} 2^{N-1} D_\lambda(\frac{1}{2}) \ .$$

□

Remark 3.5. For fixed partition λ and $\zeta \in SYT(\lambda)$ the sets $P(\zeta) := \overline{\zeta} \cap P(\lambda)$ are sublattices of the respective $G(\zeta)$:

On the set $P(N)$ of partitions of $N = |\lambda|$ define the *refinement order*, in which ' $\overline{\lambda}$ is covered by λ ' :*iff* ' $\overline{\lambda}$ originates from λ by subdivision of exactly one part of λ into two parts '. (Clearly, the refinement order turns $P(N)$ into a lattice, which is in general different from the dominance order lattice.)

In terms of the partitionlike content of a tableaux $\eta \in G(\lambda)$ the above 'subdivision' corresponds to 'adding one more element to the identification set'. Let $\lambda(\rho(\eta))$ be the partition of the unique element of $P(\zeta)$, which has the maximal possible identification set; then $P(\zeta)$ is embedded into $G(\zeta)$ as a poset isomorphic to the principal order ideal in $P(N)$ generated by $\lambda(\rho(\eta))$, and is therefore itself a lattice.

Remark 3.6. Let $\lambda, \mu \vdash N$, $n(\mu) := \sum_{j \geq 1} (j-1)\mu_j$, and $D^{\Sigma}_{\lambda}(t) := \sum_{\zeta \in SYT(\lambda)} t^{D^{\Sigma}(\zeta)}$ with $D^{\Sigma}(\zeta) := \sum_{\nu \in D(\zeta)} \nu$. The *Kostka-Foulkes polynomials* $K_{\lambda\mu}(t)$ appearing as coefficients in the expansion of $s_\lambda(x)$ into the HL-polynomials $P_\mu(x;t)$ (cf. [M1, Sec. III 6] and [Bu]) can be characterized combinatorially as

$$K_{\lambda\mu}(t) = \sum_{\eta \in P(\lambda,\mu)} t^{c(\eta)} \quad \text{or} \quad \tilde{K}_{\lambda\mu}(t) = \sum_{\eta \in P(\lambda,\mu)} t^{cc(\eta)} ,$$

where $c(\eta) \in \mathbb{N}_0$ is the *charge* of η, $cc(\eta) \in \mathbb{N}_0$ is the *cocharge* of η and $\tilde{K}_{\lambda\mu}(t) = t^{n(\mu)}K_{\lambda\mu}(t^{-1})$. Note that the above combinatorial characterizations (found by Lascaux and Schützenberger) imply:

$$cc(\eta) = n(\mu) - c(\eta) \text{ for all } \eta \in P(\lambda, \mu). \tag{3.4}$$

Now from [M1, III 6 Ex.2] and our above notations follows:

$$K_{\lambda(1^N)}(t) = \sum_{\zeta \in SYT(\lambda')} t^{D^\Sigma(\zeta)} = D^\Sigma_{\lambda'}(t) .$$

An argument similar to that used in the proof of Prop.3.4 a) then shows:

$$\tilde{K}_{\lambda(1^N)}(t) = D^\Sigma_{\lambda}(t) .$$

This does not genera-Lise to arbitrary $\zeta \in SYT(\lambda)$: in general one has $D^\Sigma(\zeta) \neq cc(\zeta)$, which leads to the conclusion that the Kostka-Foulkes polynomials are associated to the (weighted) τPx-expansions of HL-functions (see Section 5) in a nontrivial way.

We introduce now the *Lascaux-Schützenberger (LS) order* on tableaux ("lifting back" to tableaux the procedure given on words of tableaux in [M1,Bu]):

begin with the upper left 1 and label it with a '1';
assume that your "standpoint" is an η-box $\nu \geq 1$ with label 'q' ($1 \leq q \leq \mu_1$), then there are several possibilities for the next step:

(1) there is a unlabeled η-box $\nu + 1$ below: take the rightmost of the uppermost occurrence below and label it with 'q';
(2) there is no unlabeled η-box $\nu + 1$ below, but to the right: take the rightmost of the uppermost occurrence and label it with 'q';
(3) there is no unlabeled η-box $\nu + 1$ at all: (by the partitionlike content of η there are no unlabel η-boxes $> \nu$) if $q < \mu_1$, begin with the rightmost unlabeled 1 in row one and label it with '$q + 1$'; if $q = \mu_1$, every entry in η is labeled and the procedure stops.

The 'LS order' is the linear order given by reading the labels from 1 to μ_1 and for equal labels the boxes numbered in their natural order.

It is now easy to define $c(\eta)$ [$cc(\eta)$] for $\eta \in P(\lambda, \mu)$: run through the boxes of η in LS order and attach c-indices [cc-indices] to them according to the following rule: an η-box 1 always gets the index '0'; if a box has c-[cc-]index 'r' and the next box is below, its c-index is 'r' [cc-index is 'r+1'], otherwise (provided it is not a an η-box 1) its c-index is 'r+1' [cc-index is 'r']. Then $c(\eta)$ [$cc(\eta)$] is the sum of all c-indices [cc-indices].

Note that for any η-box ν the sum of its c-index and cc-index equals $\nu - 1$ in accordance with (3.4); note further that substituting all words 'rightmost' in the definition of LS order by 'leftmost' does not change the distribution of c- and cc-indices.

Example: For $\eta = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 4 & 4 \\ 3 & 5 & & & \end{array} \in P(5^2 2, 3^3 21)$ one has the labeling $\begin{array}{ccccc} 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 & 1 \\ 1 & 1 & & & \end{array}$, the c-indices $\begin{array}{ccccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & & & \end{array}$, and the cc-indices $\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & & & \end{array}$. Hence $c(\eta) = 6$, $cc(\eta) = 13$, and $c(\eta) + cc(\eta) = 19 = n(3^3 21)$.

The complete definition of LS order on tableaux has been given here not only for the sake of selfcontainedness, but mainly for comparison with the standard order: we suspect that the "orthogonality" of the two orders will simplify the combinatorial approach to Kostka-Foulkes polynomials.

4. An action of permutations on SSYT

Every (finite) permutation π can be decomposed as a product of the *elementary transpositions* $\sigma_\nu := (\nu, \nu+1)$ $(\nu \in \mathbb{N})$, i.e. the symmetric groups S_n on n 'letters' are generated by the σ_ν $(1 \le \nu \le n-1)$ with *relations*:

(i) $$\sigma_\nu^2 = id$$

(ii) $$\sigma_\nu \sigma_{\nu'} = \sigma_{\nu'} \sigma_\nu \ , \ \text{if } |\nu - \nu'| > 1$$

(iii) $$\sigma_\nu \sigma_{\nu+1} \sigma_\nu = \sigma_{\nu+1} \sigma_\nu \sigma_{\nu+1} \ .$$

The symmetric group $S_\infty := \bigcup_{n \ge 1} S_n$ (with the natural embedding of S_n into $S_{n'}$ for $n < n'$) acts on $\rho \in F_N$ by $\pi(\rho) := (\rho_{\pi(1)}, \rho_{\pi(2)}, \dots)$. We first discuss an extension of a single σ_ν to a mapping $\sigma_\nu^K : SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \sigma_\nu(\rho))$, which has been attributed to Knuth in [S, Prop.4.4.2]. The σ_ν^K enable an elegant combinatorial proof of the symmetry of the Schur functions s_λ and of formula (0.2), but since relation (iii) is in general not valid they do not extend to an action $\pi : SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \pi(\rho))$ on the sets $SSYT(\lambda)$. We introduce therefore in this section mappings $\sigma_\nu : SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \sigma_\nu(\rho))$, which have the desired properties, and which we suspect to play an important role in — yet to be found! — combinatorial proofs of the symmetry of HL, Jack and more general functions $\{\lambda\}_w$.

The mapping $\sigma_\nu^K : SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \sigma_\nu(\rho))$ for fixed $\eta \in SSYT(\lambda, \rho)$ and $\nu \in \mathbb{N}$ is defined as follows:

Suppose $\rho_{\nu+1} = 0$, then of course changing all η-boxes ν into $\nu+1$-boxes does the job; similarly in the case $\rho_\nu = 0$. Assume therefore $\rho_\nu, \rho_{\nu+1} \ne 0$; changes occur in the double strip $\eta^{[\nu+1,\nu]}$ (cf. Appendix), and clearly columns with *ν-paired boxes*, i.e. an η-box ν directly above a box $\nu+1$, can not be affected; apply now the following rule to all rows in $\eta^{[\nu+1,\nu]} \setminus \{$ ν-paired boxes $\}$: suppose a certain row contains α boxes ν and β boxes $\nu+1$ (necessarily directly to the right of the ν's): change $|\alpha - \beta|$ boxes of these $\alpha + \beta$ boxes, such that the row contains first β boxes ν and then (continuing to the right) α boxes $\nu+1$.

Proposition 4.1. *The mappings σ_ν^K defined above obey relations (i) and (ii).*

Proof. (i) is immediate from the definition, and (ii) is a consequence of the fact that $\eta^{[\nu+1,\nu]} \cap \eta^{[\nu'+1,\nu']} = \emptyset$, if $|\nu - \nu'| > 1$. $\square$

Corollary 4.2. *[Formula (0.2)]* $\forall \rho \in F_N(\mu): \ |SSYT(\lambda, \rho)| = |P(\lambda, \mu)|$.

Proof. (i) implies that the σ_ν^K are bijections, whence $|SSYT(\lambda,\rho)| = |SSYT(\lambda, \sigma_\nu(\rho))|$ and clearly there is a (finite) chain of σ_ν, which transforms an arbitrary $\rho \in F_N(\mu)$ into a partitionlike $\rho' \in F_N(\mu) \cap PF_N$. $\square$

Corollary 4.3. *The combinatorially defined Schur functions s_λ (or Schur polynomials $s_\lambda^{(m)}$) are symmetric.*

Proof. $SSYT(\lambda) = \biguplus_{\mu \vdash |\lambda|} SSYT(\lambda, \mu)$ and $\sum_{\eta \in SSYT(\lambda,\mu)} x^\eta = K_{\lambda\mu} m_\mu(x)$ imply the assertion. The case of polynomials is analogous. $\square$

Clearly every family of mappings $\sigma_\nu : SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \sigma_\nu(\rho))$, which obeys (i) and (ii), yields the results of the above corollaries. On the other hand the mappings σ_ν^K have some deficiencies: first, if for some $\eta \in SSYT(\lambda)$ one has $\rho_\nu = \rho_{\nu+1}$, then one should have $\sigma_\nu \eta = \eta$, but for example $\sigma_2^K \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&3\\ \hline 2 \\ \cline{1-1}\end{array}$, and second, the σ_ν should obey also relation (iii), in order to have welldefined compositions $\pi: \ SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \pi(\rho))$, but for example:

$$\sigma_1^K \sigma_2^K \sigma_1^K \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3 \\ \cline{1-2}\end{array} = \sigma_1^K \sigma_2^K \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3 \\ \cline{1-2}\end{array} = \sigma_1^K \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\ \hline 2&3 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&3&3\\ \hline 2&3 \\ \cline{1-2}\end{array}, \text{ and}$$

$$\sigma_2^K \sigma_1^K \sigma_2^K \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3 \\ \cline{1-2}\end{array} = \sigma_2^K \sigma_1^K \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2&3 \\ \cline{1-2}\end{array} = \sigma_2^K \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 2&3 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 3&3 \\ \cline{1-2}\end{array}.$$

Therefore we introduce mappings $\sigma_\nu: \ SSYT(\lambda, \rho) \longrightarrow SSYT(\lambda, \sigma_\nu(\rho))$, which have all desired properties:

Definition 4.4. *Fix $\eta \in SSYT(\lambda, \rho)$ and $\nu \in \mathbb{N}$; in case of $\rho_\nu = 0$ or $\rho_{\nu+1} = 0$ there is no difference between σ_ν and σ_ν^K. Assume therefore $\rho_\nu, \rho_{\nu+1} \neq 0$; as before changes occur only in the set $\eta^{[\nu+1,\nu]} \setminus \{$ ν-paired boxes $\}$, but now we introduce additional* pairs of (ν-)fixed boxes *as follows: examine the set $\eta^{[\nu+1,\nu]} \setminus \{$ ν-paired boxes $\}$ for '(1^{st}-order) fixed pairs': by this we mean 'a $\nu+1$-box , which has as the next neighbor to the right a ν-box' (necessarily with lower row number); remove all 1^{st}-order fixed pairs and search again for '(2^{nd}-order) fixed pairs', remove them, too, etc. until there are no further fixed pairs; we write for the set of remaining boxes*

$$M(\eta, \nu) := \eta^{[\nu+1,\nu]} \setminus \{ \ \nu\text{-paired boxes, } \nu\text{-fixed pairs (of all orders)} \ \} .$$

Suppose $M(\eta,\nu)$ contains α boxes ν and β boxes $\nu+1$; by construction the 'ν' are contained in α columns, which are all to the left of the β columns containing the '$\nu+1$'; now change $|\alpha - \beta|$ boxes, such that the 'new ν' are contained in β columns, which are all to the left of the α columns containing the 'new $\nu+1$'.

Example 4.5. We examine again the relation (iii), now with σ_ν instead of σ_ν^K:

$$\sigma_1\sigma_2\sigma_1 \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \sigma_1\sigma_2 \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \sigma_1 \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}, \text{ and}$$

$$\sigma_2\sigma_1\sigma_2 \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \sigma_2\sigma_1 \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \sigma_2 \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\ \hline 3&3& \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}.$$

Proposition 4.6. *The mappings σ_ν defined above obey the relations (i), (ii), and (iii), and for every $\eta \in SSYT(\lambda)$ one has: $\sigma_\nu\eta = \eta$, if $\rho_\nu = \rho_{\nu+1}$.*

Proof. (i) and (ii) are valid by the same arguments as in the proof of Prop.4.1; and also the last assertion is almost immediate: $\rho_\nu = \rho_{\nu+1}$ implies either $M(\eta,\nu) = \emptyset$, so there remains nothing to be changed, or $M(\eta,\nu) \neq \emptyset$, but then ν- and $\nu+1$-boxes occur with the same multiplicity.

For the proof of relation (iii) we use as a shortcut the combinatorial rule for the generation of Schubert polynomials proved in [W1].

Let $\eta \in SSYT_{[m]}(\lambda)$ and $1 \leq \nu \leq m$; then η corresponds by [W1, Sec.4] to a unique box diagram B, in signs: '$\eta \longrightarrow B$'. Further correspondences are: '$\eta^{[\nu+1,\nu]} \longrightarrow M_{n-\nu}(B)$', '$\nu$-paired boxes $\longrightarrow$ ν-paired boxes', 'ν-fixed boxes $\longrightarrow$ ν-fixed boxes', and '$M(\eta,\nu) \longrightarrow f(n-\nu, B)$'. Now '$\sigma_\nu\eta \longrightarrow$ the unique maximal (or minimal) element in the equivalence class $[B]_{\underset{\sim}{n-\nu}}$' (cf. [W1, Prop.3.3]), which itself corresponds to an term in the algebraically derived sum $\partial_{n-\nu}x^B$ (we need not care about the 'dummy box' $[1, n-\nu]$). The algebraic divided difference operators $\partial_{n-\nu}$ are easily checked to obey relation (iii), whence do their combinatorial counterparts; this finally translates back to: 'the σ_ν satisfy relation (iii)'. ☐

Remark 4.7. While finishing the present paper we learned from [GKLLRT, Sec.7.3] that Lascoux and Schützenberger have already introduced the above action of the symmetric groups on words: reading the entries of some SSYT columnwise from bottom to top and the columns from left to right one obtains a word and the translation of our action to an action on these words gives the rule of Lascoux and Schützenberger. Nevertheless we find it useful to give an independent exposition of this action with other connections made. (In [GKLLRT, Sec.7.3] it is shown that the 'ribbon Schur polynomials' introduced there, which are nonsymmetric in the ordinary sense, are 'symmetric' with respect to the LS-action.)

5. Hall-Littlewood polynomials

Hall-Littlewood (HL) functions $Q_\lambda(x;t)$ or $P_\lambda(x;t)$ (cf. [Bu,M1,Mo]) are one-parameter extensions of Schur functions, i.e. they include Schur functions for the parameter value $t = 0$ (and also the $m_\lambda(x)$ for $t = 1$). They appear in the enumeration of subgroups of abelian p-groups, and in the theory of ordinary, projective ($t = -1$) and modular representations of the symmetric and general linear groups. Moreover the HL functions appear in [J] as a basis of a representation of the Virasoro algebra.

Our starting point is the combinatorial definition of Hall-Littlewood functions given in [M1, Sec.III 5], which is of the form (0.5). The weight $w(\eta) \in \mathbb{Z}[t]$ for $\eta \in SSYT(\lambda)$ occurs in two different versions: as $\varphi_\eta(t) \equiv w^Q(\eta)$ and as $\psi_\eta(t) \equiv w^P(\eta)$, i.e.

$$Q_\lambda(x;t) := \sum_{\eta \in SSYT(\lambda)} \varphi_\eta(t)\ x^\eta \quad \text{and} \quad P_\lambda(x;t) := \sum_{\eta \in SSYT(\lambda)} \psi_\eta(t)\ x^\eta\ .$$

According to (0.5) the HL polynomials $Q_\lambda^{(m)}$ and $P_\lambda^{(m)}$ are defined similarly with summation over $SSYT_{(m)}(\lambda)$. Recall from the Appendix that $RH(\eta)$ and $LH(\eta)$ are the right and left boundary boxes, respectively, of the H-components of η and that $m_j(\lambda)$ is the multiplicity of j as a part of λ; let $j(\nu)$ denote the column number of an η-box[1] ν and set $L_0H(\eta) := \{\ \nu \in LH(\eta) \mid j(\nu) > 1\ \}$. Then the characterization of the HL-weights given in [M1, Sec.III 5] can be re-casted in our terminology as:

$$\varphi_\eta(t) := \prod_{\nu \in RH(\eta)} (1 - t^{m_{j(\nu)}(\eta^{(\nu)})}) \quad \text{and} \quad \psi_\eta(t) := \prod_{\nu \in L_0H(\eta)} (1 - t^{m_{j(\nu)-1}(\eta^{(\nu-1)})})\ .$$

Obviously the weights φ and ψ do not depend on the absolute values of the entries of η, and it is therefore sufficient to investigate them only for $\eta \in G(\lambda)$. Notice that from the combinatorial definition of HL functions the equalities $Q_\lambda(x;0) = P_\lambda(x;0) = s_\lambda(x)$ are immediate.

Example 5.1. Let $\eta = \begin{array}{|c|c|c|c|}\hline 1 & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \hline 2 & 2 \\ \cline{1-2} \mathbf{3} & \mathbf{4} \\ \cline{1-2} \end{array} \in G(42^2)$, where the boxes $\nu \in RH(\eta)$ are printed bold; then $\varphi_\eta(t) = (1-t)^4(1-t^2)$ and $\psi_\eta(t) = (1-t)^2$.

The functions $Q_\lambda(x;t)$ and $P_\lambda(x;t)$ are related by a multiplicative factor ([M1, III (2.11-12)]):

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t) \quad \text{with} \quad b_\lambda(t) := \prod_{j \geq 1} \prod_{i=1}^{m_j(\lambda)} (1 - t^i)\ . \tag{5.1}$$

Note that $b_\lambda(t) = \varphi_\lambda(\eta)$, if η is the superstandard tableaux (see Appendix), and that $\psi_\lambda(\eta) = 1$ for this η; note further that $b_\lambda(t)$ by definition has $\sum_{j \geq 1} m_j(\lambda) = l(\lambda)$ factors and that on the other hand $|RH(\eta)| = |H(\eta)| = |LH(\eta)|$ and $|LH(\eta)| = |L_0H(\eta)| - l(\lambda)$ imply $|RH(\eta)| = |L_0H(\eta)| + l(\lambda)$. From formulas (0.7-8) one sees that $Q_\lambda = b_\lambda P_\lambda$ is equivalent to $\forall \mu \leq \lambda :\ w_\lambda^Q(\mu) = b_\lambda(t)\ w_\lambda^P(\mu)$, but we will prove the stronger result: $\varphi_\eta(t) = b_\lambda(t) \psi_\eta(t)$ for all $\eta \in SSYT(\lambda)$ with the help of the following

Lemmma 5.2. *Let $\zeta \in SYT(\lambda)$ and η_1, η_2 elements of the lattice $G(\zeta)$ (cf. Def.3.3). ('a | b' means 'a divides b'.) Then:*
a) the relation 'η_1 covers η_2 in $G(\zeta)$', i.e. $I(\eta_1) \setminus I(\eta_2) = \{s\}$, implies the following alternative:

[1] Subsequently we write "ν" as an abbreviation for the tripel (ν, i, j), if (i,j) is the position of an η-box with label ν.

$\varphi_{\eta_2}(t)$ *contains one more factor than* $\varphi_{\eta_1}(t)$, *if* $s \in I_0(\eta_1)$, or
$\varphi_{\eta_2}(t) = \varphi_{\eta_1}(t)$, *if* $s \in I_1(\eta_1)$;

b) $\eta_1 > \eta_2 \Longrightarrow \varphi_{\eta_2}(t) \mid \varphi_{\eta_1}(t)$, $\varphi_{\eta_1\wedge\eta_2}(t) \mid \gcd\,(\varphi_{\eta_1}(t), \varphi_{\eta_2}(t))$ *and* $\mathrm{lcm}\,(\varphi_{\eta_1}(t), \varphi_{\eta_2}(t)) \mid \varphi_{\eta_1\vee\eta_2}(t)$.

a) and b) are valid also with ψ *instead of* φ.

Proof. a) $\eta_1 > \eta_2 \Longrightarrow I(\eta_2) \subset I(\eta_1) \Longrightarrow RH(\eta_1) \subset RH(\eta_2)$. Assume now that η_1 covers η_2 in $G(\zeta)$, then there are two possibilities: the removal of s splits a H-component of η_1, i.e. $s \in I_0(\eta)$, then $RH(\eta_2)$ contains a new element and $\varphi_{\eta_2}(t)$ a new factor, *or* the removal of s splits a certain stripe $\eta_1^{[\nu]}$ without creating a new element of $H(\eta_1)$, i.e. $s \in I_1(\eta)$, then $\varphi_{\eta_2}(t) = \varphi_{\eta_1}(t)$.
b) the first assertion is immediate from a); for the other two note: $\varphi_{\eta_1\wedge\eta_2}(t) \mid \varphi_{\eta_1}(t), \varphi_{\eta_2}(t)$ and $\varphi_{\eta_1}(t), \varphi_{\eta_2}(t) \mid \varphi_{\eta_1\vee\eta_2}(t)$.
The proof with ψ instead of φ is completely analogous; observe that the boxes in the first column can not be reached by P-steps. $\square$

Proposition 5.3. $\varphi_\eta(t) = b_\lambda(t)\psi_\eta(t)$ *for all* $\eta \in SSYT(\lambda)$.

Proof. Of course it is sufficient to prove the assertion for all $\eta \in G(\lambda)$. Assume that η_1 covers η_2 in $G(\zeta)$, i.e. $I(\eta_1) \setminus I(\eta_2) = \{s\}$, and $\varphi_{\eta_1}(t) = b_\lambda(t)\psi_{\eta_1}(t)$. Then $s \in I_1(\eta_1)$ implies by Lemma 5.2, that $\varphi_{\eta_2}(t) = \varphi_{\eta_1}(t)$ and $\psi_{\eta_2}(t) = \psi_{\eta_1}(t)$, hence $\varphi_{\eta_2}(t) = b_\lambda(t)\psi_{\eta_2}(t)$. The case '$s \in I_0(\eta_1)$' is only slightly more difficult: let $\theta \in H(\eta_1)$ consist of boxes ν and let s be the step from column j to column $j+1$ in θ, then the new element in $RH(\eta_2)$ is the box ν in column j and the new element in $L_0H(\eta_2)$ is the box $\nu+1$ in column $j+1$; consequently in both cases the new factor generated is $(1 - t^{m_j(\eta_2^{(\nu)})})$. We conclude that it is already sufficient to prove the assertion for all maximal elements in the lattices $G(\zeta)$, i.e.

$$\eta \in \max(\lambda) := \{\eta = \widehat{\zeta} \mid \zeta \in SYT(\lambda)\}\ .$$

Assume $\varphi_\eta(t) = b_\lambda(t)\psi_\eta(t)$ to be true for all λ of length $r \geq 1$, i.e. $\lambda = \lambda_1 \dots \lambda_r$; the case $r = 1$ is trivial for $\eta \in \max(\lambda)$. Let now $\overline{\lambda} = \lambda_1 \dots \lambda_r\lambda_{r+1}$ and $\overline{\eta} \in \max(\overline{\lambda})$. In case of $\lambda_r > \lambda_{r+1}$ one has $b_{\overline{\lambda}} = b_\lambda(t) \cdot (1-t)$; the new row $r+1$ in $\overline{\eta}$ consists of the same kind of boxes, say ν, because $\overline{\eta} \in \max(\overline{\lambda})$; the uniquely determined new element of $RH(\overline{\eta})$ in row $r+1$ and column λ_{r+1} generates a new factor $(1-t)$ in $\varphi_{\overline{\eta}}(t)$, which is not contained in $\psi_{\overline{\eta}}(t)$, since $L_0H(\overline{\eta}) = L_0H(\eta)$; now $\varphi_{\overline{\eta}}(t) = b_\lambda(t)\psi_{\overline{\eta}}(t)$ follows by induction hypothesis. In case of $\lambda_r = \lambda_{r+1}$ similarly there is a new factor $(1 - t^{m_{\lambda_r}(\lambda)+1})$ in $\varphi_{\overline{\eta}}(t)$, which does not appear in $\psi_{\overline{\eta}}(t)$. $\square$

For the derivation of the weighted τPx-formulas we need a refinement of Thm.1.5, which is based on the notions introduced in Def.3.3; this gives also the promised alternative proof of the τPx-expansions for graded Schur functions.

Theorem 5.4. *For* $\lambda \vdash N$, $\zeta \in SYT(\lambda)$, $SSYT(\zeta) \equiv \overline{\zeta}$ *and* $\emptyset \subset I, J \subset I(\zeta)$ *with* $I \cap J = \emptyset$, *let (*${}^{gr}\sum$ *stands for the grading as in* (0.4) *of the terms in* $\sum$*)*

$$SSYT(\zeta, I) := \{\ \eta \in SSYT(\zeta) \mid pr_G\ \eta = \widehat{\zeta}^{\ I}\ \}\ ,\quad B^I(\zeta)(x) := {}^{gr}\textstyle\sum_{\eta\in SSYT(\zeta,I)} x^\eta\ ,$$

$$B^{I|J}(\zeta)(x) := \sum_{J'\subset J} B^{I\cup J'}(\zeta)(x)\ , \quad \textit{and}\ \ B(\zeta)(x) := \sum_{I\subset I(\zeta)} B^{I}(\zeta)(x)\ .$$

Then one has

$$(5.2) \qquad B^{I|J}(\zeta)(x) = xB^{I|J}_{N-1}x \dots B^{I|J}_{1}x \quad \textit{with}\ \ B^{I|J}_{\nu} = \begin{cases} S & ,\ \textit{if}\ \nu \notin I\cup J \\ 1 & ,\ \textit{if}\ \nu\in I \\ P & ,\ \textit{if}\ \nu \in J \end{cases}\ ,$$

and as special cases: $B^I(x) = B^{I|\emptyset}(x)$, $B(\zeta)(x) = B^{\emptyset|I(\zeta)}(x)$*; moreover*

$$s_{[\lambda]}(x) = \sum_{\zeta\in SYT(\lambda)} B(\zeta)(x)\ .$$

Proof. The special cases: $B^I(x) = B^{I|\emptyset}(x)$ and $B(\zeta)(x) = B^{\emptyset|I(\zeta)}(x)$ are immediate from the definitions; and the τPx-expansion of $s_{[\lambda]}(x)$ follows from $SSYT(\lambda) = \biguplus_{\zeta\in SYT(\lambda)} SSYT(\zeta)$; since $D(\zeta) = \{1,\dots,N-1\}\setminus I(\zeta)$, formula (5.2) gives for $B(\zeta)(x)$ the same expression as already encountered in Thm.1.5 .

Hence it remains to show (5.2): in case of $J=\emptyset$ one sees directly from the definition that

$$B^I(\zeta)(x) = {}^{gr}\textstyle\sum_{1\le i_1\le\dots\le i_N} x_{i_1}\dots x_{i_N}\ ,$$

where $i_\nu = i_{\nu+1} \Longleftrightarrow \nu\in I$, and $i_\nu < i_{\nu+1} \Longleftrightarrow \nu\notin I$; therefore $B^I(x) = xB^I_{N-1}x\dots B^I_1 x$ with $B^I_\nu = 1 \Longleftrightarrow \nu\in I$, and $B^I_\nu = S \Longleftrightarrow \nu\notin I$ in accordance with (5.2). Assume now that (5.2) is valid for some $J\subset I(\zeta)$ with $0\le |J| < N$ and let $\overline{J}\subset I(\zeta)$ be such that $\overline{J}\setminus J = \{j\}$; then for every $I\subset I(\zeta)$ with $I\cap\overline{J} = \emptyset$ one computes (omitting the arguments '$(\zeta)(x)$')

$$B^{I|\overline{J}} = \sum_{J'\subset\overline{J}} B^{I\cup J'} = \sum_{J''\subset J} (B^{I\cup J''} + B^{I\cup J''\cup\{j\}}) = B^{I|J} + B^{I\cup\{j\}|J}\ ;$$

by induction hypothesis $B^{I|J}_\nu = B^{I\cup\{j\}|J}_\nu$ for all $\nu\ne j$ and $B^{I|J}_j + B^{I\cup\{j\}|J}_j = S+1 = P$, which gives the desired result. □

Remark 5.5. Theorem 5.4 shows that general weighted τPx-formulas are created by summing up the 'building blocks' $B^I_w(\zeta)$. In order to get the *weighted form of Thm.5.4* simply attach a subscript 'w' to every 'B' and define $B^I_w(\zeta)(x) := {}^{gr}\sum_{\eta\in SSYT(\zeta,I)} w(\eta)x^\eta$. In case of the graded Schur function and in the HL case (cf. Thm.5.6 below) the summation for a fixed $\zeta\in SYT(\lambda)$ leads to a complete collapse of the hypercube $\mathcal{B}(I(\zeta))$ (cf. Def.3.3) to a single chain of weighted operators P (and S). In Sections 6 and 7 we will see that for the graded Jack and Macdonald functions this collapsing process leads only to a chain of in general smaller hypercubes associated to H-strips in $\widehat{\zeta}$ or, equivalently, maximal sequences of consecutive P-steps.

For the moment we simply emphasize that property (0.9) is crucial for the existence of reasonable 'building blocks' $B^I_w(\zeta)$, namely

$$[w(\eta) = w(pr_G\eta)\ \forall\eta\in SSYT(\zeta,I)] \Longrightarrow B^I_w(\zeta) = w(\widehat{\zeta}^{\,I})B^I(\zeta)\ .$$

Theorem 5.6. *For $\lambda \vdash N$ and $\zeta \in SYT(\lambda)$ let $\nu \in \{1,\dots,N-1\}$ denote the box ν or the step ν in ζ (cf. Def.3.3) depending on the context; for fixed $\zeta \in SYT(\lambda)$ let $j(\nu)$ be the column number of the ζ-box ν. Define $Q(\zeta;t)$ by the following rules:*

(1) *write down $x\ Q_{N-1}x \dots Q_1 x$, where $Q_\nu = P$, if ν is a P-step or (equivalently) $\nu \in I(\zeta)$, and $Q_\nu = S$, if ν is a S-step or (equivalently) $\nu \notin I(\zeta)$;*

(2) *substitute Q_ν by $\left\{ \begin{array}{ll} Su_\nu(\zeta), & \text{if } \nu \notin I(\zeta) \\ Su_\nu(\zeta)+1, & \text{if } \nu \in I_0(\zeta) \\ Pu_\nu(\zeta), & \text{if } \nu \in I_1(\zeta) \end{array} \right\}$, where $u_\nu(\zeta) = (1-t^{m_{j(\nu)}(\zeta^{(\nu)})})$;*

(3) *multiply with $(1-t^{m_{j(N)}(\zeta)})$ on the left.*

Then

$$Q_{[\lambda]}(x;t) = \sum_{\zeta \in SYT(\lambda)} Q(\zeta;t)(x), \qquad \text{(HL)}$$

where $Q_{[\lambda]}(x;t)$ is the sequence of the $Q_\lambda^{[m]}(x;t) := \sum_{\eta \in SSYT_{[m]}} \varphi_\eta(t)\ x^\eta$ with $m \in \mathbb{N}$.

Proof. The τPx-formula for $Q_{[\lambda]}(x;t)$ is a weighted form of the one given for $s_{[\lambda]}(x)$; we use the notations of Thm.5.4 above with 'Q' instead of 'B' to indicate that weights $\varphi_\eta(t)$ are involved.

For fixed $I \subset I(\zeta)$ let $\eta := \widehat{\zeta}^{\,I} \in G(\zeta)$. Then from the various definitions one sees

$$Q^I(\zeta;t)(x) = xQ^I_{N-1}x \dots Q^I_1 x \quad \text{with} \quad Q^I_\nu = \left\{ \begin{array}{lll} 1 & , & \text{if } \nu \in I_0(\eta) \\ (1-t^{m_{j(\nu)}(\eta^{(n(\nu))})}) & , & \text{if } \nu \in I_1(\eta) \\ (1-t^{m_{j(\nu)}(\eta^{(n(\nu))})})S & , & \text{if } \nu \notin I(\eta) \end{array} \right. ,$$

where the '$n(\nu)$' in $\eta^{(n(\nu))}$ denotes the number of the ζ-box ν in η. In case of $\nu \in I_0(\eta)$ the shape $\lambda(\eta^{(n(\nu))})$ contains properly $\lambda(\zeta^{(\nu)})$; for $\nu \notin I(\eta)$ one has $\lambda(\eta^{(n(\nu))}) = \lambda(\zeta^{(\nu)})$, and for $\nu \in I_1(\eta)$ at least $m_{j(\nu)}(\eta^{(n(\nu))}) = m_{j(\nu)}(\zeta^{(\nu)})$; therefore the weights $(1-t^{m_{j(\nu)}(\eta^{(n(\nu))})})$ are in fact independent of η resp. I and can be abbreviated $u_\nu \equiv u_\nu(\zeta) := (1-t^{m_{j(\nu)}(\zeta^{(\nu)})})$. (Note that this independence explains combinatorially why it is nessecary to distinguish between P_0- and P_1-steps or in other words the choice of just the set $RH(\eta)$ in the definition of $\varphi_\eta(t)$.)

Now the summation $\sum_{I \subset I(\zeta)} Q^I(\zeta;t)(x)$ yields similarly as in the proof of Thm.5.4 the terms $Q(\zeta;t)(x)$ as specified above and hence the τPx-formula for $Q_{[\lambda]}(x;t)$. $\square$

Example 5.7. For $\lambda = 3\ 2$ we compute the $Q(\zeta;t)(x)$, the sum of which gives the (weighted) τPx-formula for $Q_{[32]}(x;t)$:

$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$: $(1-t)x\ (S(1-t)+1)x\ S(1-t)x\ (S(1-t)+1)x\ (S(1-t)+1)x$

$\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$: $(1-t)x\ (S(1-t^2)+1)x\ (S(1-t)+1)x\ S(1-t)x\ (S(1-t)+1)x$

$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$: $(1-t)x\ S(1-t)x\ P(1-t)x\ S(1-t)x\ (S(1-t)+1)x$

$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$: $(1-t)x\ S(1-t)x\ (S(1-t)+1)x\ (S(1-t^2)+1)x\ S(1-t)x$

$\begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array}$: $\quad (1-t)x\ (S(1-t^2)+1)x\ S(1-t)x\ (S(1-t^2)+1)x\ S(1-t)x$

Remark 5.8. The nice combinatorial argument of Cor.4.3 showing the symmetry of the Schur functions does not apply in the case of HL functions, because in general $\varphi_\eta(t) \neq \varphi_{\sigma_\nu \eta}(t)$; for example $\eta_1 = \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 2&3 \\ \cline{1-2}\end{array}$ and $\eta_2 = \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&2 \\ \cline{1-2}\end{array}$ in $SSYT(32,(2,2,1))$ have φ-weights $(1-t)^4$ and $(1-t)^2(1-t^2)$, respectively, but their images under σ_2 in $SSYT(32,(2,1,2))$, i.e. $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 3&3 \\ \cline{1-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3 \\ \cline{1-2}\end{array}$, have both the φ-weights $(1-t)^3$. Of course the sum of weights remains constant, but this is a non-combinatorial argument.

Remark 5.9. (*Continuation of Rem.3.6*) Let $\lambda \vdash N$, $K_{\lambda\mu}$ the Kostka numbers and $K_{\lambda\mu}(t)$ the Kostka-Foulkes polynomials; let moreover

$$\psi_{\mu\overline{\lambda}}(t) := \sum_{\overline{\eta} \in P(\mu,\overline{\lambda})} \psi_{\overline{\eta}}(t) \ ,$$

then

$$s_\lambda(x) = \sum_{\mu \leq \lambda} K_{\lambda\mu}(t)\ P_\mu(x;t) \quad \Longleftrightarrow \quad \forall \overline{\lambda} \leq \lambda :\ K_{\lambda\overline{\lambda}} = \sum_{\overline{\lambda} \leq \mu \leq \lambda} K_{\lambda\mu}(t)\ \psi_{\mu\overline{\lambda}}(t) \ . \tag{5.3}$$

Proof. (of (5.3)) Collecting in $s_\lambda(x) = \sum_{\mu \leq \lambda} K_{\lambda\mu}(t)\ P_\mu(x;t)$ all terms with a non-increasing gapless sequence of exponents gives

$$\sum_{\eta \in P(\lambda)} x^\eta = \sum_{\mu \leq \lambda} K_{\lambda\mu}(t) \sum_{\overline{\eta} \in P(\mu)} \psi_{\overline{\eta}}(t)\ x^{\overline{\eta}} \ .$$

Using $P(\lambda) = \biguplus_{\overline{\lambda} \leq \lambda} P(\lambda, \overline{\lambda})$ and (0.1-2) one concludes

$$K_{\lambda\overline{\lambda}} x^{\overline{\lambda}} = \sum_{\eta \in P(\lambda,\overline{\lambda})} x^\eta = \sum_{\overline{\lambda} \leq \mu \leq \lambda} K_{\lambda\mu}(t) \sum_{\overline{\eta} \in P(\mu,\overline{\lambda})} \psi_{\overline{\eta}}(t)\ x^{\overline{\lambda}} \ \text{ for all } \overline{\lambda} \leq \lambda.$$

For the other direction reverse the arguments and finally apply the symmetry of Schur and HL functions. ☐

From the charge definition of the Kostka Foulkes polynomials it is not hard to see that $K_{\lambda\lambda}(t) = 1$ and that t divides $K_{\lambda\mu}(t)$ for all $\mu < \lambda$; consequently the r.h.s. of (5.3) shows that the summand $K_{\lambda\lambda}(t)\ \psi_{\lambda\overline{\lambda}}(t) = \psi_{\lambda\overline{\lambda}}(t) = K_{\lambda\overline{\lambda}} \pm \ldots$ alone yields the Kostka number $K_{\lambda\overline{\lambda}}$ and all subsequent summands with $\overline{\lambda} \leq \mu < \lambda$ are necessary to cancel the other contributions to $\psi_{\lambda\overline{\lambda}}(t)$.

Remark 5.10. The *skew Hall-Littlewood functions* are defined for skew shapes λ/μ (with $\lambda \vdash N,\ \mu \vdash M, \mu \subset \lambda,\ M \leq N$) by ([M1, III (5.11)]):

$$Q_{\lambda/\mu}(x;t) := \sum_{\eta \in SSYT(\lambda/\mu)} \varphi_\eta(t)\ x^\eta \ .$$

In complete analogy to Section 2 the sequences $Q_{\lambda/\mu}(\zeta;t)(x)$ of $Q_{[\lambda/\mu]}(x;t)$ can be computed from the sequences $Q_\lambda(\zeta;t)(x)$ of $Q_{[\lambda]}(x;t)$: single out all $\zeta \in SYT(\lambda)$ with $\lambda(\zeta^{(M)}) = \mu$ and delete everything to the right of the $(N-M)^{th}$ x, e.g. for $\lambda = 32$ and $\mu = 2$ the first three SYT in Ex.5.7 are used to compute:

$$\begin{aligned} Q_{[32,2]}(x;t) &= (1-t)x\ (S(1-t)+1)x\ S(1-t)x \\ &+\ (1-t)x\ (S(1-t^2)+1)x\ (S(1-t)+1)x\ +\ (1-t)x\ S(1-t)x\ P(1-t)x\ . \end{aligned}$$

Remark 5.11. Generalizing Rem.2.3 to the HL case we discuss again the recursive structure τPx-expansions with respect to the Young lattice $\mathcal{Y}$:

(1) Suppose the τPx-expansion of some $Q_{[\lambda]}(x;t)$ is available; then the τPx-expansions of all $Q_{[\overline{\lambda}]}(x;t)$ with $\overline{\lambda} \subset \lambda$ can be computed similarly as in the Schur case: let $\lambda \vdash N$, $\overline{\lambda} \vdash M$ and $M < N$; single out a subset of $SYT(\lambda)$ representing $SYT(\overline{\lambda})$; in the corresponding $Q(\zeta;t)(x)$ delete everything to the left of the M^{th} x counted from the right except the factor $u_M(\zeta)$, e.g. for $\lambda = 32$ and $\mu = 21$ we choose the second and the fourth SYT in Ex.5.7 to compute:
$Q_{[21]}(x;t) = (1-t)x\ S(1-t)x\ (S(1-t)+1)x\ +\ (1-t)x\ (S(1-t^2)+1)x\ S(1-t)x\ .$

(2) Suppose that for all $\lambda \vdash N$ the τPx-expansions of $Q_{[\lambda]}(x;t)$ have been computed; then $Q_{[\overline{\lambda}]}(x;t)$ for $\overline{\lambda} \vdash N+1$ can be determined as the sum of the τPx-expansions of the $Q_{[\lambda]}(x;t)$ with $\lambda \in C(\overline{\lambda})$, where the leftmost factor u_N of the $Q(\zeta;t)(x)$ can be reused in the execution of 2. and 3. of Thm.5.6 . As an example study again Ex.5.7 and observe that $C(32) = \{31, 2^2\}$.

6. Jack polynomials

Jack functions $J_\lambda(x;\alpha)$ (cf. [M2,St2]) are one-parameter extensions of Schur functions, i.e. they include Schur functions for the parameter value $\alpha = 1$ (and also the λ'-product of elementary symmetric functions for $\alpha = 0$ and the monomial symmetric functions for $\alpha \longrightarrow \infty$). For $\alpha = 2$ they specialize to the *zonal polynomials* used in multivariate statistics ([MPH]). Moreover every exited state in the Calogero-Sutherland model describing the long-range interaction of n quantum particles on a circle can be written as a linear combination of Jack polynomials and also the singular vectors of a conformal field theory are given by Jack functions with rectangular Ferrer diagram (cf. [LV, AKOS1]).

Our starting point is the combinatorial definition of Jack functions given in [St2, Thm.6.3], which is of the form (0.5). The weight $w^J(\eta) \in \mathbb{Q}(\alpha)$ for $\eta \in SSYT(\lambda)$ can be computed as follows:

Let $\lambda \vdash N$ and $s = (i,j) \in \lambda$ be the box in row i and column j of λ; then

$$H_\lambda(s) := \{\ (i',j) \in \lambda \mid i' \geq i\ \} \cup \{\ (i,j') \in \lambda \mid j' \geq j\ \}$$

is *the hook based at s in* λ with *arm length* $a_\lambda(i,j) := \lambda_i - j$ and *leg length* $l_\lambda(i,j) := \lambda'_j - i$. To every hook $H_\lambda(s)$ one associates two important 'linear factors' in $\mathbb{Z}[\alpha]$:

> the *lower hooklength* $h^+_\lambda(i,j) := (l_\lambda(i,j)+1) + a_\lambda(i,j)\alpha$, if $(i,j)\in\lambda$, and $= 1$ otherwise; the *upper hooklength* $h^-_\lambda(i,j) := l_\lambda(i,j) + (1 + a_\lambda(i,j)\alpha)$, if $(i,j)\in\lambda$, and $= 1$ otherwise;

i.e. the 'base box' s is taken up to the leg in the 'lower' case (corresponding to the vertical line in '+') or to the arm in the 'upper' case (corresponding to the horizontal line in '−').

Let $(\lambda|\ \mathbf{j}) \equiv (\lambda|\ j_1,\dots,j_r)$ be a pair consisting of a partition λ and an r-tuple $\mathbf{j}$ with $1 \le j_1 < \dots < j_r \le \lambda_1$ of 'column numbers in λ' and set $\{\mathbf{j}\} := \{j_1,\dots,j_r\}$; then one associates a polynomial $w^J(\lambda|\ \mathbf{j}) \in \mathbb{Z}[\alpha]$ to such a pair by:

$$w^J(\lambda|\ \mathbf{j}) := \prod_{\substack{(i,j)\in\lambda\\ j\in\{\mathbf{j}\}}} h^+_\lambda(i,j) \prod_{\substack{(i,j)\in\lambda\\ j\notin\{\mathbf{j}\}}} h^-_\lambda(i,j)\ . \tag{6.1}$$

For arbitrary $\eta\in SSYT(\lambda)$ set $\eta^{[0]} := \emptyset$ and let $\mathbf{j}(\eta^{[\nu]})$ be the ordered tuple of column numbers of the boxes in the H-strip $\eta^{[\nu]}$; then the 'Jack weight' w^J as a rational function in $\mathbb{Q}(\alpha)$ is defined by:

$$w^J(\eta) := \prod_{\nu\ge1} w^J(\eta^{(\nu)};\mathbf{j}(\eta^{[\nu]})) \quad \text{with} \quad w^J(\eta^{(\nu)};\mathbf{j}(\eta^{[\nu]})) := \frac{w^J(\eta^{(\nu)}|\ \mathbf{j}(\eta^{[\nu]})\ )}{w^J(\eta^{(\nu-1)}|\ \mathbf{j}(\eta^{[\nu]})\ )}\ . \tag{6.2}$$

In the case of $\eta^{[\nu]} = \emptyset$ for some ν one has $\mathbf{j}(\eta^{[\nu]}) = \emptyset$ and $\eta^{(\nu)} = \eta^{(\nu-1)}$ and consequently $w^J(\eta^{(\nu)};\mathbf{j}(\eta^{[\nu]})) = 1$; hence

$$w^J(\eta) = w^J(pr_G\ \eta) \quad \text{for all } \eta\in SSYT(\lambda). \tag{6.3}$$

The Jack functions $J_\lambda(x;\alpha) := \sum_{\eta\in SSYT(\lambda)} w^J(\eta)\ x^\eta$ by (0.7-8) have expansions of the form:

$$J_\lambda(x;\alpha) = \sum_{\mu\le\lambda} v_{\lambda\mu}(\alpha)\ m_\mu(x) \quad \text{with} \quad v_{\lambda\mu}(\alpha) \equiv w^J_\lambda(\mu)(\alpha) := \sum_{\eta\in P(\lambda,\mu)} w^J(\eta)\ . \tag{6.4}$$

Stanley and Macdonald have conjectured ([M2,St2]) that the $v_{\lambda\mu}(\alpha)$ are polynomials $\mathbb{Z}[\alpha]$ with non-negative coefficients (compare the Introduction).

Directly from the definition one computes $J_{(1^N)}(x;\alpha) = N!\ e_N(x)$: in the case of $\lambda = 1^N$ one has $P(\lambda) = SYT(\lambda) = \{\zeta\}$ and $v_{1^N,1^N}(\alpha) = w^J(\zeta) = \prod_{i=1}^N \frac{w^J(\zeta^{(i)}|1)}{w^J(\zeta^{(i-1)}|1)} = w^J(\zeta^{(N)}|1) = w^J(\zeta|1) = \prod_{i=1}^N h^+_{(1^i)}(\alpha) = N!$. Slightly more involved combinatorial arguments yield the formula for $J_{(N)}(x;\alpha)$ given in [St2, Prop.2.2 a)].

In case of $\alpha = 1$ one has $h^+_\lambda(s) = h^-_\lambda(s) = |H_\lambda(s)| =: h_\lambda(s)$ for $s = (i,j)\in\lambda$, i.e. upper and lower hooklength specialize to the ordinary hooklength; consequently $w^J(\eta) = \prod_{s\in\lambda} h_\lambda(s) \equiv h_\lambda(\lambda)$ for arbitrary $\eta\in SSYT(\lambda)$ and

$$J_\lambda(x;1) = h_\lambda(\lambda)s_\lambda(x)\ .$$

Signed Diagrams (for human calculations): The product of hooklength's $h^-_\lambda(s)$ resp. $h^+_\lambda(s)$ relative to a fixed shape λ can be represented diagrammatically by marking the 'base box' s of every 'hook factor' in the Ferrer diagram λ with a '$-$' resp. a '$+$' ; boxes in λ for which no hook factor occurs are marked with '0'. We call Ferrer diagrams, in which every box has a 'sign'$\in \{+,0,-\}$, *signed (Ferrer) diagrams.* For example:

$$\begin{array}{ccccc} 0&0&+&0&+\\ -&-&+&-&0\\ 0&0&&& \end{array} \quad = \quad (2+2\alpha)\cdot 2\cdot(1+5\alpha)(1+4\alpha)(1+2\alpha)\cdot 2\alpha \ .$$

Since it is possible to cancel '$+$'-hooks or '$-$'-hooks of the same shape occurring in a quotient of signed diagrams, one can comfortably compute $w^J(\eta)$ for every $\eta\in SSYT(\lambda)$. *Example*:

$$w^J\Big(\begin{array}{cccc}1&1&1&3\\2&3&&\end{array}\Big) = \begin{array}{ccc}+&+&+\end{array} \cdot \frac{\begin{array}{ccc}+&-&-\\+&&\end{array}}{\begin{array}{ccc}+&-&-\end{array}} \cdot \frac{\begin{array}{cccc}-&+&-&+\\-&+&&\end{array}}{\begin{array}{ccc}-&+&-\\-&&\end{array}} = \begin{array}{ccc}0&+&0\end{array} \cdot \frac{\begin{array}{ccc}+&0&0\\0&&\end{array}}{\begin{array}{ccc}0&0&0\end{array}} \cdot\ 4\ \frac{\begin{array}{cccc}-&+&0&0\\0&0&&\end{array}}{\begin{array}{ccc}-&+&0\\0&&\end{array}}$$

$= 16(1+\alpha)^2(1+4\alpha)/(1+3\alpha)$. Using the method of signed diagrams it not very hard to give combinatorial proofs of many of the formulas for the $v_{\lambda\mu}(\alpha)$ contained in [St2]; we mention the formulas for $v_{N\mu}(\alpha)$ [St2, Prop.2.2 a], $v_{\lambda\lambda}(\alpha)$ [St2, Thm.5.6], and $v_{\lambda\mu}(\alpha)$ [St2, Prop.7.1], where $\lambda=\lambda_1\dots\lambda_l$ and $\mu=\mu_1\dots\mu_j 1^{\lambda_{j+1}+\dots+\lambda_l}$ for $1\le j\le l$. □

Proposition 6.1. *(for computer calculations): Let $\eta\in SSYT(\overline{\lambda})$ for some $\overline{\lambda}\vdash N$, $\nu\in\mathbb{N}$ with $|\eta^{[\nu]}|=r>0$, $\lambda:=\lambda(\eta^{(\nu)})$, λ' its conjugate defined by $\lambda'_i:=\sum_{j\ge i} m_j(\lambda)$ and set*

$$\eta^{[\nu]} = \{\ (i_v,j_v)\in\lambda \mid v=1,\dots,r\ \}$$

with $l(\lambda)\ge i_1\ge\dots\ge i_r\ge 1$ and $1\le j_1<\dots<j_r\le\lambda_1$. Then:

$$w^J(\eta,\nu) = \varphi^0(\eta,\nu)\ \varphi^+(\eta,\nu)\ \varphi^-(\eta,\nu) \equiv \prod_{v=1}^r \varphi^0_v \cdot \prod_{v=1}^r \varphi^+_v \cdot \prod_{v\in p(\eta,\nu)} \varphi^-_v \tag{6.5}$$

with

$$p(\eta,\nu) := \{\ p' \mid p'=1 \text{ or } i_{p'-1}>i_{p'}\ \}\ ,\quad \varphi^0_v = 1+(\lambda_{i_v}-j_v)\alpha\ ,$$

$$\varphi^+_v = \prod_{p_v=1}^{i_v-1} \frac{p_v+(\lambda_{p_v}-j_v)\alpha}{p_v-1+(\lambda_{p_v}-j_v)\alpha}\ ,\quad \varphi^-_v = \prod_{\substack{q_v=1\\ q_v\ne j_1,\dots,j_{v-1}}}^{j_v-1} \frac{(\lambda'_{q_v}-i_v)+(j_v-q_v)\alpha}{(\lambda'_{q_v}-i_v)+(\lambda_{i_v}-q_v+1)\alpha}\ . \tag{6.6}$$

Proof. The above formulas are simply a recasting of the previous 'diagrammatic method'. The main observation is that in general only some hook factors survive cancelation, namely the factors φ^+_v based in columns $j_1,\dots,j_r$, because their leg length changes, and the factors φ^-_v based in rows $i_1,\dots,i_r$, because their arm length changes, and of course the factors φ^0_v of the boxes of $\eta^{[\nu]}$. Notice that $p(\eta,\nu)$ is the set of the leftmost boxes in every row of $\eta^{[\nu]}$ and that the boxes with column numbers $\in\{j_1,\dots,j_r\}$ do not contribute to the φ^-_v. □

Definition 6.2. *Let $[\lambda]_w$ be a graded weighted symmetric function (as in (0.5)) fulfilling (0.9), i.e. the basic 'building blocks'*

$$B_w^I(\zeta)(x) := {}^{gr}\sum\nolimits_{\eta\in SSYT(\zeta,I)} w(\eta)x^\eta$$

are well defined for all $\zeta \in SYT(\lambda)$ (see also Rem.5.5). Assume moreover that $w(\eta)$ is a product of factors $w(\eta,\nu)$, where each factor $w(\eta,\nu)$ depends only on the shapes of $\eta^{(\nu)}$ and $\eta^{(\nu-1)}$ or in other words: on the position of $\eta^{[\nu]}$ in $\zeta(\eta)$. Then the weight w is called S-insected and $[\lambda]_w$ has a S-insected τPx-expansion.

Clearly by (6.2-3) w^J is S-insected (and so is w^M as we will see in Sec.7); but also w^S and w^{HL} are S-insected: trivially $w^S(\eta) = \prod w^S(\eta,\nu)$ with $w^S(\eta,\nu) = 1$ for all η and ν, and $w^{HL}(\eta) \equiv \varphi_\eta(t) = \prod w^{HL}(\eta,\nu)$ with

$$w^{HL}(\eta,\nu) = \prod_{\nu\in RH(\eta^{[\nu]})} (1 - t^{m_{j(\nu)}(\eta^{(\nu)})}) \quad \text{(see Section 5)}.$$

Proposition 6.3. *Let w be a S-insected weight as in Def.6.2 above and fix $\zeta \in SYT(\lambda)$. Partition the set $I(\zeta)$ of P-steps for ζ into maximal subsets $M_1,\ldots,M_s$ of consecutive integers, i.e. if $M_k = \{\nu+1,\nu+2,\ldots,\nu+r\} \subset I(\zeta)$ for appropriate $k,\nu,r \in \mathbb{N}_0$, then $\nu, \nu+r+1 \notin I(\zeta)$. Let the notation '$(M_1',\ldots,M_s') \prec M_1\times\cdots\times M_s$' express the fact that $M_k' \subset M_k$ for all $k \in \{1,\ldots,s\}$ in the s-tuple.*

Then with the notations of the 'weighted Thm.5.4' (cf. Rem.5.5) one has

$$B_w(\zeta)(x) := \sum_{I\subset I(\zeta)} B_w^I(\zeta)(x) = \sum_{(M_1',\ldots,M_s')\prec M_1\times\cdots\times M_s} B_w^{M_1'\cup\cdots\cup M_s'}(\zeta)(x)\ .$$

Therefore $B_w(\zeta)(x)$ is represented by a chain of '$|M_k|$-dimensional ($k=1,\ldots,s$) hypercubes' of weighted operators separated by weighted S-operators.

Proof. Similar to Thm.5.4 . □

We describe now the weighted τPx-formulas for graded Jack functions $J_{[\lambda]}(x;\alpha)$.

Theorem 6.4. *For $\lambda \vdash N$ and $\zeta \in SYT(\lambda)$ let $\nu \in \{1,\ldots,N-1\}$ denote the box ν or the step ν in ζ (cf. Def.3.3) depending on the context; for fixed $\zeta \in SYT(\lambda)$ let $j(\nu)$ be the column number of the ζ-box ν and for a sequence of r consecutive natural numbers $1 \le \nu+1,\nu+2,\ldots,\nu+r \le N-1$ let $\mathbf{j} = (j_1,\ldots,j_r) \equiv (j(\nu+1),\ldots,j(\nu+r))$. Note that $1 \le j_1 < \cdots < j_r \le \lambda(\zeta_1^{(\nu+r)})$, if $\{\nu+1,\nu+2,\ldots,\nu+r\} \subset I(\zeta)$, i.e. $\nu+1,\nu+2,\ldots,\nu+r$ are consecutive P-steps in ζ, whence for k with $0 \le k \le r-1$ and fixed ζ the following notation is well defined:*

$$[\ r; r-k,\ldots,r]_\nu := w^J(\lambda(\zeta^{(\nu+r)});\ j_{r-k},\ldots,j_r) \quad \text{(cf. (6.2))}.$$

One uses the following rules to compute $J(\zeta;t)$:

(1) *write down $xB_{N-1}x\ldots B_1x$ for ζ as in Thm.1.5;*
(2) *if $B_\nu = S$, then substitute B_ν by $[1;1]_\nu\ S$;*

(3) *assume* $\{\nu+1, \nu+2, \ldots, \nu+r\} \subset I(\zeta)$ *is* maximal, *i.e.* $\nu, \nu+r+1 \notin I(\zeta)$, *then substitute the expression* $xB_{\nu+r}x \ldots xB_\nu$ *by* $xP_{(r+1)}(\zeta, \nu)$, *where*

$$P_{(2)}(\zeta,\nu) = [2;2]_\nu \ S + \frac{[2;1,2]_\nu}{[1;1]_\nu} \quad \textit{and recursively for } r \geq 3\textit{:}$$

$$P_{(r)}(\zeta,\nu) = \sum_{k=0}^{r-3} x^k [r;\ r-k, \ldots, r]_\nu \ SxP_{(r-k-1)}(\zeta,\nu) + x^{r-2}\left([r;2,\ldots,r]_\nu \ S + \frac{[r;1,\ldots,r]_\nu}{[1;1]_\nu}\right) \ .$$

Then

$$J_{[\lambda]}(x;\alpha) = \sum_{\zeta \in SYT(\lambda)} J(\zeta;\alpha)(x), \tag{J}$$

where $J_{[\lambda]}(x;t)$ *is the sequence of the* $J_\lambda^{[m]}(x;\alpha) := \sum_{\eta \in SSYT_{[m]}} w^J(\eta) \ x^\eta$ *with* $m \in \mathbb{N}$.

Proof. The τPx-formula for $J_{[\lambda]}(x;\alpha)$ is a weighted form of the one given for $s_{[\lambda]}(x)$; we use Thm.5.4 with 'J' instead of 'B' to indicate that weights w^J are involved.

It is easy to see that the sequence '$\ldots x$' of step 1 translates to $\ldots x \ [1;1]_0 = \ldots x \ w^J(1;1) = \ldots x$ and that $w^J(\zeta^{(\nu+1)};\ j(\nu+1)) = [1;1]_\nu$ gives the weight for an S-step. Therefore it is only necessary to proof the validity of step 3. Since w^J is S-insected, Prop.6.3 applies and it is sufficient to show the validity of step 3 in the case of a single set $M_{\overline{\nu}} := \{\nu+1, \ldots, \nu+r\}$. Note that such a set corresponds to a unique H-strip in the maximal identification $\widehat{\zeta}$ of ζ; clearly the special values of the row and column numbers of the boxes in this strip are not important for the translation step 3, whence it is possible to restrict to the case of $\zeta^r \in SYT(r)$, i.e. the study of the unique SYT of shape (r).

Subsequently let $[\lambda;\ \mathbf{j}] \equiv [\lambda;\ \mathbf{j}]_0$ and $P_{(r)} \equiv P_{(r)}(\zeta^r, 0)$. We have to show:

$$J(\zeta^r)(x) = \sum_{I \subset \{1,\ldots,r-1\}} J^I(\zeta^r)(x) \overset{(!)}{=} xP_{(r)}x \ .$$

For $r=2$ the assertion is true:

$$J(\zeta^2)(x) = J^{\{1\}}(\zeta^2)(x) + J^{\emptyset}(\zeta^2)(x) = x \ [2;2] \ S \ x + x \ [2;1,2]x = x([2;2] \ S + \frac{[2;1,2]}{[1;1]})x \ .$$

Now let $r \geq 3$; then

$$J(\zeta^{r+1})(x) = \sum_{I \subset \{1,\ldots,r-1\}} J^I(\zeta^{r+1})(x) + \sum_{I \subset \{1,\ldots,r-1\}} J^{\{r\}\cup I}(\zeta^{r+1})(x) \ .$$

The first summand is by induction hypothesis

$$x \ [r+1;r+1]S \sum_{I \subset \{1,\ldots,r-1\}} J^I(\zeta^r)(x) = x \ [r+1;r+1]S \ xP_{(r)}x$$

and the second equals

$$\sum_{k=0}^{r-2} \sum_{I\subset\{1,\ldots,r-k-2\}} J^{\{r-k,\ldots,r\}\cup I}(\zeta^{r+1})(x) + J^{\{1,\ldots,r\}}(\zeta^{r+1})(x) \ .$$

For $0 \le k \le r-3$ one has

$$\sum_{I\subset\{1,\ldots,r-k-2\}} J^{\{r-k,\ldots,r\}\cup I}(\zeta^{r+1})(x) =$$

$$x^{k+2}\ [r+1;\ r-k,\ldots,r+1]\ S \sum_{I\subset\{1,\ldots,r-k-2\}} J^{I}(\zeta^{r-k})(x) =$$

$$x^{k+2}\ [r+1;\ r-k,\ldots,r+1]\ Sx\ P_{(r-k-1)}x$$

and for $k=r-2$: $J^{\{2,\ldots,r\}}(\zeta^{r+1})(x) = x^r\ [r+1;\ 2,\ldots,r+1]Sx$. Finally

$$J^{\{1,\ldots,r\}}(\zeta^{r+1})(x) = x^r\ [r+1;1,\ldots,r]\ \frac{[r;1,\ldots,r]}{[1;1]}\ x$$

and summation gives $J(\zeta^{r+1})(x) = x\ P_{(r+1)}x$ as desired. $\square$

Example 6.5. For $\lambda = 3\ 2$ we compute the $J(\zeta;\alpha)(x)$, which sum up to the weighted τPx-expansion of $J_{[32]}(x;\alpha)$. (We omit the 'w^J' and the 'ζ' in $P_r(\zeta,\nu)$.):

For $\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5\\ \cline{1-2}\end{array}$ step 1.) gives $xPxSxPxPx$, which translates to
$xP_{(2)}(3)\ x[1;1]_3\ Sx\ P_{(3)}(0)x =$
$x\left([2;2]_3\ S + \frac{[2;1,2]_3}{[1;1]_3}\right) x[1;1]_3\ Sx\ \left[[3;3]_0 Sx\left([2;2]_0\ S + \frac{[2;1,2]_0}{[1;1]_0}\right) + x\left([3;2,3]_0\ S + \frac{[3;1,2,3]_0}{[1;1]_0}\right)\right] =$

$x\left((32;2)\ S + \frac{(32;1,2)}{(31;1)}\right) x(31;1)\ Sx\ [(3;3)Sx\,((2;2)\ S + (2;1,2)) + x\,((3;2,3)\ S + (3;1,2,3))];$

Similarly one computes:

$w^J(\ \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4\\ \cline{1-2}\end{array}\ ) = xP_{(3)}(2)\ x[1;1]_2\ Sx\ P_{(2)}(0)x =$
$x\ \left[(32;3)Sx\left((2^2;2)\ S + \frac{(2^2;1,2)}{(21;1)}\right) + x\left((32;2,3)\ S + \frac{(32;1,2,3)}{(21;1)}\right)\right] x(21;1)Sx\,((2;2)\ S + (2;1,2));$

$w^J(\ \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5\\ \cline{1-2}\end{array}\ ) = x[1;1]_4\ SxP_{(2)}(2)\ x[1;1]_2\ Sx\ P_{(2)}(0)x =$
$x(32;2)Sx\left((31;3)\ S + \frac{(31;1,3)}{(21;1)}\right) x(21;1)Sx\,((2;2)\ S + (2;1,2));$

$w^J(\ \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5\\ \cline{1-2}\end{array}\ ) = x[1;1]_4\ SxP_{(3)}(1)\ x[1;1]_1\ Sx =$
$x(3;2,3)\ Sx\ \left[(31;3)Sx\left((21;2)\ S + \frac{(21;1,2)}{(1^2;1)}\right) + x\left((31;2,3)\ S + \frac{(31;1,2,3)}{(1^2;1)}\right)\right] x(1^2;1)Sx;$

$w^J(\ \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4\\ \cline{1-2}\end{array}\ ) = xP_{(2)}(3)\ x[1;1]_3\ SxP_{(2)}(1)\ x[1;1]_1\ Sx =$
$x\left((32;2,3)\ S + \frac{(32;2,3)}{(2^2;2)}\right) x(2^2;2)Sx\left((21;2)\ S + \frac{(21;1,2)}{(1^2;1)}\right) x(1^2;1)Sx.$

Remark 6.6. The *Skew Jack funtions* for skew shapes λ/μ (with $\lambda \vdash N,\ \mu \vdash M, \mu \subset \lambda$, $M \leq N$) are combinatorially defined by ([St2, Thm.6.3, Thm.5.8])

$$J_{\lambda/\mu}(x;\alpha) := j_\mu \sum_{\eta \in SSYT(\lambda/\mu)} w^J(\eta)\ x^\eta\ , \text{ where }\ j_\mu := \prod_{s\in\mu} h^+(s)\ h^-(s)\ .$$

The $J_{[\lambda/\mu]}(x;\alpha)$ can be computed similarly to the Schur and HL case from the $J(\zeta;\alpha)(x)$ of $J_{[\lambda]}(x;\alpha)$: single out all $\zeta \in SYT(\lambda)$ with $\lambda(\zeta^{(M)}) = \mu$ and delete everything to the right of the $(N-M)^{th}$ x except an 'appropriate weight', i.e. if $M \notin I(\zeta)$, then use $[1;1]_M$, if $M \in I(\zeta)$, then one has to single out from the weight $P_{(r)}(\nu)$ $(\nu < M)$ a specific part in accordance with the length $r' < r$ of the remaining sequence of consecutive numbers in "$I(\zeta/\mu)$".

As an example consider $\lambda = 32$ and $\mu = 2$ and the $J(\zeta)(x)$ for the first three SYT in Ex.6.5: $j_\mu = 2\alpha^2(1+\alpha)$; the second and the third SYT give $xP_{(3)}(\zeta,2)\ x[1;1]_2$ and $x[1;1]_4\ SxP_{(2)}(\zeta,2)\ x[1;1]_2$, respectively; the first SYT yields $xP_{(2)}(\zeta,3)\ x[1;1]_3\ Sx\ [1;1]_2$.

Remark 6.7. In analogy to the HL case the Jack functions have a recursive structure with respect to the Young lattice $\mathcal{Y}$ (compare Rem.5.11); it is only necessary to adjust the weights appropriately.

7. Macdonald polynomials and 'Super-orthogonality'

Macdonald functions $Q_\lambda(x;q,t)$ (cf. [M3]) are two-parameter extensions of the previous functions, i.e. they include Schur functions for $q = t$, HL functions for $q = 0$, and Jack functions for $q = t^\alpha$ $(\alpha \in \mathbb{R}, \alpha > 0)$ and $t \to 1$ (and also the λ'-product of elementary symmetric functions for $q = 1$ and the monomial symmetric functions for $t = 1$). In the Macdonald case there are functions $P_\lambda(x;q,t)$, too, which are related to the $Q_\lambda(x;q,t)$ by a factor $b_\lambda(q,t)$ ([M3, (4.12), (5.9)]). Moreover Macdonald polynomials appear as singular vectors in representations of quantum deformations of both the Virasoro algebra ([AKOS1, AKOS2]) and the sl_n ([K2]). The Macdonald polynomials discussed in this section are in fact 'case A' specializations of general orthogonal polynomials associated to root systems of irreducible Weyl groups ([M4]), which have been extensively studied and further generalized for example by A.A. Kirillov Jr. ([K1, K2]) and I. Cherednik ([C1, C2]).

Our starting point is the combinatorial definition of Macdonald functions[2] given in [M3, (4.10-11), (5.10-13)], which is of the form (0.5). For every $\eta \in SSYT(\lambda)$ the weight $\varphi(\eta) \equiv w^M(\eta) \in \mathbb{Q}(q,t)$ can be computed completely as in the Jack case (6.1-5), except that the analog of (6.1) now reads:

$$w^M(\lambda|\ \mathbf{j}) := \prod_{\substack{s=(i,j)\in\lambda \\ j\in\{\mathbf{j}\}}} \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}\ . \tag{7.1}$$

[2]The author is indebted to I.G. Macdonald for explaining him in a letter some intricacies of the paper [M3].

Therefore w^M is S-insected and 'signed diagrams', Thm.6.4, Ex.6.5 and Rem.6.6-7 apply as in the Jack case: simply substitute w^J by w^M.

We finish our short treatment of Macdonald functions by proving from the combinatorial definitions:

$$\text{(a)} \qquad Q_\lambda(x;0,t) = Q_\lambda(x;t) \quad \text{and}$$

$$\text{(b)} \qquad \lim_{t\to 1} Q_\lambda(x;t^\alpha,t) = \frac{1}{h_\lambda^-(\lambda)}\ J_\lambda(x;\alpha)\ ,$$

where $h_\lambda^-(S) := \prod_{s\in S} h_\lambda^-(s)$ for $S \subset \mathbb{N}\times\mathbb{N}$.

Proof. a) Set $q=0$, fix $\eta \in G(\lambda)$, $\nu\in\mathbb{N}$, $j \in \{\mathbf{j}(\eta)^{[\nu]}\}$ and let $\lambda \equiv \lambda(\eta^{(\nu)})$; then

$$w^M(\lambda|\mathbf{j}) = \prod_{\substack{s=(i,j)\in\lambda\\ j\in\{\mathbf{j}\}}} \left(1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}\right)\ ,$$

i.e. only $(i,j)\in\lambda$ with armlength $a(i,j)=0$ contribute to the product. Assume $(i,j),(i-1,j),\ldots,(i-r,j)\in\eta^{(\nu)}$ have all armlength zero relative to λ and $(i+1,j),(i-r-1,j)\notin\eta^{(\nu)}$, then the boxes $(i,j),(i-1,j),\ldots,(i-r,j)$ contribute a factor $(1-t)\ldots(1-t^{r+1})$ to the numerator of $w^M(\eta,\nu)$. In case of $j+1\in\{\mathbf{j}(\eta)^{[\nu]}\} \iff (i,j)\notin RH(\eta^{[\nu]})$ it follows that $(i-1,j),\ldots,(i-r-1,j)$ are boxes with armlength zero relative to $\lambda(\eta^{(\nu-1)})$ and numerator and denominator of $w^M(\eta,\nu)$ cancel completely; if on the other hand $j+1\notin\{\mathbf{j}(\eta)^{[\nu]}\} \iff (i,j)\in RH(\eta^{[\nu]})$, then $a(i-r-1,j)\neq 0$ relative to $\lambda(\eta^{(\nu-1)})$ and there will 'survive' a factor $(1-t^{r+1}) = (1-t^{m_j(\eta^{(\nu)})})$ as desired.

b) Let

$$b_\lambda(s)\equiv b_\lambda(s;q,t) := \frac{1-q^{a_\lambda(s)}t^{l_\lambda(s)+1}}{1-q^{a_\lambda(s)+1}t^{l_\lambda(s)}}\ ,\ \text{if } s\in\lambda, \text{ and } =1 \text{ otherwise.}$$

Clearly $\lim_{t\to 1} b_\lambda(s;t^\alpha,t) = h_\lambda^+(s)/h_\lambda^-(s)$.

Now fix $\eta\in SSYT(\lambda)$ with maximal entry r. Using the notations

$$h_\lambda^+(S) := \prod_{s\in S} h_\lambda^+(s)\ ,\quad b_\lambda(S) := \prod_{s\in S} b_\lambda(s)\ \text{ for } S\subset\mathbb{N}\times\mathbb{N} \text{ and}$$

$$C_\nu \equiv C_\nu(\eta) := \{(i,j)\in\eta^{(\nu)} \mid j\in\{\mathbf{j}(\eta^{[\nu]})\}\}\ ,\quad \overline{C_\nu}\equiv\overline{C_\nu(\eta)} := \{(i,j)\in\eta^{(\nu)} \mid j\notin\{\mathbf{j}(\eta^{[\nu]})\}\}$$

one computes:

$$
\begin{aligned}
\lim_{t\to 1} w^M(\eta)|_{q=t^\alpha} &= \lim_{t\to 1}\prod_{\nu=1}^{r}\frac{b_{\eta^{(\nu)}}(C_\nu)}{b_{\eta^{(\nu-1)}}(C_\nu)} \\
&= \prod_{\nu=1}^{r}\frac{h^+_{\eta^{(\nu)}}(C_\nu)}{h^-_{\eta^{(\nu)}}(C_\nu)}\cdot\frac{h^-_{\eta^{(\nu-1)}}(C_\nu)}{h^+_{\eta^{(\nu-1)}}(C_\nu)} \\
&= \prod_{\nu=1}^{r}\frac{h^+_{\eta^{(\nu)}}(C_\nu)\ h^-_{\eta^{(\nu)}}(\overline{C_\nu})}{h^-_{\eta^{(\nu)}}(\eta^{(\nu)})}\cdot\frac{h^-_{\eta^{(\nu-1)}}(\eta^{(\nu)})}{h^+_{\eta^{(\nu-1)}}(C_\nu)\ h^-_{\eta^{(\nu-1)}}(\overline{C_\nu})} \\
&= \prod_{\nu=1}^{r}\frac{h^-_{\eta^{(\nu-1)}}(\eta^{(\nu-1)})}{h^-_{\eta^{(\nu)}}(\eta^{(\nu)})}\cdot\prod_{\nu=1}^{r} w^J(\eta,\nu) = \frac{1}{h^-_\lambda(\lambda)}w^J(\eta)\ ,
\end{aligned}
$$

which implies the desired result. □

Definition 7.1. *The weighted symmetric functions $\{\lambda\}_w(x)$ resp. the weight w is called* super-orthogonal*, if for all partitions λ there exist factors c_λ in the ring containing the weight such that*

$$\{\lambda\}^{(n)}_w(x) = c_\lambda\ x_1\ldots x_n\ \{\lambda^-\}^{(n)}_w \quad ,$$

where $n = l(\lambda)$ and $\lambda^- := \lambda_1 - 1, \ldots \lambda_n - 1$.

Observing that the deletion of the first column defines a bijection from $SSYT_{(n)}(\lambda)$ to $SSYT_{(n)}(\lambda^-)$ it is not hard to see that Schur and HL polynomials are are super-orthogonal with all $c_\lambda = 1$; using signed diagrams one similarly verifies super-orthogonality in the Jack case with factors $c_\lambda(\alpha) \in \mathbb{Z}[\alpha]$ as given in [St2, Prop.5.5]; and in case of Macdonald functions $P_\lambda(x; q, t)$ one has again all factors $c_\lambda = 1$ (cf. [M3, (5.8)]). In fact this is a nearly complete list of super-orthogonal symmetric functions as S.V. Kerov has shown using the theory of orthogonal polynomials:

Theorem 7.2. ([Ke, Thm.2]) *Suppose the weighted symmetric functions $\{\lambda\}_w$ are super-orthogonal for $n = 2$, then these functions are (specializations of) Macdonald, Jack or slightly generalized Jack functions.*

Appendix

Let $N \in \mathbb{N}$ and $\lambda \equiv \lambda_1 \ldots \lambda_s$ with $\lambda_1 \geq \cdots \geq \lambda_s \geq 1$ be a *partition of* N: $\lambda \vdash N$. Alternatively λ can be written as $\lambda = 1^{m_1(\lambda)}2^{m_2(\lambda)}\ldots$, where $m_j(\lambda)$ denotes the number of parts of λ with size j. Depending on the context the symbol λ will denote a partition, its (english style) *Ferrer diagram*, or a mapping, which associates in an obvious way a partition to an object. As usual let $s = l(\lambda)$ be the *lenght* of λ, λ' the *conjugate* of λ, λ/μ for $\mu \subset \lambda$ the *skew shape* of 'λ without μ', and for $\lambda, \mu \vdash N$: $\mu \leq \lambda \ :\Longleftrightarrow \forall j : \sum_{i=1}^{j}\mu_i \leq \sum_{i=1}^{j}\lambda_i$ the *dominance order* (cf. [M]).

F_N: $= \{\rho = (\rho_1, \rho_2, \rho_3, \dots) \mid \forall \nu \in \mathbb{N} : \ \rho_\nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ , |\rho| := \sum_{\nu=1}^{\infty} \rho_\nu = N \ \}$ is the set of all finite sequences of non-negative integers with sum of components $= N$ (usually the sequence of end zeroes will be omitted);

$F_N^{(m)}$: $= \{\rho \in F_N \mid \forall \nu > m : \ \rho_\nu = 0 \ \}$;

$F_N^{[m]}$: $= \{\rho \in F_N \mid \forall \nu > m : \ \rho_\nu = 0 \ , \ \rho_m \neq 0 \ \} = \{\rho \in F_N^{(m)} \mid \rho_m \neq 0 \ \} = F_N^{(m)} \setminus F_N^{(m-1)}$;

GF_N: $= \{\rho \in F_N \mid \rho_\nu = 0 \Longrightarrow \rho_{\nu+1} = 0 \ \forall \nu \in \mathbb{N} \ \}$ the set of *gapless* elements of F_N;

PF_N: $= \{\rho \in F_N \mid \rho_1 \geq \rho_2 \geq \dots \ \}$ can be identified with the set of partitions of N: $P(N) := \{\lambda \vdash N\}$;

pr_G: $F_N \longrightarrow GF_N$ is the mapping, which 'projects' every $\rho \in F_N$ to an element of GF_N by deleting all '0' between nonzero entries (without changing their order);

pr_P: $F_N \longrightarrow PF_N$ is the mapping, which 'projects' every $\rho \in F_N$ to an element of PF_N by ordering all nonzero entries in ρ nonincreasingly into the initial part of the sequence;

$F_N(\mu)$: $= \{\rho \in F_N \mid \lambda(\rho) \equiv \lambda(pr_P \rho) = \mu \ \}$ for some given $\mu \vdash N$.

$SSYT(\lambda)$ denotes the set of *semistandard* or *columnstrict Young tableaux of shape* λ, i.e. $\eta \in SSYT(\lambda)$ is a numbering of the boxes of the Ferrer diagram λ with natural numbers, such that the numbers increase weakly in the rows from left to right and increase strictly in the columns from top to bottom; then the *content* $\rho \equiv \rho(\eta)$ of η is defined as the sequence $(\rho_1, \rho_2, \dots) \in F_N$, where ρ_ν is the multiplicity of the value ν in η.

$SSYT_{(m)}(\lambda)$: $= \{\eta \in SSYT(\lambda) \mid \rho(\eta) \in F_N^{(m)}\}$;

$SSYT_{[m]}(\lambda)$: $= \{\eta \in SSYT(\lambda) \mid \rho(\eta) \in F_N^{[m]}\} = SSYT_{(m)}(\lambda) \setminus SSYT_{(m-1)}(\lambda)$;

$SSYT(\lambda, \rho)$: $= \{\eta \in SSYT(\lambda) \mid \rho(\eta) = \rho \ \}$, where '$\rho(\eta) = \rho$' should be understand as 'the content of η is equal to the given $\rho \in F_N$';

$SSYT(\lambda, \mu)$: $= \{\eta \in SSYT(\lambda) \mid \lambda(\rho(\eta)) = \mu \ \}$, where '$\lambda, \mu \vdash N$' and $\lambda(\rho(\eta)) = \mu$ should be read: 'the partition $pr_P \ \rho(\eta)$ is equal to the given $\mu \vdash N$'; clearly: $SSYT(\lambda, \mu) = \biguplus_{\rho \in F_N(\mu)} SSYT(\lambda, \rho)$, where $\biguplus$ stands for 'disjoint union';

$G(\lambda)$: $= \{\eta \in SSYT(\lambda) \mid \rho(\eta) \in GF_N \ \}$ is the set of *gapless* $SSYT(\lambda)$; $G(\lambda, \rho) := SSYT(\lambda, \rho) \cap G(\lambda)$, $G(\lambda, \mu) := SSYT(\lambda, \mu) \cap G(\lambda)$;

$P(\lambda)$: $= \{\eta \in SSYT(\lambda) \mid \rho(\eta) \in PF_N \ \}$ is the set of $SSYT(\lambda)$ with *partitionlike content*; $P(\lambda, \mu) := SSYT(\lambda, \mu) \cap P(\lambda)$; the unique element of $P(\lambda, \lambda)$ is called the *superstandard tableaux*;

pr_G: $SSYT(\lambda) \longrightarrow G(\lambda)$ is the mapping, which 'projects' every $\eta \in SSYT(\lambda)$ to an element of $G(\lambda)$: let $\rho_{\nu_1}, \rho_{\nu_2}, \dots$ with $\nu_1 < \nu_2 < \dots$ be the subsequence of nonzero entries in $\rho(\eta)$, then $pr_G(\eta)$ is the same as η, but with 1 instead of ν_1, 2 instead of ν_2, etc. ;

pr_P: $SSYT(\lambda) \longrightarrow P(\lambda)$ analogous to pr_G;

$SYT(\lambda)$: $= G(\lambda, 1^N) = P(\lambda, 1^N) = SSYT_{[N]}(\lambda, 1^N) = SSYT_{(N)}(\lambda, 1^N)$ is the set of *standard Young tableaux of shape* λ, i.e. the subset of all $\eta \in SSYT(\lambda)$, which take every number from $\{1, \dots, N\}$ exactly once.

$\eta^{(\nu)}$**:** for some $\eta \in SSYT(\lambda)$ is the sub-SSYT of η, which containes exactly the boxes with entries $\leq \nu$;

$\eta^{[\nu]}$**:** $= \eta^{(\nu)} - \eta^{(\nu-1)}$ is called the *horizontal strip* or H-*strip of* ν*-boxes in* η, because it contains at most one box per column (by the columnstrictness of SSYT);

$\eta^{[\nu,\nu-k]}$**:** $= \eta^{(\nu)} - \eta^{(\nu-k-1)}$ for $1 \leq k \leq \nu - 2$ is a *multistrip*; this includes as special case the *double strip* $\eta^{[\nu,\nu-1]}$.

The no(ta)tions for *skew tableaux* $SSYT(\lambda/\mu)$ are analogous.

Let $\mathcal{Y}$ denote the (distributive) *Young lattice* of all partitions ordered by inclusion of Ferrer diagrams with bottom element $\emptyset$ and rank function $rk : \mathcal{Y} \longrightarrow \mathbb{N}_0$ given by $rk(\lambda) := |\lambda|$. We are interested in chains $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$, where in every step is added at most one box per column, i.e. the growth of diagrams proceeds by adding *horizontal stripes* or H-*stripes*; these chains or multichains, i.e. chains with repeated elements, in $\mathcal{Y}$ are called H-*(multi)chains*. In case of $\mu = \emptyset$ one has a bijection between the set of all H-multichains from $\emptyset$ to λ and $SSYT(\lambda)$: all boxes added in one step '$\lambda^{(\nu-1)} \subset \lambda^{(\nu)}$' are numbered with '$\nu$' in the corresponding $\eta \in SSYT(\lambda)$ and conversely the sequence of shapes $\lambda(\eta^{(\nu)})$ is clearly a H-multichain. Similarly there are the bijections: 'H-chains $\longleftrightarrow G(\lambda)$' and 'saturated H-chains (i.e. H-chains with minimal steps) $\longleftrightarrow SYT(\lambda)$'.

For $\emptyset \neq \mu \subset \lambda$ the skew analogues are obtained. Using vertical instead of horizontal stripes gives essentially only $SSYT(\lambda')$, but the use of 'rim hooks' instead of H-stripes for example yields important other informations: compare the 'Murnaghan-Nakayama Rule' ([K,S]).

Let $\eta^{[\nu]}$ be the H-strip of ν-boxes for some $\eta \in SSYT(\lambda)$ as above, then the set $H(\eta^{[\nu]})$ of *horizontal* or H-*components* of $\eta^{[\nu]}$ contains all subsets of boxes of $\eta^{[\nu]}$, which are horizontally connected, i.e. not separated by empty columns. Furthermore $H(\eta) := \biguplus_{\nu \geq 1} H(\eta^{[\nu]})$ is the set of all H-components of η and $RH(\eta) \equiv \biguplus_{\nu \geq 1} RH(\eta^{[\nu]})$ the set of all rightmost boxes in the H-components of η.

Of course one can define similarly the sets: $LH(\eta)$ of leftmost boxes in the H-components of η, $V(\eta^{[\nu]})$ of *vertical* or V-*components* of $\eta^{[\nu]}$ (no separation by empty rows), and $C(\eta^{[\nu]})$ of *complete* or C-*components* of $\eta^{[\nu]}$ (no separation by empty columns and rows), etc. .

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