A COMBINATORIAL DERIVATION OF THE POINCARÉ POLYNOMIALS OF THE FINITE IRREDUCIBLE COXETER GROUPS

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Abstract. (1) The Poincaré polynomials of the finite irreducible Coxeter groups are derived by an elementary combinatorial method avoiding the use of Lie theory and invariant theory. (2) Non-recursive methods for the computation of 'standard reduced words' for (signed) permutations are described. The algebraic basis for both (1) and (2) is a simple partition property of the weak Bruhat order of Coxeter groups into isomorphic parts.

The combinatorial properties of weak Bruhat order of Coxeter groups, especially of the finite irreducible and affine ones, have been investigated for a long time (see [BjW, Sm] and the references therein). But many results in this subject are scattered over hard to find journals, unpublished, or part of the "folklore" — known only to a few experts. Therefore the present paper seems to fill a gap in the literature.

More specifically, we describe a partitioning property of the weak Bruhat order of Coxeter groups into isomorphic parts (Theorem 0.1) and demonstrate its usefulness by (1) giving a pictorial combinatorial derivation — in contrast to the usual invariant theoretic or Lie theoretic derivation (cf. [B, Hu]) — of the Poincaré polynomials for the finite irreducible Coxeter groups and by (2) deriving simple non-recursive schemes for the computation of 'standard reduced words' for both unsigned and signed permutations. Some of the pictures of the labeled Hasse diagrams in Section 1 have appeared also in connection with Verma modules and Schubert cells [GM], as a graphical device for calculating the homology of the most elementary Artin groups [Sq], and most recently together with the partition property in [CP]. Related theoretical results are also contained in [D], results concerning reduced words in [Cl1, Cl2]. For the sporadic and affine Coxeter groups the unpublished work [E] of Eriksson is especially relevant [1†].

Below in the introduction we recall some well known facts about Coxeter groups, weak Bruhat order, and Poincaré series. We state and prove the Partitioning Theorem 0.1 about the weak Bruhat order of Coxeter groups into order-isomorphic parts and derive the basic combinatorial Algorithm 0.5 used subsequently. Sections 1 contains pictures of the labeled Hasse diagrams for those parts of finite Coxeter groups, which are induced by the maximal parabolic subgroups with connected sub-Coxeter graphs. These pictures are derived with the help of Algorithm 0.5 directly from the group relations, and elementary calculations than yield the Poincaré polynomials or series for each of these Coxeter groups.

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[1†] I am indebted to the referees for pointing out the references [Cl1, Cl2, CP, D, E, Lc2].

In case of the infinite families of type A, B, D the labeled Hasse diagrams of the isomorphic parts are chains or almost chains, which suggests the idea of 'standard reduced words'. This has been (independently) observed by J.R. Stembridge in [Sm, Sec.1.3], but in Section 2 we go one step further and describe simple non-recursive methods, which yield these standard reduced words from the list form of (signed) permutations via 'codes' of permutations. In order to expound the symmetries of the subject we discuss four kinds of codes and standard reduced words in the case of ordinary permutations and one kind (out of two) in the case of signed permutations of type B and D, respectively (Propositions 2.4, 2.9, and 2.13). Possible further directions of research are indicated in Section 3. — It is important to note that the codes are in fact nothing but the well known 'inversion tables' in a different form. For type A appearently the first description of this connection is due to Rothe [Ro] in 1800 !

A *Coxeter system* (W, S) is a group W together with a subset $S \subset W$ of generators (we write: $W = \langle S \rangle$) subject only to relations of the form

$$\forall\ s, s' \in S\ \exists\ m(s, s') \in \mathbb{N} \cup \{\infty\}\ :\quad (ss')^{m(s,s')} = \mathbf{1}\ ,$$

where $m(s, s) = 1$ and $m(s, s') \geq 2$ for $s \neq s'$. We recall below some basic facts about Coxeter groups a full account of which can be found for example in the books [B, Hu, Hi].

For every $I \subset S$ let $W_I := \langle I \rangle$ be the *parabolic subgroup* generated by I. Every $w \in W$ can be written as a product or *word* in the generators: $w = s_1 \dots s_r$. Words of minimal length are called *reduced words*, the length of a reduced word for $w \in W$ is the *length* $l(w)$ of w, and $R(w)$ denotes the set of all reduced words for w. Usually the generators in S are numbered by some subset of the non-negative integers $\mathbb{N}_0$ so that we can write $w = s_{a_1} s_{a_2} \dots s_{a_r} \equiv s_{a_1 a_2 \dots a_r} \equiv s_a$ for $a = a_1 a_2 \dots a_r$; moreover, in a slight abuse of notation we write $a \in R(w)$ iff $s_a \in R(w)$ for 'reduced' $a = a_1 a_2 \dots a_r$.

For $N \in \mathbb{N}_0$ let $W^{[N]} := \{w \in W \mid l(w) = N\}$ and $|W^{[N]}|$ the cardinality of the set $W^{[N]}$; then the 'generating function' of the length of elements in W

$$P_W(t) := \sum_{w \in W} t^{l(w)} = \sum_{N \geq 0} |W^{[N]}|\ t^N$$

is called the *Poincaré series*, if W is infinite, and the *Poincaré polynomial*, if W is finite.

The *right weak Bruhat order* or *right order* '$\leq$' on (W, S) is defined as the transitive closure of the following *covering relation*:

$$w \prec w' :\Longleftrightarrow\ \exists s \in S : w' = ws \text{ and } l(w') = l(w) + 1\ .$$

The above defined partial order is called *right order*, because s acts on the right; if s is placed on the left in the definition of the covering relation, one gets the *left order*, which is clearly order isomorphic to the right order under reading words backwards (cf. [BjW]).

The right order on W can be visualized using the *Hasse diagram*: the elements of $W^{[N]}$ are depicted as points on *level* N — the *root* on level 0 is the identity $\mathbf{1}$ — with edges drawn according to the covering relation. If one in addition labels the edges with their respective generators s, then the resulting *labeled Hasse diagram* (LHD) contains the full information about the sets $R(w)$ for all $w \in W$: $R(w)$ is the set of all sequences of edge labels for ascending (maximal/saturated) chains from the root to w.

Given a Coxeter system (W, S) and a parabolic subgroup W_I for $I \subset S$, the set of *minimal coset representatives* is defined as

$$W^I := \{w \in W \mid l(ws) > l(w) \text{ for all } s \in I\} \ ,$$

which is the set of elements of minimal length in all cosets wW_I. We are interested in the sets defined very similarly as

$$V^I := \{w \in W \mid l(sw) > l(w) \text{ for all } s \in I\} \ .$$

In fact, $V^I = (W^I)^{-1}$ as sets, but usually we think of W^I and V^I without further mentioning as partially ordered sets with respect to the right order.

For arbitrary $I, I' \subset S$ it follows immediately from the definition that $V^{I \cup I'} = V^I \cap V^{I'}$, whereas LHD$(V^{I \cap I'})$ is an interesting kind of quotient of LHD$(V^I \times V^{I'})$ in the order theoretic (not in the group theoretic) sense.

The sets V^I are *convex* w.r.t. right order, i.e. $v, v' \in V^I$ and $v \leq w \leq v'$ implies $w \in V^I$, the sets W^I except for the trivial one element case never (see Ex.1.2 below). Therefore the LHD(V^I) are constructible with ease (cf. Algorithm 0.5), contrary to the LHD(W^I).

Clearly the cosets wW_I partition the set W and a well known result (cf. [Hu, Prop.1.10 c]) says that for every $w \in W$ and $I \subset S$ there exists a unique $u \in W^I$ and a unique $v \in W_I$, such that $w = uv$. Therefore the following theorem does not come as a surprise:

Theorem 0.1. *For every $w \in W$ and $I \subset S$ there exists a unique $u \in W_I$ and a unique $v \in V^I$, such that $w = uv$; in other words: W is the disjoint union of the sets uV^I:*

$$W = \biguplus_{u \in W_I} uV^I \ .$$

But more can be said: for every $I \subset S$ the LHD(W) can be partitioned into convex parts in right weak Bruhat order, such that all parts are of the same form LHD(V^I) and every $u \in W_I$ occurs exactly once as the bottom element of one of these parts.

Proof. Observe first that every $w \equiv s_a \in W$ can be written as $w = uv$ with $u \in W_I$ and $v \in V^I$: investigate all representation $s_a \in R(w)$ and choose one with r maximal, such that $a = a_1 \dots a_r a_{r+1} \dots a_t$ with $\{a_1, \dots, a_r\} \subset I$ and $a_{r+1} \notin I$; then $u = s_{a_1 \dots a_r} \in W_I$ and $v = s_{a_{r+1} \dots a_t} \in V^I$.

It remains to be shown that $u \neq u'$ for $u, u' \in W_I$ implies $uV^I \cap u'V^I = \emptyset$: assume there exist $v, v' \in V^I$, such that $uv = u'v'$; since W_I is a group, we conclude $v = u^{-1}u'v'$ with $u^{-1}u' \in W_I$, but by the definition of V^I the first generator s in every representation of $v = s \dots$ is not in I; hence $u^{-1}u' = id$ (and $v = v'$) contradicting our assumption.

The remaining assertion is now immediate from our previous discussion of the labeled Hasse diagrams of the right weak Bruhat order on Coxeter groups. $\square$

Let $P_X(t) := \sum_{w \in X} t^{l(w)}$ for an arbitrary $X \subset W$; then $P_W(t) = P_{W_I}(t) P_{W^I}(t)$ (cf. [Hu, Sec.1.11]).

Corollary 0.2. *$P_W(t) = P_{W_I}(t)P_{V^I}(t)$ and $|V^I| = |W^I| = [W : W_I]$, which is the index of W_I in W.*

Example 0.3. We investigate the symmetric group on 3 letters $W = A_2 \equiv S_3$ with $I = \{1\}$. The two figures below both show the LHD of S_3 together with the permutation representations of the elements $w \in S_3$. (The elementary transpositions $s_1 = (1,2)$ and $s_2 = (2,3)$ act on the right, i.e. on the places.) In the left figure double lines connect elements of the same coset, where minimal coset representatives are printed bold; in the right figure the double lines connect the elements of V^I and s_1V^I, respectively.

Notice that W^I and V^I as posets with respect to right order are not isomorphic.

In case of a finite Coxeter group W every parabolic subgroup W_I has an unique element w_I (w_S in case of $I = S$) of maximal length, which is characterized by the condition $l(w_Is) < l(w_I)$ for all $s \in I$.

Corollary 0.4. *With the preceding notations*

$$V^I \cong [\mathbf{1}, w_I^{-1}w_S]$$

in the right order.

Proof. By Theorem 0.1 one has that $V^I \cong [w_I, w_S]$, where the latter interval is isomorphic to $[\mathbf{1}, w_I^{-1}w_S]$ by [BjW, Prop.2.3]. $\square$

From Theorem 0.1 and its proof one deduces for finite Coxeter groups the following

Algorithm 0.5. (*for the computation of the* LHD(V^I)):
Let $V^I_{(N)}$ be the part of V^I up to level N. In particular: $V^I_{(0)} = \{\mathbf{1}\}$.
Set $V^I_{[N]} := V^I_{(N)} \setminus V^I_{(N-1)}$ for $N \geq 1$, i.e. $V^I_{[N]}$ contains the elements of rank N of V^I.
For $m_{ij} \equiv m(s_i, s_j) \geq 2$ we call the sequences $ijij\ldots$ and $jiji\ldots$ of length $m_{ij}-1$ *primitive for* (i,j). (The empty sequence, in particular, is not primitive.)

$V^I_{[1]}$ is constructed by attaching all possible edges to the root $V^I_{[0]}$ with labels from $S \setminus I$.
Assume that LHD($V^I_{(N)}$) for $N \geq 1$ has already been constructed.
Let v be any vertex in $V^I_{[N]}$ and let $j \in S$ be such that there is no edge j leading upwards to v. For every such pair (v, j) determine, whether there is a label i and a vertex $\alpha(v,i,j) \in V^I_{[N-m_{ij}+1]}$, such that the sequence of edge labels of a chain beginning in $\alpha(v,i,j)$ with edge

label i or j and ending in v with edge label i is primitive for (i,j). Consider the following three possibilities:

(1) There is no such primitive sequence: then the edge j is accepted as 'continuation' from v to $V^I_{[N+1]}$.

(2) There exists such a primitive sequence for (i,j) and v, and moreover there is a second primitive sequence for (i,j) of edge labels of a chain beginning at $\alpha(v,i,j)$ and ending in some $v' \in V^I_{[N]}$ with edge label j: then we can 'continue' from v with an edge j and from v' with an edge i to a new common vertex $\omega(v,i,j)$ in $V^I_{[N+1]}$.

(3) For v there exists a primitive sequence for (i,j) beginning at $\alpha(v,i,j)$, but no second primitive sequence for (i,j) beginning at $\alpha(v,i,j)$: j is rejected as the label of a possible upward edge from v.

Proof. In accordance with Theorem 0.1 (or the proof of Corollary 0.4) we construct V^I as the interval $[w_I, w_S]$: by the maximality of the element w_I in W_I the edges leading from the root of V^I to the first level $V^I_{[1]}$ are exactly those labeled by the elements in $S \setminus I$.

Henceforth we have to guarantee only that we move *upwards* from level N to $N+1$ in right order. Since we do everything in an interval contained in the full group, no further mentioning of the fixed set I is necessary. By the relation $(s_i s_j)^{m_{ij}} = \mathbf{1}$ a sequence is primitive for (i,j) iff by appending appropriately i or j to the right gives one side of the equality $iji\cdots = jij\ldots$. In cases (1) and (2) therefore the addition of an edge j at v leads indeed upwards: in case (1) the relation between i and j does not apply and the condition "there is no edge j leading upwards to v" excludes conflicts with other relations involving j; in case (2) the new chain fits together with the second chain given by the relation.

The only remaining possibility is case (3): assume that the single primitive sequence for (i,j) begins with an edge label i at $\alpha(v,i,j)$ (the argument for beginning label j is analogous). Then the addition of an edge j in v would lead by application of the relation between i and j to an edge j at vertex $\alpha(v,i,j)$. But by assumption such an upward edge j and a second primitive sequence for (i,j) does not exist, whence the edge j at $\alpha(v,i,j)$ leads downward and the sequence of i's and j's has already length m_{ij} in contradiction to primitivity. $\square$

Remark 0.6. For infinite Coxeter groups W the Algorithm 0.5 works exactly as in the finite case as long as the parabolic subgroup W_I is finite: then W_I has a unique maximal element w_I and V^I can be computed as right order on the set $\{w \in W \mid w \geq w_I\}$.

1. The Poincaré polynomials of the finite irreducible Coxeter groups

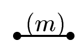
The Poincaré polynomials of the finite irreducible Coxeter groups have the form

$$P_W(t) = \prod_i \frac{1 - t^{d_i}}{1 - t} \ ,$$

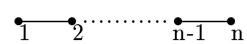
where the product is necessarily finite and the numbers d_i are called the *degrees* of the Coxeter group. Usually the above formula is proved and the degrees are computed with the help of invariant theory and Lie theory (see [Hu, Section 2.4, Chapter 3]); in this section we accomplish the same thing in a completely elementary way by explicitly computing the labeled Hasse diagrams of appropriate V^I's.

For every item in the case list of finite irreducible Coxeter groups (see e.g. [Hu, Figure 2.1],[Hi]) we collect the following facts:

1. The name of the group and it's degrees.
2. Its *Coxeter graph*, which is simply a graphical representation of the group relations: the labeled vertices represent generators and the edges between vertices i and j are labeled with $m(s_i, s_j)$ in parenthesis. We follow the usual convention that in case of $m(s_i, s_j) = 2$ resp. $s_i s_j = s_j s_i$ no edge is drawn between vertices i and j, and in case of $m(s_i, s_j) = 3$ the label of the edge (i, j) is omitted. (The labeling of the vertices is strictly speaking not part of the Coxeter graph, but essential for the next steps.)
3. The set I for which we compute V^I; usually I is the set of all labels of the vertices appearing in the Coxeter graph except the greatest.
4. The labeled Hasse diagram of V^I, which we draw left to right instead of bottom to top to save space.
5. A final calculation of the Poincaré polynomial in accordance with Corollary 0.2, where we use the abbreviations $\langle m\rangle := \frac{1-t^m}{1-t} = 1 + t + t^2 + \cdots + t^{m-1}$ and $m_n \equiv m \ldots m$ (n-fold) in the description and manipulation of the sequence of *Whitney numbers of the second kind*: $|V^I_{[0]}| \quad |V^I_{[1]}| \quad |V^I_{[2]}| \ \ldots$.

$\mathbf{I_2(m)}$ with degrees: $2, m$ and Coxeter graph 

has clearly the '$2m$-gon' as LHD and the sequence of Whitney numbers: $1\ 2_{m-1}\ 1$ corresponding to $(1+t)(1+t+\ldots t^{m-1}) = (1-t^2)(1-t^m)/(1-t)^2 = \langle 2\rangle\langle m\rangle$.

$\mathbf{A_n}$ ($n \geq 1$) with degrees: $2, 3, \ldots, n+1$, Coxeter graph 

. For $I = \{1, \ldots, n-1\}$ one computes V^I as:

and for $I = \{2, \ldots, n\}$ as:

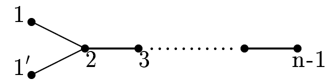
.

Since trivially $P_{A_1}(t) = P_{S_2}(t) = 1 + t$ and $A_2 = I_2(3)$ it follows in both cases by induction that $P_{A_n}(t) = P_{A_{n-1}}(t)\langle n+1\rangle = \langle 2\rangle\langle 3\rangle \ldots \langle n\rangle\langle n+1\rangle$. For a combinatorial derivation of of $P_{A_n}(t)$ using the inversion index of permutations see [St, Cor.1.3.10]. Compare also [Lc2].

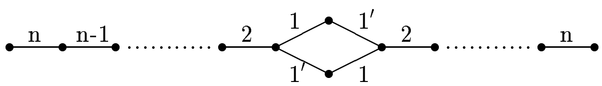
$\mathbf{B_n}$ ($n \geq 2$) has degrees: $2, 4, \ldots, 2n$, and Coxeter graph 

.
For $W = B_{n+1}$ and $I = \{0, \ldots, n-1\}$ one computes the following V^I:

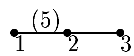
Since $B_2 = I_2(4)$ it follows by induction that $P_{B_n}(t) = P_{B_{n-1}}(t)\ (1 + t + \cdots + t^{2n-1}) = \langle 2\rangle\langle 4\rangle \ldots \langle 2n\rangle$.

$\mathbf{D_n}$ has degrees: $2, 4, \ldots, 2n-2, n$, and Coxeter graph 

.
For $W = D_{n+1}$ and $I = \{1', 1, 2, \ldots, n-1\}$ one computes the following V^I:



Since $P_{D_2}(t) = 1 + 2t + t^2$, it follows by induction that $P_{D_n}(t) = P_{D_{n-1}}(t) \cdot (1 + t^{n-1})\langle n\rangle = \langle 2\rangle\langle 4\rangle \ldots \langle 2n-4\rangle\langle n-1\rangle \cdot (1 + t^{n-1})\langle n\rangle = \langle 2\rangle\langle 4\rangle \ldots \langle 2n-4\rangle\langle 2n-2\rangle\langle n\rangle$.

$\mathbf{H_3}$ has degrees: $2, 6, 10$, and Coxeter graph 

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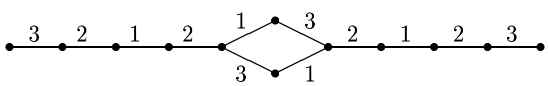
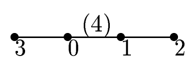
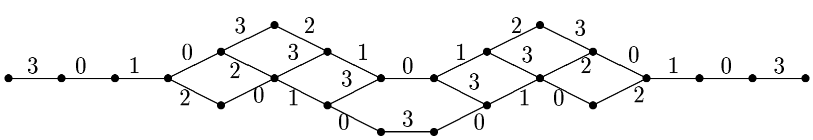
For $I = \{1, 2\}$ one has $W_I = I_2(5)$ and the following V^I:



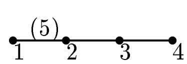
Therefore the sequence of Whitney numbers for V^I is $1_5\ 2\ 1_5$ and hence $P_{H_3}(t) = P_{I_2(5)}(t) \cdot (1 + t^5)(1 + t + \cdots + t^5) = \langle 2\rangle\langle 5\rangle(1 + t^5)\langle 6\rangle = \langle 2\rangle\langle 6\rangle\langle 10\rangle$. Leclerc [Lc1] has drawn the LHD of H_3 and the reader is invited to check the above V^I and the Theorem with the help of this drawing ($I = \{b, c\}$ in [L]); moreover it is interesting to see that other choices of I lead to more complicated V^I's, which is clearly a consequence of $|W_I|$ being smaller.

$\mathbf{F_4}$ has degrees: $2, 6, 8, 12$, and Coxeter graph 

.
For $I = \{0, 1, 2\}$ one has $W_I = B_3$ and the following V^I:



Therefore the sequence of Whitney numbers for V^I is $1_4\ 2_8\ 1_4$ and hence $P_{F_4}(t) = P_{B_3}(t) \cdot (1 + t^4)\ \langle 12\rangle = \langle 2\rangle\langle 4\rangle(1 + t^4)\langle 6\rangle\langle 12\rangle = \langle 2\rangle\langle 6\rangle\langle 8\rangle\langle 12\rangle$.

$\mathbf{H_4}$ has degrees: $2, 12, 20, 30$, and Coxeter graph 

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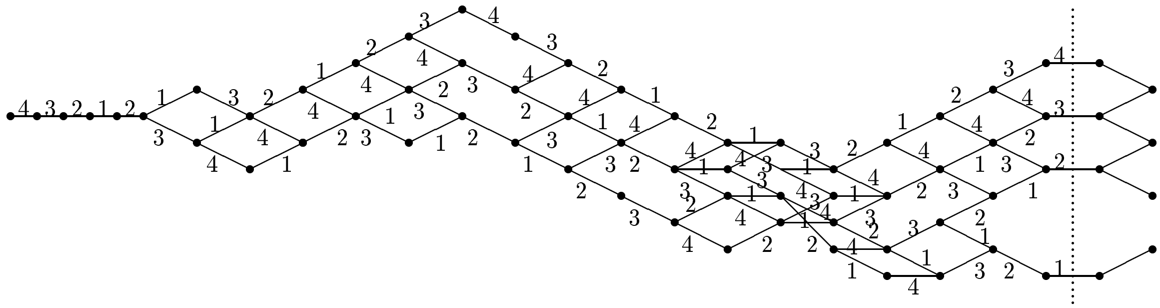
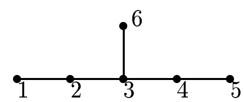
For $I = \{1, 2, 3\}$ one has $W_I = H_3$ and the following V^I (reflected at the dotted line):



Therefore the sequence of Whitney numbers for V^I is $1_6\ 2_4\ 3_6\ 4_{14}\ 3_6\ 2_4\ 1_6$ and hence $P_{H_4}(t) = P_{H_3}(t) \cdot (1 + t^6)(1 + t^{10})\langle 30\rangle = \langle 2\rangle\langle 6\rangle(1 + t^6)\langle 10\rangle(1 + t^{10})\langle 30\rangle = \langle 2\rangle\langle 12\rangle\langle 20\rangle\langle 30\rangle$.

$\mathbf{E_6}$ has degrees: $2, 5, 6, 8, 9, 12$, and Coxeter graph



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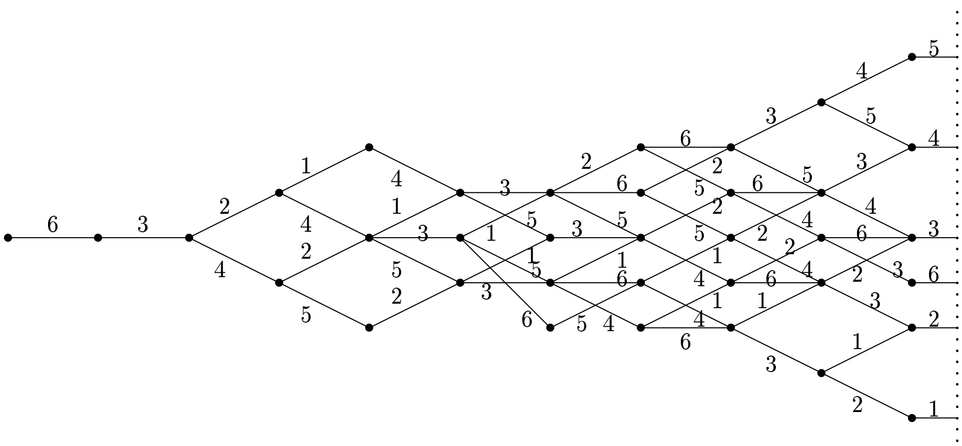
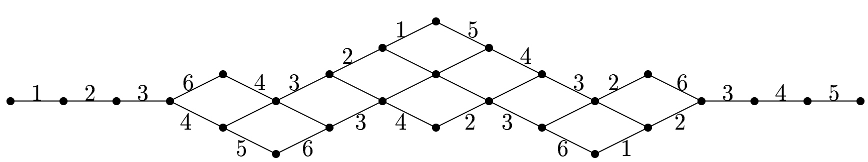
For $I = \{1, \ldots, 5\}$ one has $W_I = A_5$ and the following V^I (reflected at the dotted line):

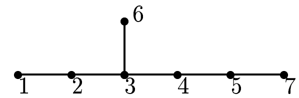


Therefore the sequence of Whitney numbers for V^I is $1_3\ 2\ 3_2\ 4\ 5_3\ 6_2\ 5_3\ 4\ 3_2\ 2\ 1_3$ and hence $P_{E_6}(t) = P_{A_5}(t) \cdot (1 + t^3)(1 + t^4)(1 + t^6)\langle 9\rangle = \langle 2\rangle\langle 3\rangle(1 + t^3)\langle 4\rangle(1 + t^4)\langle 5\rangle\ \langle 6\rangle(1 + t^6)\langle 9\rangle = \langle 2\rangle\langle 6\rangle\langle 8\rangle\langle 5\rangle\langle 12\rangle\langle 9\rangle$. Notice that the LHD of $V^{\{1,2,4,5\}}$ relative to $W = A_5$ is embedded into V^I between the edges '6' at the beginning and at the dotted line.

Alternatively, for $I = \{2, \ldots, 6\}$ one has $W_I = D_5$ and the following V^I (the edge labeling of the 'outer edges' is to be continued to the translates in between):



Therefore the sequence of Whitney numbers for V^I is $1_4\ 2_4\ 3\ 2_4\ 1_4$ and hence $P_{E_6}(t) = P_{D_5}(t) \cdot (1 + t^4 + t^8)\langle 9\rangle = \langle 2\rangle\langle 4\rangle(1 + t^4 + t^8)\langle 5\rangle\langle 6\rangle\langle 8\rangle\langle 9\rangle = \langle 2\rangle\ \langle 12\rangle\langle 5\rangle\langle 6\rangle\langle 8\rangle\langle 9\rangle$.

E₇ has degrees: $2, 6, 8, 10, 12, 14, 18$, and Coxeter graph 

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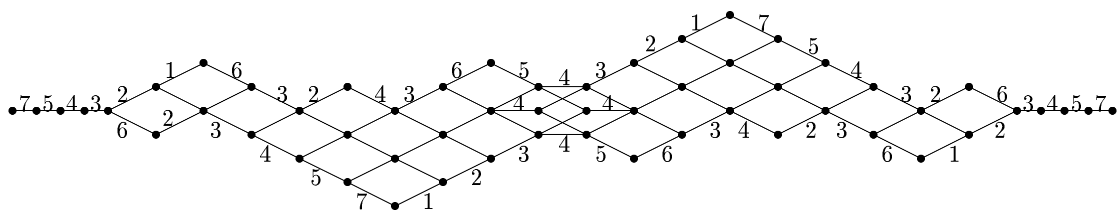
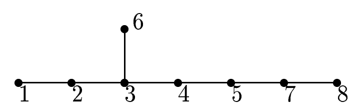
For $I = \{1, \ldots, 6\}$ one has $W_I = E_6$ and the following point symmetric V^I (the edge labeling of the 'outer edges' is to be continued to the translates in between):

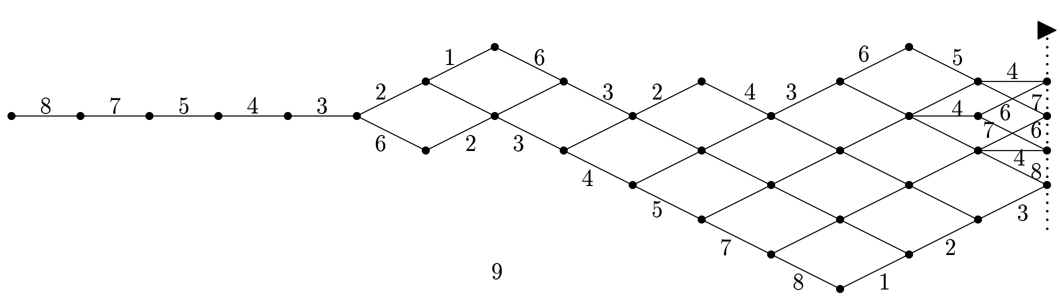


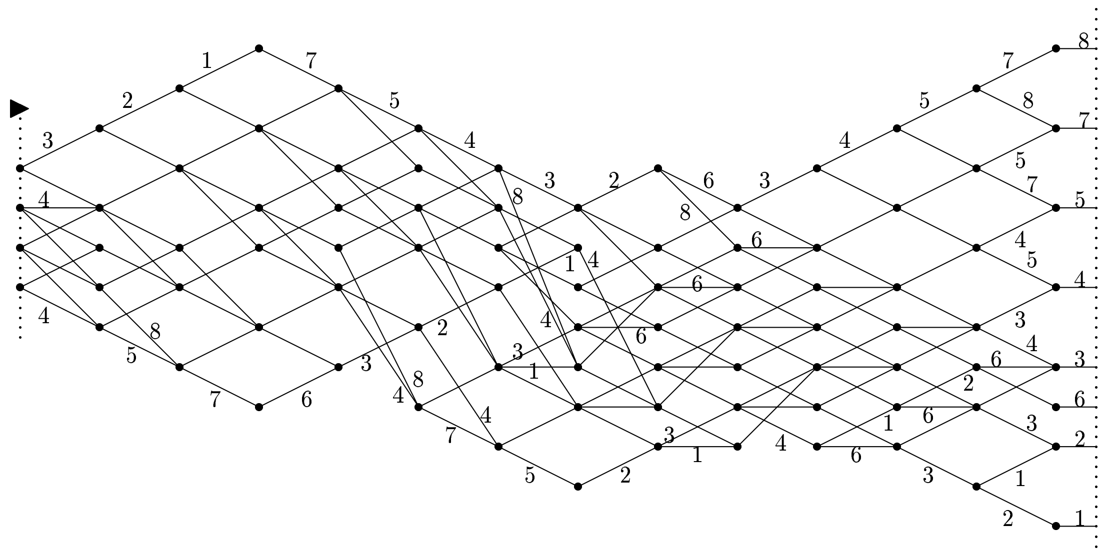
Therefore the sequence of Whitney numbers for V^I is $1_5\ 2_4\ 3_{10}\ 2_4\ 1_5$ and hence $P_{E_7}(t) = P_{E_6}(t) \cdot (1+t^5)(1+t^9)\langle 14\rangle = \langle 2\rangle\langle 5\rangle(1+t^5)\langle 6\rangle\langle 8\rangle\langle 9\rangle(1+t^9)\langle 12\rangle\ \langle 14\rangle = \langle 2\rangle\langle 10\rangle\langle 6\rangle\langle 8\rangle\langle 18\rangle\langle 12\rangle\langle 14\rangle$.

E₈ has degrees: $2, 8, 12, 14, 18, 20, 24, 30$, and Coxeter graph 

.

For $I = \{1, \ldots, 7\}$ one has $W_I = E_7$ and the following V^I (the two parts have to be pasted at the vertical lines marked with ▶, and the resulting LHD has to be reflected at the dotted line; the labels have to be 'continued' to parallel edges in the obvious way):





Therefore the sequence of Whitney numbers for V^I is
$1_6\ 2_4\ 3_2\ 4_4\ 5_2\ 6_4\ 7_6\ 8_2\ 7_6\ 6_4\ 5_2\ 4_4\ 3_2\ 2_4\ 1_6$ and hence $P_{E_8}(t) = P_{E_7}(t)\cdot(1+t^6)(1+t^{10})(1+t^{12})\langle 30\rangle = \langle 2\rangle\langle 6\rangle(1+t^6)\langle 8\rangle\ \langle 10\rangle(1+t^{10})\ \langle 12\rangle(1+t^{12})\langle 14\rangle\langle 18\rangle\langle 30\rangle = \langle 2\rangle\langle 12\rangle\langle 8\rangle\langle 20\rangle\langle 24\rangle\ \langle 14\rangle\langle 18\rangle\langle 30\rangle$.
Notice that the LHD of $V^{\{1,\dots,6\}}$ relative to $W = E_7$ is embedded into V^I between the edges '8' at the beginning and at the dotted line; similarly the LHD of $V^{\{1,\dots,5\}}$ relative to $W = E_6$ is embedded symmetrically w.r.t. the dotted line of reflection.

2. Codes for the Coxeter groups of type A, B, D and standard reduced words for signed and unsigned permutations

Since the poset (partially ordered set) of parabolic subgroups for a given Coxeter system (W, S) ordered by the property 'subgroup' is isomorphic to the Boolean lattice $\mathcal{B}(S)$ of subsets of S ordered by inclusion [Hu, Thm.5.5], it is a natural idea to investigate the meaning of elementary order theoretic notions for $\mathcal{B}(S)$ in terms of parabolic subgroups. In the present section we investigate the notion of a 'chain'.

Our Theorem shows that a step downward in $\mathcal{B}(S)$ induces a partition of LHD(W) into isomorphic posets, which by the results of Section 1 are especially simple in case of the Coxeter families A, B, and D. Therefore we investigate maximal chains in $\mathcal{B}(S)$ for these families yielding codes (=inversion tables) and standard reduced sequences.

2.1. **Case A.** It is well known that the abstract Coxeter groups A_{n-1} can be represented as the symmetric groups S_n of permutations of the *letters* in $\underline{n} := \{1, \dots, n\}$, i.e. every $w \in A_{n-1}$ corresponds to a permutation $\pi \equiv \pi(1)\dots\pi(n) \in S_n$; this π can be computed with the help of the elementary transpositions $\sigma_i = (i, i+1)$ as the product $\pi = \sigma_{a_1}\dots\sigma_{a_l}$ for arbitrary $a \equiv a_1 \dots a_l \in R(w)$, i.e. every generator s_i corresponds to the transposition

σ_i. Clearly $R(\pi)$ can be identified with $R(w)$ under this correspondence and $l = l(\pi)$ is the *length* of the permutation.

For $n \geq 2$ there are two obvious embeddings of the Coxeter graph of A_{n-1} into the Coxeter graph of A_n, which we call the *left* and *right embedding*, respectively. The corresponding left and right embeddings of S_n into S_{n+1} are then given by $\pi \mapsto \pi\ n+1 := \pi(1)\dots\pi(n)\ n+1$ and $\pi \mapsto 1\ 1_+(\pi) := 1\ \pi(1)+1\dots\pi(n)+1$, respectively.

Definition 2.1. Let $A = \{a_1, a_2, \dots\}$ be an alphabet and

$$a = a^1_1 \dots a^1_{i_1} \mid a^2_1 \dots a^2_{i_2} \mid \ \dots \mid a^n_1 \dots a^n_{i_n}$$

an *n-sectioned word* in A with n *sections* indicated by the vertical lines and the superscripts. Let furthermore $\nu = (\nu_1, \dots, \nu_n)$ be an n-tuple of natural numbers such that $0 \leq \nu_j \leq i_j$ for $j = 1, \dots, n$; then a subword b of the n-sectioned word a is called *ν-initial in a* [or *ν-terminal in a*], if it is of the form

$$\begin{aligned} b =& a^1_1 \dots a^1_{\nu_1} \mid a^2_1 \dots a^2_{\nu_2} \mid \ \dots \mid a^n_1 \dots a^n_{\nu_n} \quad \text{or} \\ [\quad b =& a^1_{i_1-\nu_1+1} \dots a^1_{i_1} \mid a^2_{i_2-\nu_2+1} \dots a^2_{i_2} \mid \ \dots \mid a^n_{i_n-\nu_n+1} \dots a^n_{i_n} \quad]\ , \end{aligned}$$

where the subword of section j is empty iff $\nu_j = 0$.

Definition 2.2. Let $Y_n := \{y = [y_{n-1} \dots y_1 y_0] \mid 0 \leq y_j \leq j,\ j = 0, \dots, n-1\ \}$; then for every $\pi \in S_n$ one can compute the following images in Y_n:

$$\begin{aligned} L(\pi)\ &:\quad l_{n-i}\ := \sharp\{\ j \mid j > i,\ \pi j < \pi i\ \}\ , \\ H(\pi)\ &:\quad h_{n-i}\ := \sharp\{\ j \mid j < \pi^{-1} i,\ \pi j > i\ \}\ , \\ K(\pi)\ &:\quad k_{i-1}\ := \sharp\{\ j \mid j < i,\ \pi j > \pi i\ \}\ , \\ G(\pi)\ &:\quad g_{i-1}\ := \sharp\{\ j \mid j > \pi^{-1} i,\ \pi j < i\ \}\ . \end{aligned}$$

We call these mappings *codes* of permutations; $L(\pi)$ is known as the 'Lehmer code' of $\pi \in S_n$, but goes back in fact to Rothe [Ro]. Notice that 'i' in the cases G and H designates a letter and in the cases L and K a place.

It has been shown in [W1, Sec.2] that the mappings L, H, K, G are in fact bijections from S_n to Y_n for all $n \in \mathbb{N}$, which are interrelated as follows:

$$H(\pi) = L(\pi^{-1}), \quad K(\pi) = L(\omega_n \pi \omega_n), \quad G(\pi) = K(\pi^{-1})\ , \tag{2.1}$$

where $\omega_n := n \dots 1$ is the permutation of maximal length in S_n. Moreover:

$$|L(\pi)| = |K(\pi)| = |H(\pi)| = |G(\pi)| = l(\pi) \text{ for all } \pi \in S_n\ . \tag{2.2}$$

Example 2.3. Let $\pi = 3417625 \in S_7$ with $l(\pi) = 9$, then

$$\begin{aligned} L(\pi) &= [2203200] \\ H(\pi) &= [2400210] = L(\pi^{-1}) = L(3612754) \\ K(\pi) &= [2410200] = L(\omega_7 \pi \omega_7) = L(3621745) \\ G(\pi) &= [3202200] = L(\omega_7 \pi^{-1} \omega_7) = L(4316725)\ . \end{aligned}$$

The following proposition shows how the above defined *codes* can be used to compute reduced words for arbitrary finite permutations. Since these reduced words correspond to the natural chains of left or right embeddings of the symmetric groups, we call them *standard reduced words.*

Proposition 2.4. *Let $n \in \mathbb{N}$, $\pi \in S_{n+1}$ (' $n+1$ ' for convenient formulas!), $\pi \equiv \sigma_a$ for $a \in R(\pi)$, $G(\pi)^* := g_1 \dots g_n$, and $H(\pi)^* := h_1 \dots h_n$. Then:*
$\Phi^G G(\pi) := \Phi^G(g_1) \dots \Phi^G(g_n)$ with $\Phi^G(g_j) := j \ \dots \ j+1-g_j$, if $g_j > 0$, and $:= \emptyset$, if $g_j = 0$, is the unique $a \in R(\pi)$, which is initial and necessarily $G(\pi)^*$-initial in

$$a(G,n) \equiv 1 \mid 2\ 1 \mid 3\ 2\ 1 \mid \ \dots \mid n\ \dots 1 \quad ;$$

$\Phi^H H(\pi) := \Phi^H(h_1) \dots \Phi^H(h_n)$ with $\Phi^H(h_j) := n+1-j \ \dots \ n-j+h_j$, if $h_j > 0$, and $:= \emptyset$, if $h_{i-1} = 0$, is the unique $a \in R(\pi)$, which is initial and necessarily $H(\pi)^*$-initial in

$$a(H,n) \equiv n \mid n-1\ n \mid \ \dots \mid 1\ \dots n \quad ;$$

$\Phi^L L(\pi) := \Phi^L(l_n) \dots \Phi^L(l_1)$ with $\Phi^L(l_j) := n-j+l_j \ \dots \ n+1-j$, if $l_j > 0$, and $:= \emptyset$, if $l_j = 0$, is the unique $a \in R(\pi)$, which is terminal and necessarily $L(\pi)$-terminal in

$$a(L,n) \equiv n\ \dots 1 \mid n\ \dots 2 \mid \ \dots \mid n\ n-1 \mid n \quad ;$$

$\Phi^K K(\pi) := \Phi^K(k_n) \dots \Phi^K(k_1)$ with $\Phi^K(k_j) := j+1-k_j \ \dots \ j$, if $k_j > 0$, and $:= \emptyset$, if $k_j = 0$, is the unique $a \in R(\pi)$, which is terminal and necessarily $K(\pi)$-terminal in

$$a(K,n) \equiv 1\ \dots n \mid 1\ \dots n-1 \mid \ \dots \mid 1\ 2 \mid 1 \quad .$$

The proof is not hard and is omitted.

Example 2.5. Consider again $\pi = 3417625 \in S_7$ from Ex.2.3 above (for clarity we have included the vertical sectioning bars):

$$\begin{aligned}
\Phi^G G(\pi) = \quad & \Phi^G[3202200] & = \quad & |21|32||54|654\ , \\
\Phi^H H(\pi) = \quad & \Phi^H[2400210] & = \quad & 6|56|||2345|12\ , \\
\Phi^L L(\pi) = \quad & \Phi^L[2203200] & = \quad & 21|32||654|65|\ , \\
\Phi^K K(\pi) = \quad & \Phi^K[2410200] & = \quad & 56|2345|4||12|\ .
\end{aligned}$$

Remark 2.6. Proposition 2.4 can be interpreted in two ways: the obvious one is that it shows different possibilities to compute in a simple way reduced words for a given permutation; the other one is that it enables the easy reconstruction of the permutation from one of its codes, the latter appearing naturally in the context of Schubert polynomials (see [W1] and [W2, Sec.5]).

2.2. **Case B.** It is well known that the abstract Coxeter groups B_{n-1} can be represented as groups $S^B := \mathbb{Z}_2 \wr_{\underline{n}} S_n$ of *signed permutations*, i.e. $\bar{\pi} \in S^B$ is represented as $\bar{\pi} \equiv (\pi, u)$ with $\pi \in S_n$, and $u \in \mathbb{Z}_2^n := ((\mathbb{Z}/2\mathbb{Z})^n, +)$. The generators $s_1, s_2, \ldots$ are again represented by the elementary transpositions $\sigma_1, \sigma_2, \ldots$ and s_0 by the operator σ_0 changing the sign at place 1. The group multiplication is defined by $\bar{\pi} \circ \bar{\mu} \equiv (\pi, u) \circ (\mu, v) := (\pi \circ \mu, u + \pi(v))$ with $\pi(v) := (w_{\pi(1)}, \ldots, w_{\pi(n)})$; for example: $\overline{13}2 \circ 31\overline{2} \equiv (132, (1,1,0)) \circ (312, (0,0,1)) = (213, (1,1,0) + (0,1,0)) = (213, (1,0,0)) = (\overline{2}13)$. The inverse $\bar{\pi}^{-1}$ of some $\bar{\pi} \equiv (\pi, u)$ is clearly $(\pi^{-1}, \pi^{-1}(v))$.

In the case of B there is only one possibility to embed the Coxeter graph of B_{n-1} into the Coxeter graph of B_n, namely the 'left' embedding. This restricts the possible codes and standard reduced words to the analogs of cases G and K. We discuss only case G:

Definition 2.7. Let $Y^B := \{\bar{y} = \bar{y}_{n-1} \ldots \bar{y}_0 \mid 0 \leq \bar{y}_j \leq 2j+1,\ j = 0, \ldots, n-1\ \}$; then for every $\bar{\pi} \equiv (\pi, u) \in S^B$ one can compute the following images in Y^B (note that i a letter):

$$\bar{g}_{i-1} := \begin{cases} \sharp\{\ j \mid j > \pi^{-1}i,\ \pi j < i\ \} & \text{if } u(\pi^{-1}i) = 0, \\ i + \sharp\{\ j \mid j < \pi^{-1}i,\ \pi j < i\ \} & \text{if } u(\pi^{-1}i) = 1. \end{cases}$$

We call this mapping the *G-code* of signed permutations.

Proposition 2.8. *Let $n \in \mathbb{N}$, $\bar{\pi} \in S_{n+1}^B$, $\bar{\pi} \equiv \sigma_a$ for $a \in R(\bar{\pi})$, and $G^B(\bar{\pi})^* := \bar{g}_0 \ldots \bar{g}_n$. Then:*
$\bar{\Phi}^G G^B(\bar{\pi}) := \bar{\Phi}^G(\bar{g}_0) \ldots \bar{\Phi}^G(\bar{g}_n)$ with

$$\bar{\Phi}^G(\bar{g}_j) := \begin{cases} \emptyset & \text{if } g_j = 0, \\ j\ \ldots\ j+1-g_j & \text{if } 0 < g_j \leq j, \\ j\ \ldots\ 1\ 0\ \ldots\ g_j - j - 1 & \text{if } j < g_j \end{cases}$$

is the unique $a \in R(\bar{\pi})$, which is initial and necessarily $G^B(\bar{\pi})^*$-initial in

$$a(G^B, n) \equiv 0 \mid 1\ 0\ 1 \mid 2\ 1\ 0\ 1\ 2 \mid \ \ldots \mid n\ \ldots 1\ 0\ 1\ \ldots\ n \quad .$$

Proof. Using Theorem 0.1 and the results of Section 1 for the sequence of left embeddings $S_1^B \hookrightarrow S_2^B \hookrightarrow \cdots \hookrightarrow S_{n+1}^B$, it is immediate that every $\bar{\pi} \in S_{n+1}^B$ has a unique representation $\bar{\pi} = \sigma_a$ with a initial in $0 \mid 1\ 0\ 1 \mid 2\ 1\ 0\ 1\ 2 \mid \ \ldots \mid n\ \ldots 1\ 0\ 1\ \ldots\ n$.

This initial sequence is in deed $G^B(\bar{\pi})^*$-initial: assume that the letters $1, \ldots, k-1$ are placed (with their respective bars) as in $\bar{\pi}$, then the next step is to move the letter k from place k, such that the letters $1, \ldots, k$ are placed (and 'bared') as in $\bar{\pi}$. If k has no bar in $\bar{\pi}$ it moves at most $k-1$ places to the left as in the unsigned case; if k has a bar in $\bar{\pi}$ it first moves to place 1 and 'gets its bar' by application of σ_0, then it moves right to its place in $\bar{\pi}|_{\{1,\ldots,k\}}$. □

Corollary 2.9. *The mapping G^B is a bijection from S^B to Y_n^B for all $n \in \mathbb{N}$ with the property*

$$|G^B(\bar{\pi})| = l(\bar{\pi}) \text{ for all } \bar{\pi} \in S^B \ . \tag{2.3}$$

Example 2.10. Let $\bar{\pi} = 35\bar{1}2\bar{4} \in S_5^B$ with $l(\pi) = 13$, then $G^B(\bar{\pi}) = \overline{37201}$ and $\bar{\Phi}^G\overline{37201} = 0 \mid\mid 21 \mid 3210123 \mid 432$.

2.3. **Case D.** It is well known that the abstract Coxeter groups D_{n-1} can be represented as subgroups S_n^D of index two of the S^B consisting of the signed permutations with an even number of bars, i.e. $\bar{\bar{\pi}} \equiv (\pi, u)$ and $|u| = u(1) + \cdots + u(n) \equiv 0 \pmod 2$. The generators $s_1, s_2, \dots$ are again represented by the elementary transpositions $\sigma_1, \sigma_2, \dots,$ and $s_{1'}$ by the operator $\sigma_{1'}$ changing simultaneously the numbers at places 1 and 2 and their signs, i.e. "$\sigma_{1'} = \sigma_0\sigma_1\sigma_0$". Again from the two possible cases we discuss only the case of G:

Definition 2.11. Let $Y_n^D := \{\bar{\bar{y}} = \bar{\bar{y}}_{n-1} \dots \bar{\bar{y}}_0 \mid 0 \le \bar{\bar{y}}_j \le 2j,\ j = 0, \dots, n-1, \text{ or } \bar{\bar{y}}_j = j' \}$ $(0' := 0)$; then for every $\bar{\bar{\pi}} \equiv (\pi, u) \in S_n^D$ the element $G^D(\bar{\bar{\pi}}) \in Y_n^D$ is given by the prescription (note that i a letter):

$$\bar{\bar{g}}_{i-1} := \begin{cases} \sharp\{\ j \mid j > \pi^{-1}i,\ \pi j < i\ \} & \text{if } i \notin E(\pi),\ u(\pi^{-1}i) = 0, \\ i - 1 + \sharp\{\ j \mid j < \pi^{-1}i,\ \pi j < i\ \} & \text{if } i \notin E(\pi),\ u(\pi^{-1}i) = 1, \\ (i-1) & \text{if } i \in E(\pi),\ D(i) \text{ even}, \\ (i-1)' & \text{if } i \in E(\pi),\ D(i) \text{ odd}, \end{cases}$$

where $E(\pi) := \{i \mid \neg\exists j :\ j < \pi^{-1}i,\ \pi j < i\}$ and $D(i) \equiv D(\bar{\bar{\pi}}, i) = \sum_{j \ge i} u(\pi^{-1}j)$.

The mapping is called the G-*code* of the signed permutations in S_n^D.

Proposition 2.12. *Let $n \in \mathbb{N}$, $\bar{\bar{\pi}} \in S_{n+1}^D$, $\bar{\bar{\pi}} \equiv \sigma_a$ for $a \in R(\bar{\bar{\pi}})$, and $G^D(\bar{\bar{\pi}})^* := \bar{\bar{g}}_1 \dots \bar{\bar{g}}_n$. Then:*
$\bar{\bar{\Phi}}^G G^D(\bar{\bar{\pi}}) := \bar{\bar{\Phi}}^G(\bar{\bar{g}}_1) \dots \bar{\bar{\Phi}}^G(\bar{\bar{g}}_n)$ *with*

$$\bar{\bar{\Phi}}^G(\bar{\bar{g}}_j) := \begin{cases} \emptyset & \textit{if } g_j = 0, \\ j\ \dots\ j+1-g_j & \textit{if } 0 < g_j < j, \\ j\ \dots\ 1 & \textit{if } g_j = j, \\ j\ \dots\ 2\ 1' & \textit{if } g_j = j', \\ j\ \dots\ 2\ 1'\ 1\ \dots\ g_j - j & \textit{if } j < g_j \end{cases}$$

is the unique $a \in R(\bar{\bar{\pi}})$, which is initial and necessarily $G^D(\bar{\bar{\pi}})^*$-initial in

$$a(G^D, n) \equiv (1\ 1') \mid 2\ (1\ 1')\ 2 \mid 3\ 2\ (1\ 1')\ 2\ 3 \mid \ \dots \mid n\ \dots 2\ (1\ 1')\ 2\ \dots\ n \quad ,$$

where the notation $(1\ 1')$ indicates that the numbers 1 and $1'$ commute.

Proof. First notice that in case of $i \notin E(\pi)$ the letter i behaves essentially as in case B above; the only difference is that $1\ 0\ 1$ is replaced by the shorter $(1\ 1')$. Let $i_s\ \dots\ i_1$ be the subword of $\bar{\bar{\pi}}$ consisting of the elements of $E(\pi)$; then necessarily $i_s > \cdots > i_1 = 1$ and $\pi^{-1}(i_s) = 1$. Assume that

$$\bar{\bar{\pi}}^{(i_s)} := (\pi|_{\{1,\dots,i_s\}}\quad i_s + 1\ \dots\ n+1\ ,\quad u^{(i_s)})$$

(where $\pi|_M$ is the subword of π containing exactly the letters from the set M) has already been constructed, then every time one of the numbers $i_s+1,\dots,n+1$ 'gets its bar' the sign of i_s changes so that for $\bar{\bar{\pi}}=(\pi,u)$ one has

$$u(1)=u(\pi^{-1}(i_s))\equiv u^{(i_s)}(1)+u(\pi^{-1}(i_s+1))+\cdots+u(\pi^{-1}(n+1))\pmod 2$$

or $u^{(i_s)}(1)\equiv D(i_s)\pmod 2$. Now let $1\le\nu<s$, and $\bar{\bar{\pi}}^{(i_\nu)}:=(\pi|_{\{1,\dots,i_\nu\}}\quad i_\nu+1\ \dots\ n+1\ ,\quad u^{(i_\nu)})$; the construction of $\bar{\bar{\pi}}^{(i_{\nu+1})}$ from $\bar{\bar{\pi}}^{(i_\nu)}$ then yields the congruence

$$u^{(i_{\nu+1})}(1)+u(\pi^{-1}(i_\nu))\equiv u^{(i_\nu)}(1)+u(\pi^{-1}(i_\nu+1))+\cdots+u(\pi^{-1}(i_{\nu+1}-1))\pmod 2$$

or by induction hypothesis

$$u^{(i_\nu)}(1)\equiv u(\pi^{-1}(i_\nu))+\cdots+u(\pi^{-1}(i_{\nu+1}-1))+u^{(i_{\nu+1})}(1)\equiv D(i_\nu)\pmod 2$$

as desired. $\square$

Corollary 2.13. *The mapping G^D is a bijection from S_n^D to Y_n^D for all $n\in\mathbb{N}$ with the property*

$$|G^D(\bar{\bar{\pi}})|=l(\bar{\bar{\pi}})\text{ for all }\bar{\bar{\pi}}\in S_n^D\ . \tag{2.4}$$

Example 2.14. Let $\bar{\bar{\pi}}=35\overline{1}2\overline{4}\in S_5^D$ with $l(\pi)=11$, then $G^D(\bar{\bar{\pi}})=\overline{\overline{362'00}}$ and $\bar{\bar{\Phi}}^G\,\overline{\overline{362'00}}=\ ||\ 21'\ |\ 321'123\ |\ 432$.

3. Some further directions

The Coxeter graphs of the irreducible affine Coxeter groups $\widetilde{W}$ are found by adding exactly one new generator '0' to the crystallographic irreducible finite Coxeter groups W and their graphs (cf. [Hu, Sec.4.7]). Therefore it is natural to view W as the parabolic subgroup W_I of $\widetilde{W}$ for the set $I=\{$ all generators except 0 $\}$. Bott's Theorem on the Poincaré series of $\widetilde{W}$ (cf. [Hu, Sec.8.9], [Hi, Sec.5.6]) then says that

$$P_{\widetilde{W}}(t)=P_W(t)\prod\frac{1}{1-t^{d_i-1}}\ ,$$

where $P_W(t)$ and the d_i's are as in Section 1. Though it does not seem possible to give a general elementary proof of Bott's Theorem in the style of Section 1, it is possible (according to Remark 0.6) to depict LHD(V^I) as planar labeled graphs in case of the affine Coxeter groups on at most 3 generators: $\widetilde{A_1}$, $\widetilde{A_2}$, $\widetilde{B_2}=\widetilde{C_2}$, and $\widetilde{G_2}=\widetilde{I_2(6)}$. The posets V^I in these cases are all of the following form: take (semi)infinite chains of edges – corresponding to the factor $(1-t)^{-1}$ – and join them appropriately, such that for $d=\min\{d_i>2\}$ there starts a new chain on all levels $(d-1)k,\ k\in N_0$ – giving a factor $(1-t^{d-1})^{-1}$. By Bott's Theorem this approach should yield pictures of 'generic 2-dim. slices' of the V^I's in the remaining affine cases, too; it is also interesting to choose $d\in\{d_i>2\}$ arbitrarily.

The combinatorial study of the affine Coxeter groups of type A is the subject of a recent paper by A. Björner and F. Brenti [BjB], in which the list representation of permutations, inversion tables, and other combinatorial notions of the finite case are generalized. It should be possible to extend the results of Section 2.1 to these affine permutations.

Hyperbolic Coxeter groups (cf. [Hu, Sec.6.7-8]) have the property that the Coxeter graphs obtained by the deletion of any vertex are of finite type; moreover every Coxeter group on three generators not being finite or affine is hyperbolic and admits a planar picture of LHD($V^{\{1,2\}}$), which of course is not as nicely representable as in the affine cases above (compare the book [R], Chapter 7), but it should be possible to find alternative ways to depict their labeled Hasse diagrams.

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