ON THE MULTIPLICATION OF SCHUBERT POLYNOMIALS

RUDOLF WINKEL*

ABSTRACT: Finding a combinatorial rule for the multiplication of Schubert polynomials is a long standing problem. In this paper we give a combinatorial proof of the extended Pieri rule as conjectured by N. Bergeron and S. Billey, which says how to multiply a Schubert polynomial by a complete or elementary symmetric polynomial, and describe some observations in the direction of a general rule.

To every finite permutation π contained in one of the symmetric groups S_n there is associated a Schubert polynomial $X_\pi \in \mathbb{Z}[x] \equiv \mathbb{Z}[x_1, x_2, x_3, \ldots]$, which is invariant under the natural embedding $S_n \hookrightarrow S_{n+1}$ given by $\pi \equiv \pi(1) \ldots \pi(n) \mapsto \pi(1) \ldots \pi(n)\ (n{+}1)$. We write S_∞ for the direct limit of all the S_n under this inclusion so that e.g. $\pi = 231 = 2314 = 23145 = \ldots \in S_\infty$. The multiplication of Schubert polynomials is well known to represent faithfully the 'Schubert calculus' of cohomology classes of Schubert varieties of flag manifolds with respect to the cup-product, i.e. intersection. Since the set of all Schubert polynomials forms a $\mathbb{Z}$-basis of the algebra $\mathbb{Z}[x]$ this geometric interpretation shows that the structure constants $c_{\pi\pi'}^{\pi''}$ occurring in

$$X_\pi X_{\pi'} = \sum_{\pi''} c_{\pi\pi'}^{\pi''} X_{\pi''} \tag{0.1}$$

are all non-negative integers. This suggests the possibility of a combinatorial rule for the determination of these constants, but over the years only little progress has been made in solving this problem, which quite aptly can be called the most important open problem in the "elementary" theory of Schubert polynomials. (It is of course possible to compute the product $X_\pi X_{\pi'}$ explicitly and expand it afterwards using the property that every Schubert polynomial has the form $X_\pi = x^{L(\pi)} + \ldots$, where the Lehmer code $L(\pi)$ of π (see below) is the smallest exponent with respect to the lexicographic order induced by $0 < 1 < 2 < \ldots$.)

The existence of Schubert polynomials and many of their properties have been established in a sequence of papers by A. Borel (1953), I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand (1973), M. Demazure (1973-74), and finally A. Lascoux and M.-P. Schützenberger (1982-87). For comprehensive accounts on the geometry and combinatorics of Schubert polynomials see [Hi] and [LS, M1, M2, W]. We expose here only those parts of the theory, which are strictly necessary for our presentation.

For $1 \le k < l$ let $\sigma_{kl} \equiv (k, l)$ be the transposition interchanging k and l, and $\sigma_k := (k, k{+}1)$. The complete and elementary symmetric functions in m variables of degree N are denoted by $h_N^{(m)} = h_N(x_1, \ldots, x_m)$ and $e_N^{(m)} = e_N(x_1, \ldots, x_m)$, respectively. The Schubert polynomial X_{σ_k} is then given by $X_{\sigma_k} = x_1 + \cdots + x_k = h_1^{(k)} = e_1^{(k)}$. Monk has established in [Mo] a

Date: Oktober 1996, revised January 1997.

1991 *Mathematics Subject Classification.* 14M15, 05E15.

* Supported by Deutsche Forschungsgemeinschaft (DFG).

combinatorial rule for the multiplication of a Schubert polynomial by a X_{σ_k}, which we will describe next. (In fact Chevalley [C] has shown a more general result probably earlier, but his paper was unpublished until recently.)

For a given $\pi \in S_\infty$ and $1 \le k < l$ we write

$$(0.2) \qquad \pi(k) \lhd \pi(l) :\Longleftrightarrow \pi(k) < \pi(l) \text{ and } \sharp\{\nu \mid k < \nu < l,\ \pi(k) < \pi(\nu) < \pi(l)\} = 0 \ .$$

It is well known (see e.g. [W, Prop.2.3]) that

$$(0.3) \qquad l(\pi\sigma_{kl}) = l(\pi) + 1 \Longleftrightarrow \pi(k) \lhd \pi(l) \ ,$$

where $l(\pi)$ is the length of the permutation π. The notation

$$(0.4) \qquad \pi(k) \overset{b}{\lhd} \pi(l) :\Longleftrightarrow \pi(k) \lhd \pi(l) \text{ and } 1 \le k \le b < l$$

is helpful in defining the (π, b)-*admissible set*:

$$(0.5) \qquad J(\pi, b) := \{(k,l) \mid \pi(k) \overset{b}{\lhd} \pi(l)\} \ .$$

Theorem 0.1. (Monk) *With the above notations one has for every $\pi \in S_\infty$ and $b \in \mathbb{N}$:*

$$(0.6) \qquad X_{\sigma_b} X_\pi = \sum_{(k,l)\, \in J(\pi,b)} X_{\pi\,(k,l)} \ .$$

Simple proofs can be found in [M2, W]. Notice that '$\lhd$' is the covering relation for the Bruhat order on S_∞ and '$\overset{b}{\lhd}$' for the b-Bruhat order, so that Monk's rule can be rephrased by saying: "The product of X_π with X_{σ_b} is the sum over all Schubert polynomials indexed by all permutations covering π in b-Bruhat order."

Example 0.2. Let $\pi = 25413$ and $b = 3$; in other words; we want to compute $X_{\sigma_3} X_{25413}$. Since 35412, 264135, and 256134 cover 25413 in 3-Bruhat order the result is $X_{35412} + X_{264135} + X_{256134}$.

Only recently N. Bergeron and S. Billey [BB] have, unaware of the fact that A. Lascoux and M.-P. Schützenberger had stated an equivalent rule in a different form in [LS2], formulated a conjecture (described in Section 1 below), which extends the Monk's rule from the case $h_1^{(m)} = e_1^{(m)}$ to the cases of arbitrary complete and elementary symmetric polynomials $h_N^{(m)}$ and $e_N^{(m)}$. These conjectures have been proven by F. Sottile [So] using an explicit geometric description of certain intersections of Schubert varieties. We will instead give in Section 1 a combinatorial proof of these facts departing directly from Monk's rule and using only manipulations of permutations. Despite its conceptual simplicity the checking of the details is quite tedious and tricky – an appearend feature already of Sottile's geometric proof.

In the subsequent sections we describe some observations and "approximative rules" of increasing generality, how to multiply a Schubert polynomial by an Schur polynomial (Sec.2), by a Schubert polynomial associated to a L-unimodal permutation (Sec.3), and by a general Schubert polynomial (Sec.4). We speak of "approximative rules", because these observations give the correct results for many examples, but are not as yet reliable rules or even steady conjectures. The appearent difficulty of the problem may serve as an excuse for including this increasingly more speculative material.

The 'L' in L-unimodal stands for the Lehmer code $L(\pi)$ of a permutation $\pi \in S_n$, i.e.

$$L(\pi) \in \{\ \overline{l_{n-1}, \ldots, l_0} \mid 0 \le l_{n-\nu} \le n - \nu,\ \nu = 1, \ldots, n\ \} \ ,$$

where $l_{n-\nu}(\pi) := \sharp\{\ j \mid \nu < j,\ \pi(\nu) > \pi(j)\}$ for all $\nu \in \{1, \ldots, n\}$, e.g. $L(361542) = \overline{240210}$ or $L(1257346) = \overline{0023000}$. It is not hard to see that π can be reconstructed from its Lehmer code so that L can be regarded as a bijection between permutations and codes.

A permutation π is called *Grassmannian* iff there is a partition $\lambda \equiv \lambda_1 \ldots\ \lambda_s$ ($\lambda_1 \geq \ldots \geq \lambda_s$) and a natural number $m \geq l(\lambda) = s$ such that $L(\pi) = \overline{0 \ldots\ 0\ \lambda_s\ \ldots\ \lambda_1\ 0 \ldots\ 0}$ with $m - s \geq 0$ zeros on the left and (at least) λ_1 zeros on the right. An alternative definition is: π is called Grassmannian iff π has a at most one descent, i.e. there is at most one i with $\pi(i) > \pi(i+1)$. Anyway

$$\pi(\lambda, m) := L^{-1}(\overline{\underbrace{0 \ldots\ 0\ \lambda_s\ \ldots\ \lambda_1}_{m}\ 0 \ldots\ 0})\ . \tag{0.7}$$

Then

$$X_{\pi(\lambda,m)} = s_\lambda^{(m)}(x)\ ; \tag{0.8}$$

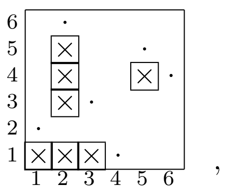
in other words: a Schubert polynomial X_π is a Schur polynomial exactly when π is Grassmannian. A permutation is called *dominant* iff its Lehmer code is non-increasing; then $X_\pi = x^{L(\pi)} = x_1^{l_{n-1}(\pi)} x_2^{l_{n-2}(\pi)} \ldots$. (For proofs of the above facts see [M2] or [W]).

Obviously both Grassmannian and dominant permutations are special cases of *L-unimodal* permutations π, which are permutations having an unimodal Lehmer code: $l_{n-1} \leq \cdots \leq l_{n-\nu} \geq \cdots \geq l_0$ for some ν.

The general case, where the permutation π is not necessarily L-unimodal, is treated in Section 4. Hereby and already in Sections 2 and 3 we rely strongly upon the notion of the *diagram* $D(\pi)$ of a permutation $\pi \in S_n$, which is a subset of an $n \times n$-array of unit squares or boxes in the plane: $D(\pi) \subset \{[i,j] \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i,j \leq n\}$ such that $D(\pi)$ originates from $\{[i,j] \mid 1 \leq i,j \leq n\}$ by cancelation of the 'hooks' of boxes ($i = 1, \ldots, n$):

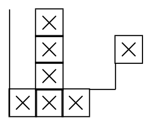
$$\{[i', \pi(i)] \mid i' \geq i\} \cup \{[i, j'] \mid j' \geq \pi(i)\}\ .$$

For example $\pi = 263154$ has the diagram



,

where we have added dots in the positions $(i, \pi(i))$ indicating the 'corners' of the hooks removed. We will use the notation (i,j) for the position 'column i, row j' and $[i,j]$ for the box in position (i,j). Note that the number of boxes in column i of $D(\pi)$ is equal to the i-th entry of $L(\pi)$. Subsequently we will always use the convention that empty rows are removed from the diagram $D(\pi)$ and we reduce the $n \times n$ frame box to some lines, which indicate the left border and the relative positions of boxes; e.g.

$D(263154) =$ 

Moreover the observations in Sec.3 and 4 will be based on the notion of a *component* of a diagram $D(\pi)$: two boxes in $D(\pi)$ are said to be adjacent, if they share a common edge in $D(\pi)$ (without empty rows !); then a component of $D(\pi)$ is defined as an element of the partition of boxes of $D(\pi)$ induced by adjacency as equivalence relation. It is not hard to see (cf. Lem.3.1) that $D(\pi)$ contains exactly one component iff π is L-unimodal; and in Sec.4 we will argue that the 'interaction' of several components in $D(\pi)$ is the cause, which makes it difficult to find a *simple* combinatorial rule for the multiplication of Schubert polynomials.

Finally we remark that the examples in the present paper (and many more not included here) are checked with the help of the package SYMMETRICA maintained at the University of Bayreuth, Germany.

1. The extended Pieri rule

In [BB] N. Bergeron and S. Billey conjectured a combinatorial rule, which says how to multiply a Schubert polynomial by a complete symmetric or elementary symmetric polynomial. For the description and proof of this rule we have to extend the notations (0.2-5) given in the introduction:

$$X_m := X_{\sigma_m} \ , \tag{1.1}$$

$$(k,l;N) := ((k_1,l_1),\ldots,(k_N,l_N)) \text{ with } 1 \le k_i < l_i,\ i=1,\ldots,N \ , \tag{1.2}$$

$$\pi^{(i)} \equiv \pi^{(i)}(k,l;N) := \pi(k_1,l_1)\ldots(k_i,l_i) \text{ for given } \pi\in S_\infty,\ 0\le i\le N\ (\pi^{(0)} := \pi), \tag{1.3}$$

$$\beta := (\beta_1,\ldots,\beta_N) \text{ with } \beta_i\in\mathbb{N},\ i=1,\ldots,N \ , \tag{1.4}$$

$$J(\pi,\beta) := \{(k,l;N) \mid (k_i,l_i)\in J(\pi^{(i-1)},\beta_i) \text{ for } i=1,\ldots,N\} \ , \tag{1.5}$$

$$J(\lambda,\pi,\beta) := \{(k,l;N)\in J(\pi,\beta) \mid \ \lambda\text{-specific condition } \} \ , \tag{1.6}$$

where the 'λ-specific condition' is

$$1 \le \pi^{(i)}(k_i) < \pi^{(i+1)}(k_{i+1}) \text{ for } i=1,\ldots,N-1, \text{ if } \lambda = N \ , \tag{1.7}$$

$$\pi^{(i)}(k_i) > \pi^{(i+1)}(k_{i+1}) \ge 1, \text{ for } i=1,\ldots,N-1, \text{ if } \lambda = 1^N \ . \tag{1.8}$$

(1.1) is an abbreviation, (1.2) the notation for an arbitrary sequence of length N of transpositions, (1.3-4) select more specific sequences in the spirit of (0.4), and the sets $J(\pi,\beta)$ and $J(\lambda,\pi,\beta)$ of (1.5-6) are of course called the (π,β)- and (λ,π,β)-admissible sets in generalization of (0.5).

Theorem 1.1. *Let $\pi\in S_\infty$, $m,N\in\mathbb{N}$, $\beta=(\beta_1,\ldots,\beta_N)=(m,\ldots,m)\equiv m^N$, and $\lambda\in\{N,1^N\}$ ($m\ge N$ for $\lambda=1^N$). Then with the above notations:*

$$X_{\pi(\lambda,m)}X_\pi = \sum_{(k,l;N)\in J(\lambda,\pi,\beta)} X_{\pi^{(N)}} \ . \tag{1.9}$$

Proof. [1] We concentrate on the 'complete symmetric' case, because the 'elementary symmetric' case is analogous using (1.8) instead of (1.7); alternatively and more conveniently one can

[1] I am endebted to Sara Billey (MIT) for many suggestions, which helped to improve the presentation of the proof.

argue with 'duality' as in [So, Lem.2]. For $h_N^{(m)} = h_N(x_1, \dots, x_m) = X_{\pi(N,m)}$ one has the simple recursion

$$h_N^{(m)} = h_N^{(m-1)} + x_m h_{N-1}^{(m)} = h_N^{(m-1)} + (X_m - X_{m-1}) h_{N-1}^{(m)} \ , \tag{1.10}$$

which is of course equivalent to

$$X_{m-1} h_{N-1}^{(m)} - h_N^{(m-1)} = X_m h_{N-1}^{(m)} - h_N^{(m)} \ . \tag{1.11}$$

The latter formula (1.11) is the "best" version of several algebraically equivalent ones as we will see shortly.

The proof proceeds by simultaneous induction over m and N: for $m \in \mathbb{N}$ and $N = 1$ equation (1.9) is Monk's rule and for $m = 1$ and $N \in \mathbb{N}$ one computes by repeated application of Monk's rule

$$x_1^N X_\pi = \sum x_1^{N-1} X_{\pi(1,l_1)} = \dots = \sum X_{\pi(1,l_1)\dots(1,l_N)} \ ,$$

where the sums are taken according to Monk's rule and $l_1, \dots, l_N$ are sequences of natural numbers (depending on π) with the property $2 \le l_1 < \dots < l_N$. But because

$$\pi^{(i)}(k_i) = \pi^{(i)}(1) \lhd \pi^{(i)}(l_i) = \pi^{(i+1)}(1) = \pi^{(i+1)}(k_{i+1})$$

one sees that the N-tuples $((1,l_1), \dots, (1,l_N))$ of transpositions fulfill the condition (1.7), which establishes (1.9) in this case.

We assume now that (1.9) is true for the multiplication by $h_{N-1}^{(m)}$ and $h_N^{(m-1)}$ and use the notation

$$(k_N, l_N) \overset{b}{\rightharpoondown} J(N-1, \pi, m^{N-1}) \tag{1.12}$$

to indicate the concatenation of the pair (k_N, l_N) with the property $\pi^{(N-1)}(k_N) \overset{b}{\lhd} \pi^{(N-1)}(l_N)$ to the $(N-1)$-tuples $(k, l; N-1) \in J(N-1, \pi, m^{N-1})$. A useful and necessary notation is

$$\Pi S := \{\pi^{(N)} \mid (k, l; N) \in S\} \text{ for } S = J(\pi, \beta), J(\lambda, \pi, \beta), \text{ etc. } . \tag{1.13}$$

for the multiset of 'final' permutations $\pi^{(N)}$ originating from some π by a sequence of N transpositions. The main reason for introducing this notation is that we have to speak of a multiset of 'final' permutations independently of the chain of steps through which these are reached.

Taking into account the induction hypothesis, Monk's rule, equation (1.11), and Lemma 1.2 below we are left to prove the following equality of sets:

$$\begin{aligned} \Pi\{(k, l; N) \mid (k_N, l_N) \overset{m-1}{\rightharpoondown} J(N-1, \pi, m^{N-1})\} \setminus \Pi J(N, \pi, (m-1)^N) = \\ \Pi\{(p, q; N) \mid (p_N, q_N) \overset{m}{\rightharpoondown} J(N-1, \pi, m^{N-1})\} \setminus \Pi J(N, \pi, m^N) \ . \end{aligned} \tag{1.14}$$

Lemma 1.2 below shows that (1.11) and (1.14) are equivalent: we have no "negative sets".

The proof of Lemma 1.2 below shows that the r.h.s. of (1.14) equals

$$\Pi\{(p, q; N) \mid (p_N, q_N) \overset{m}{\rightharpoondown} J(N-1, \pi, m^{N-1}),\ \pi^{(N-1)}(p_{N-1}) > \pi^{(N-1)}(q_N)\} \ ,$$

where the inequality is strict, because $p_{N-1} \le m < q_N$. Since the two sets in (1.14) are different only by virtue of the step N, we can collect our knowledge for the two pairs (p_N, q_N)

and (k_N, l_N) as follows:

(1.15) r.h.s. (1.14): $\pi^{(N-1)}(p_N) \overset{m}{\triangleleft} \pi^{(N-1)}(q_N) \qquad \pi^{(N-1)}(p_{N-1}) > \pi^{(N)}(p_N)$

(1.16) l.h.s. (1.14): $\pi^{(N-1)}(k_N) \overset{m-1}{\triangleleft} \pi^{(N-1)}(l_N) \qquad \pi^{(N-1)}(k_{N-1}) \geq \pi^{(N)}(k_N)$,

where the last assertion follows, because we remove permutations related to the condition $\pi^{(N-1)}(k_{N-1}) < \pi^{(N)}(k_N)$.

We investigate conditions (1.15) and (1.16) more closely: in case of $p_N, l_N \neq m$ (and $\pi^{(N-1)}(k_{N-1}) > \pi^{(N)}(k_N)$) the two conditions are clearly equivalent (observe in particular that $\pi^{(N-1)}(k_{N-1}) > \pi^{(N-1)}(l_N)$, because $k_{N-1} \leq m < l_N$). Therefore the problematic cases are $l_N = m$ (together with $\pi^{(N-1)}(k_{N-1}) = \pi^{(N)}(k_N)$) and $p_N = m$. The remaining proof has two parts: we show first that every $\pi^{(N)}$, which obeys (1.16) and $l_N = m$, can be found in the set of all $\pi^{(N)}$, which obey (1.15) and $p_N = m$; and second that the reverse is true. In fact the arguments below reveal that the two "critical cases" in (1.15) and (1.16) are the same: "$l_N = m \Longleftrightarrow p_N = m$". For both inclusions the overall tactic is first to show that a $\pi^{(N)}$ belonging to one critical case has special properties, which allow it to be reached through a chain of transpositions belonging to the other critical case, and second one has to check that the other path is indeed admissible in terms of the other set.

In the subsequent proof and in the Lemmas of this Section we will use the following notations:

$\pi^{(i)}$ $(i = 1, \ldots, N)$ for the sequence of permutations associated to $(k, l; N)$ in accordance with (1.2-3);

$\widetilde{\pi}^{(i)}$ $(i = 1, \ldots, N)$ for the sequence of permutations associated to $(p, q; N)$ in accordance with (1.2-3); and

'$\equiv$' for the componentwise equality of two sequences of transpositions.

Assume that (1.16) and $l_N = m$ are valid for some $\pi^{(N)}$; then Lem.1.3 below shows that $k_{N-1} = m$.

Since $k_{N-1} = m$, there must exist a $j \in \mathbb{N}$ such that $m = k_{N-1} = k_{N-2} = \cdots = k_{N-j} > k_{N-j-1}$. Note once more that $l_{N-1}, l_{N-2}, \ldots, l_{N-j} > m$ are all distinct by the λ-condition. Now with

$$(m, l_{N-j}) \ldots (m, l_{N-1})(k_N, m) = (k_N, l_{N-j}) \ldots (k_N, l_{N-1})(m, l_{N-j})$$

we define

(1.17) $$(p, q; N) :\equiv (k_1, l_1) \ldots (k_{N-j-1}, l_{N-j-1})(k_N, l_{N-j}) \ldots (k_N, l_{N-1})(m, l_{N-j}) ,$$

which says especially that $p_N = m$!! Lemma 1.5 below now shows that the permutation $\widetilde{\pi}^{(N)}$ associated to the sequence $(p, q; N)$ of (1.17) is in fact an element of the r.h.s. of (1.14) fulfilling (1.15).

So far we have proved the inclusion 'l.h.s. (1.14)' $\subset$ 'r.h.s. (1.14)'. For the reverse inclusion we have to show that every $\pi^{(N)} \in$ 'r.h.s. (1.14)', which obeys (1.15) and $p_N = m$, can be found in the set 'l.h.s. (1.14)' and obeys (1.16) and $l_N = m$. Note first that $p_N = m$ implies $p_{N-1} < m$, because otherwise

$$\widetilde{\pi}^{(N-1)}(p_{N-1}) = \widetilde{\pi}^{(N-1)}(m) \triangleleft \widetilde{\pi}^{(N-1)}(q_{N-1}) = \widetilde{\pi}^{(N)}(p_{N-1}) = \widetilde{\pi}^{(N)}(p_N) ,$$

in contradiction to (1.15). Hence $p_N = m$ implies the existence of a $j \in \mathbb{N}$, such that $(p_{N-j-1} \neq \ ) \ p_{N-j} = \cdots = p_{N-1} < m$. Lemma 1.4 below shows that we furthermore may assume $q_{N-j} = q_N$. Therefore

$$
(1.18)\quad \begin{aligned}
(p,q;N) =&(p_1,q_1)\dots(p_{N-j-1},q_{N-j-1})(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})(m,q_N)\\
=&(p_1,q_1)\dots(p_{N-j-1},q_{N-j-1})(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})(m,q_{N-j})\\
=&(p_1,q_1)\dots(p_{N-j-1},q_{N-j-1})(m,q_{N-j})\dots(m,q_{N-1})(p_{N-1},m)\\
\equiv&(k_1,l_1)\dots(k_{N-j-1},l_{N-j-1})(m,l_{N-j})\dots(m,l_{N-1})(k_N,m) =: (k,l;N)\ ,
\end{aligned}
$$

i.e. $l_N = m$!! Lemma 1.6 below now shows that the permutation $\pi^{(N)}$ associated to the sequence $(k,l;N)$ of (1.18) is in fact an element of the l.h.s. of (1.14) fulfilling (1.16).

This finishes the proof of the equality of sets (1.14) and therefore of Theorem 1.1 . □

Lemma 1.2. *With the notations of the Theorem one has:*

$$
\begin{aligned}
\Pi J(N,\pi,(m-1)^N) \subset&\quad \Pi\{(k,l;N) \mid (k_N,l_N) \overset{m-1}{\rightharpoondown} J(N-1,\pi,m^{N-1})\}\ ,\\
J(N,\pi,m^N) \subset&\quad \{(p,q;N) \mid (p_N,q_N) \overset{m}{\rightharpoondown} J(N-1,\pi,m^{N-1})\}\ .
\end{aligned}
$$

Proof. Let first $(k,l;N) \in J(N,\pi,(m-1)^N)$; if $l_1,\dots,l_{N-1} \neq m$, then $(k,l;N-1) \in J(N-1,\pi,m^{N-1})$ and consequently $(k,l;N) \in \{(k_N,l_N) \overset{m-1}{\rightharpoondown} J(N-1,\pi,m^{N-1})\}$. Assume now that $l_i = m$ for one $i \in \{1,\dots,N-1\}$; then

$$
\pi^{(i)}(k_i) = \pi^{(i-1)}(l_i) = \pi^{(i-1)}(m) \triangleright \pi^{(i-1)}(k_i) = \pi^{(i)}(l_i) = \pi^{(i)}(m)
$$

and by the λ-specific condition (1.7) one concludes that from $\pi^{(i)}$ to $\pi^{(N)} \in \Pi J(N,\pi,(m-1)^N)$ there occurs no change on the places m and k_i:

$$
\pi^{(i)}(m) = \dots = \pi^{(N)}(m)\ .
$$

This shows $l_{i+1},\dots,l_N > m$, where all the $l_{i+1},\dots,l_N$ are distinct, and in particular that the i with $l_i = m$ is unique. If now $k_i \neq k_{i+1},\dots,k_N$, then the transposition (k_i,l_i) can be commuted to the right and consequently by Lemma 1.7

$$
((k_1,l_1),\dots,(k_{i-1},l_{i-1}),(k_{i+1},l_{i+1}),\dots,(k_N,l_N),(k_i,l_i)) \in \{(k_i,l_i) \overset{m-1}{\rightharpoondown} J(N-1,\pi,m^{N-1})\}\ .
$$

If there is some $k_{i+1},\dots,k_N = k_i$, then we choose the one with minimal index, say without loss of generality $k_{i+1} = k_i$ and observe that $(k_i,m)(k_i,l_{i+1}) = (m,l_{i+1})(k_i,m)$. We have to show that in this case

$$
\text{(a)}\ \ \pi^{(i-1)}(k_i) \triangleleft \pi^{(i-1)}(m) \text{ and (b)}\ \ \pi^{(i)}(k_i) \triangleleft \pi^{(i)}(l_{i+1})
$$

imply

$$
\text{(A)}\ \ \pi^{(i-1)}(m) \triangleleft \pi^{(i-1)}(l_{i+1}) \text{ and (B)}\ \ \pi^{(i-1)}(m,l_{i+1})(k_i) \triangleleft \pi^{(i-1)}(m,l_{i+1})(m)\ .
$$

But (A) is a direct consequence of (b), and (B) follows from

$$
\pi^{(i-1)}(m,l_{i+1})(k_i) = \pi^{(i-1)}(k_i) \overset{(a)}{\triangleleft} \pi^{(i-1)}(m) \overset{(A)}{\triangleleft} \pi^{(i-1)}(l_{i+1}) = \pi^{(i-1)}(m,l_{i+1})(m)\ .
$$

We can now repeat the arguments for a new i greater than the old i as many times as necessary.

For the proof of the second inclusion observe that the difference between the two sets is simply that for $J(N,\pi,m^N)$ there is an additional requirement on the pair (p_N,q_N) namely by the λ-specific condition: $\pi^{(N-1)}(p_{N-1}) < \pi^{(N)}(p_N) = \pi^{(N-1)}(q_N)$. □

Lemma 1.3. *Let* $\pi^{(N)} \in \Pi\{(k,l;N) \mid (k_N,l_N) \overset{m-1}{\rightharpoondown} J(N-1,\pi,m^{N-1})\} \setminus \Pi J(N,\pi,(m-1)^N)$ *with* $l_N = m$ *in the notations of the Theorem, then* $k_{N-1} = m$.

Proof. Let $k_{N-1} < m$ and $i := \max\{j \mid 1 \leq j \leq N-2,\ k_j = m\}$. Such an i must exist or otherwise we would have $\pi^{(N)} \in \Pi J(N, \pi, (m-1)^N)$. To facilitate notation we assume $i = 1$ so that $\pi^{(N)} = \pi\ (m, l_1)(k_2, l_2) \dots (k_{N-1}, l_{N-1})(k_N, m)$, where $k_2, \dots, k_N < m$ and $l_1, l_2, \dots, l_{N-1} > m$, the latter being all distinct by the λ-condition. We then show

$$(m, l_1)(k_2, l_2) \dots (k_{N-1}, l_{N-1})(k_N, m) \overset{(1)}{=} (m, l_1)(k_N, m)(k_2, l_2) \dots (k_{N-1}, l_{N-1})$$
$$= (k_N, m)(k_N, l_1)(k_2, l_2) \dots (k_{N-1}, l_{N-1}) \overset{(2)}{\in} \Pi J(N, \pi, (m-1)^N)\ ,$$

which proves $k_{N-1} = m$, because $\pi^{(N)} \notin \Pi J(N, \pi, (m-1)^N)$ by assumption.

(1) is valid, if $k_2, \dots, k_{N-1} \neq k_N$. Assume that $j := \max\{j' \mid 2 \leq j' \leq N-1,\ k_{j'} = k_N\}$. Then, using the λ-specific condition (1.7), one has

$$\pi^{(N-1)}(m) = \pi^{(1)}(m) = \pi^{(1)}(k_1) < \pi^{(j)}(k_j) = \pi^{(j)}(k_N) = \pi^{(N-1)}(k_N)\ ,$$

which is in contradiction to $\pi^{(N-1)}(k_N) \lhd \pi^{(N-1)}(m)$.

For (2) we have to check that

$$\begin{array}{ll} \text{(a)} & \pi(k_N) \lhd \pi(m)\ , \\ \text{(b)} & \pi\ (k_N, m)(k_N) \lhd \pi\ (k_N, m)(l_1)\ , \\ \text{(c)} & \pi\ (k_N, m)(k_N) < \pi\ (k_N, m)(k_N, l_1)(k_N)\ , \end{array}$$

where (c) comes from the λ-condition. Is easy to see that (b) is equivalent to $\pi(m) \lhd \pi(l_1)$, and similarly one computes for (c):

$$\pi(k_N, m)(k_N) = \pi(m) < \pi(l_1) = \pi(k_N, m)(l_1) = \pi(k_N, m)(k_N, l_1)(k_N)\ .$$

For (a) observe that

$$\pi(k_N) \lhd \pi(m) \lhd \pi(l_1) \Longrightarrow \text{(b)}\quad .$$

But it is easy to see that the negation of $\pi(k_N) \lhd \pi(m)$ together with $\pi(m) \lhd \pi(l_1)$ implies the negation of (b); since $\pi(m) \lhd \pi(l_1)$ and (b) are already seen to be true, $\pi(k_N) \lhd \pi(m)$ must be true. $\square$

Lemma 1.4. *With the notations of the Theorem one can for* $\widetilde{\pi}^{(N)}(p, q; N) \in \Pi\{(p, q; N) \mid (p_N, q_N) \overset{m}{\rightharpoonup} J(N-1, \pi, m^{N-1})\}$ *with* $p_N = m$ *assume without loss of generality that* $q_{N-j} = q_N$ *for some* $j \in \mathbb{N}$.

Proof. Assume that there is no i with $1 \leq i \leq N-1$ and $q_i = q_N$. If in addition all $p_i < m$, then the transposition (m, q_N) in (1.18) can be commuted in such a way that the λ-condition (1.7) is fulfilled for all i; in other words: $\widetilde{\pi}^{(N)} \in \Pi J(N, \pi, m^N)$. If on the other hand $p_i = m$ for some maximally chosen $i \leq N-1$, then by (1.15)

$$\widetilde{\pi}^{(i)}(p_i) = \widetilde{\pi}^{(i)}(m) = \widetilde{\pi}^{(N-1)}(m) \lhd \widetilde{\pi}^{(N-1)}(q_N) = \widetilde{\pi}^{(N)}(p_N)\ (\ < \widetilde{\pi}^{(N-1)}(p_{N-1})\ )\ ,$$

so that (m, q_N) can be commuted to one of the places $i+1, \dots, N-1$ with (1.7) again fulfilled for all i.

Therefore we conclude that $q_i = q_N$ for some maximally chosen $i \leq N-1$. We first investigate the possibility $i = N-j-1$ (the cases $1 \leq i < N-j-1$ are similar). Let $q_{N-j-1} = q_N$, then

$$\widetilde{\pi}^{(N)}(m) = \widetilde{\pi}^{(N-1)}(q_N) = \widetilde{\pi}^{(N-j-1)}(q_N) = \widetilde{\pi}^{(N-j-1)}(q_{N-j-1}) < \widetilde{\pi}^{(N-j-1)}(p_{N-j-1})\ ,$$

and

$$(p_{N-j-1},q_N)(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})(m,q_N)=$$
$$(p_{N-j-1},q_N)(m,q_N)(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})=$$
$$(m,q_N)(p_{N-j-1},m)(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})=$$
$$(m,q_N)(p_{N-1},q_{N-j})\dots(p_{N-1},q_{N-1})(p_{N-j-1},m) \quad .$$

But the latter permutation with $(p_1,q_1)\dots(p_{N-j-2},q_{N-j-2})$ in front is already an element of l.h.s. (1.15), because the λ-condition (1.7) is fulfilled for all $i \le N-2$:

$$\widetilde{\pi}^{(N-j)}(p_{N-1}) > \widetilde{\pi}^{(N-j-1)}(p_{N-j-1}) \triangleright \widetilde{\pi}^{(N-j-1)}(q_{N-j-1}) = \widetilde{\pi}^{(N-j-1)}(m,q_N)(m).$$

The above arguments for the case $i = N-j-1$ can be carried out in exactly the same way for $i = N-j$ (as representative for $N-j \le i \le N-1$) up to the point

$$(p_{N-1},q_{N-j})(p_{N-1},q_{N-j+1})\dots(p_{N-1},q_{N-1})(m,q_N) = \dots =$$
$$(m,q_N)(p_{N-1},m)(p_{N-1},q_{N-j+1})\dots(p_{N-1},q_{N-1}) \ ,$$

but now the transposition (p_{N-1},m) can not be commuted to the right, whence without loss of generality only the case $q_{N-j} = q_N$ remains to be considered. □

Lemma 1.5. *With the notations of the Theorem and $(p,q;N)$ given by (1.17) one has*

$$\widetilde{\pi}^{(N)}(p,q;N) \in \Pi\{(p,q;N) \mid (p_N,q_N) \stackrel{m}{\rightharpoonup} J(N-1,\pi,m^{N-1})\} \setminus \Pi J(N,\pi,m^N) \ .$$

Proof. Obviously $\pi^{(i)} = \widetilde{\pi}^{(i)}$ for $i = 1,\dots,N-j-1$ and $i = N$, and $\widetilde{\pi}^{(i)}(p_i) < \widetilde{\pi}^{(i+1)}(p_{i+1})$ for $i = 1,\dots,N-1$. It remains to be shown that

$$\begin{aligned}
&(1) \quad \pi^{(N-j-1)}(k_N) \triangleleft \pi^{(N-j-1)}(l_{N-j}) \ ,\\
&(2) \quad \widetilde{\pi}^{(N-1)}(m) \triangleleft \widetilde{\pi}^{(N-1)}(q_N) \ ,\\
&(3) \quad \widetilde{\pi}^{(N-1)}(p_{N-1}) > \widetilde{\pi}^{(N-1)}(q_N) \ , \text{ and}\\
&(4) \quad \pi^{(N-j-1)}(m) \triangleleft \pi^{(N-j-1)}(k_N) \ ,
\end{aligned}$$

where (4) is auxiliary for (2), (1) is necessary for having $\pi^{(N-1)}(p,q;N-1) \in \Pi J(N-1,\pi,m^{N-1})$ and (2), (3) for condition (1.15).

To show (4) assume first that $\pi^{(N-j-1)}(m) > \pi^{(N-j-1)}(k_N)$ and let

$$\gamma := 2 \cdot \sharp\{\nu \mid k_N < \nu < m, \ \pi^{(N-j-1)}(k_N) < \pi^{(N-j-1)}(\nu) < \pi^{(N-j-1)}(m)\} \ .$$

In [W, Prop.2.3] it has been shown that under this circumstances interchanging m and k_N increases the length of $\pi^{(N-j-1)}$ by $\gamma+1$: $l(\pi^{(N-j-1)}(k_N,m)) = l(\pi^{(N-j-1)}) + \gamma + 1$. We conclude therefore

$$l(\widetilde{\pi}^{(N-j)}) = l(\pi^{(N-j)}) + \gamma + 1 \Longrightarrow l(\widetilde{\pi}^{(N-1)}) = l(\pi^{(N-1)}) + j\,\gamma + j$$
$$\Longrightarrow l(\widetilde{\pi}^{(N)}) \ge l(\pi^{(N)}) + j\,\gamma + 1 \ ,$$

because during the j steps from $\pi^{(N-j-1)}$ to $\pi^{(N-1)}$ at most $j-1$ numbers greater $\pi^{(N-j-1)}(k_N)$ and less than $\pi^{(N-j-1)}(m)$ may have "immigrated" between the places k_N and m (— at least $\widetilde{\pi}^{(N-1)}(l_{N-1}) = \pi^{(N-j-1)}(l_{N-1})$ is greater than $\pi^{(N-j-1)}(m)$ —). But $l(\widetilde{\pi}^{(N)}) \ge l(\pi^{(N)}) + j\gamma + 1$ is in contradiction to $\widetilde{\pi}^{(N)} = \pi^{(N)}$. If on the other hand $\pi^{(N-j-1)}(m) < \pi^{(N-j-1)}(k_N)$, then (1) follows, because the numbers on the places between k_N and m remain unchanged for $\pi^{(N-j)}, \ldots, \pi^{(N-1)}$, whence it would be a contradiction to $\pi^{(N-1)}(k_N) \lhd \pi^{(N-1)}(m)$, if (1) were not valid.

(1) implies now (2), since

$$\widetilde{\pi}^{(N-1)}(m) = \pi^{(N-j-1)}(m) \lhd \pi^{(N-j-1)}(k_N) = \widetilde{\pi}^{(N-j)}(l_{N-j}) = \widetilde{\pi}^{(N-1)}(l_{N-j}) \ .$$

For (3) one computes similarly that

$$\begin{aligned}\widetilde{\pi}^{(N)}(k_N) = \widetilde{\pi}^{(N-1)}(k_N) = \widetilde{\pi}^{(N-1)}(p_{N-1}) > \widetilde{\pi}^{(N)}(p_N) = \widetilde{\pi}^{(N)}(l_{N-j}) \quad \Longleftrightarrow \\ \pi^{(N-j-1)}(l_{N-1}) = \pi^{(N-2)}(l_{N-1}) = \pi^{(N-1)}(m) = \pi^{(N)}(k_N) > \\ \pi^{(N)}(l_{N-j}) = \pi^{(N-j)}(l_{N-j}) = \pi^{(N-j-1)}(m) \quad \Longleftrightarrow \\ \pi^{(N-j-1)}(l_{N-1}) \geq \pi^{(N-j-1)}(l_{N-j}) \rhd \pi^{(N-j-1)}(m) \ .\end{aligned}$$

For (1) assume that there is ν with $k_N < \nu < m$ so that

$$\widetilde{\pi}^{(N-j-1)}(k_N) < \widetilde{\pi}^{(N-j-1)}(\nu) < \widetilde{\pi}^{(N-j-1)}(l_{N-j}) \ .$$

Then we have a contradiction, because $\widetilde{\pi}^{(N-j-1)}(k_N) = \widetilde{\pi}^{(N-1)}(k_N)$ and

$$\widetilde{\pi}^{(N-j-1)}(l_{N-j}) = \widetilde{\pi}^{(N-j)}(k_{N-j}) \stackrel{(1.7)}{=} \widetilde{\pi}^{(N-1)}(k_{N-1}) = \widetilde{\pi}^{(N-1)}(m) \ .$$

□

Lemma 1.6. *With the notations of the Theorem and $(k, l; N)$ given by (1.18) one has*

$$\pi^{(N)}(k, l; N) \in \Pi\{(k, l; N) \mid (k_N, l_N) \stackrel{m-1}{\rightharpoonup} J(N-1, \pi, m^{N-1})\} \setminus \Pi J(N, \pi, (m-1)^N) \ .$$

Proof. We have to check that

$$\begin{aligned} &(1) \quad \widetilde{\pi}^{(N-j-1)}(m) \lhd \widetilde{\pi}^{(N-j-1)}(q_{N-j}) \ , \\ &(2) \quad \pi^{(N-1)}(k_N) \lhd \pi^{(N-1)}(m) \ , \text{ and} \\ &(3) \quad \pi^{(N-1)}(k_{N-1}) \geq \pi^{(N)}(k_N) \ . \end{aligned}$$

For (3) simply observe $\pi^{(N-1)}(k_{N-1}) = \pi^{(N-1)}(m) = \pi^{(N)}(k_N)$.

For (1) one notes that $\widetilde{\pi}^{(N-1)}(m) \lhd \widetilde{\pi}^{(N-1)}(q_N)$ implies that there is no ν with $m < \nu < q_{N-j}$ such that

$$\widetilde{\pi}^{(N-j)}(m) = \widetilde{\pi}^{(N-1)}(m) < \widetilde{\pi}^{(N-j)}(\nu) < \widetilde{\pi}^{(N-1)}(q_N) = \widetilde{\pi}^{(N-1)}(q_{N-j}) = \widetilde{\pi}^{(N-j)}(q_{N-j}) \ ,$$

because the numbers on places ν with $m < \nu < q_N$ are not changed by the transpositions $(p_{N-1}, q_{N-j}), \ldots, (p_{N-1}, q_{N-1})$. Therefore we can conclude that

$$\widetilde{\pi}^{(N-j-1)}(m) = \widetilde{\pi}^{(N-j)}(m) < \widetilde{\pi}^{(N-j)}(q_{N-j}) = \widetilde{\pi}^{(N-j-1)}(p_{N-j}) = \widetilde{\pi}^{(N-j-1)}(p_{N-1}) \ ,$$

and that there is no ν with $m < \nu < q_{N-j}$ such that $\widetilde{\pi}^{(N-j-1)}(m) < \widetilde{\pi}^{(N-j-1)}(\nu) < \widetilde{\pi}^{(N-j-1)}(p_{N-1})$. This is exactly what is needed to conclude (1) from $\widetilde{\pi}^{(N-j-1)}(p_{N-1}) \lhd \widetilde{\pi}^{(N-j-1)}(q_{N-j})$.

With regard to (2) it is easy to see by (1.15) that $\pi^{(N-1)}(k_N) < \pi^{(N-1)}(m)$, because $\pi^{(N-1)}(k_N) = \pi^{(N)}(m) = \widetilde{\pi}^{(N)}(m) = \widetilde{\pi}^{(N)}(p_N)$ and $\pi^{(N-1)}(m) = \pi^{(N)}(k_N) = \widetilde{\pi}^{(N)}(p_{N-1}) = \widetilde{\pi}^{(N-1)}(p_{N-1})$. But for '$\lhd$' one has to work a bit harder: note first that

$$(2) \Longleftrightarrow \widetilde{\pi}^{(N)}(m) \lhd \widetilde{\pi}^{(N)}(p_{N-1}) \ ,$$

i.e. $\widetilde{\pi}^{(N)}(p_{N-1}) > \widetilde{\pi}^{(N)}(m)$ and in addition for each ν with $p_{N-1} < \nu < m$ either $\widetilde{\pi}^{(N)}(m) > \widetilde{\pi}^{(N)}(\nu)$ or $\widetilde{\pi}^{(N)}(\nu) > \widetilde{\pi}^{(N)}(p_{N-1})$. But $\widetilde{\pi}^{(N-j-1)}(p_{N-1}) \triangleleft \widetilde{\pi}^{(N-j-1)}(q_{N-j})$ so that for each ν with $p_{N-1} < \nu \leq m$ either $\widetilde{\pi}^{(N-j-1)}(p_{N-1}) > \widetilde{\pi}^{(N-j-1)}(\nu)$ or $\widetilde{\pi}^{(N-j-1)}(\nu) > \widetilde{\pi}^{(N-j-1)}(q_{N-j})$. Since moreover

$$\widetilde{\pi}^{(N)}(m) = \widetilde{\pi}^{(N-1)}(q_N) = \cdots = \widetilde{\pi}^{(N-j)}(q_N) = \widetilde{\pi}^{(N-j)}(q_{N-j}) = \widetilde{\pi}^{(N-j-1)}(p_{N-1}) \text{ , and}$$
$$\widetilde{\pi}^{(N)}(p_{N-1}) = \widetilde{\pi}^{(N-1)}(p_{N-1}) > \cdots > \widetilde{\pi}^{(N-j)}(p_{N-1}) = \widetilde{\pi}^{(N-j-1)}(q_{N-j})$$

by the λ-condition, we conclude

$$\widetilde{\pi}^{(N)}(m) = \widetilde{\pi}^{(N-j-1)}(p_{N-1}) \triangleleft \widetilde{\pi}^{(N-j-1)}(q_{N-j}) < \widetilde{\pi}^{(N)}(p_{N-1}) \ .$$

But the numbers on places ν with $p_{N-1} < \nu < m$ are not changed by the transpositions $(p_{N-1}, q_{N-j}), \ldots, (p_{N-1}, q_{N-1}), (m, q_N)$, whence the desired conclusion is valid. □

Lemma 1.7. *Let $\pi \in S_\infty$ and $k, l, m, n \in \mathbb{N}$ be all different with $k < l$ and $m < n$. (Then the transpositions (k, l) and (m, n) commute.) Assume furthermore that for $\pi^{(1)} := \pi(k, l)$ one has $l(\pi^{(1)}) = l(\pi) + 1$ and $l(\pi^{(1)}(m, n)) = l(\pi^{(1)}) + 1$, then also $l(\pi(m, n)) = l(\pi) + 1$.*

Proof. By (0.1-3) the assertion of the Lemma is that

$$\text{(a): } \pi(k) \triangleleft \pi(l) \quad \text{and} \quad \text{(b): } \pi^{(1)}(m) \triangleleft \pi^{(1)}(n) \quad \text{imply} \quad \text{(c): } \pi(m) \triangleleft \pi(n) \ .$$

We use the notation $[s, t] := \{r \in \mathbb{N} \mid s \leq r \leq t\}$ and investigate all possible orderings of the numbers k, l, m, n:

(c) is trivially true in the cases of $k < l < m < n$ and $m < n < k < l$;

in the cases $k < m < n < l$ and $m < k < l < n$ conditions (a) and (b), respectively, imply $[\pi(k), \pi(l)] \cap [\pi(m), \pi(n)] = \emptyset \Longrightarrow$ (c);

$k < m < l < n \overset{(b)}{\Longrightarrow} \pi(k) = \pi^{(1)}(l) \notin [\pi(m), \pi(n)]$, i.e. $\pi(k) > \pi(n)$ or $\pi(k) < \pi(m)$, but $\pi(k) > \pi(n) \overset{(a)}{\Longrightarrow} \pi(l) > \pi(n) \Longrightarrow$ (c), and $\pi(k) < \pi(m) \overset{(a)}{\Longrightarrow} \pi(m) > \pi(l) \Longrightarrow$ (c); similarly

$m < k < n < l \overset{(b)}{\Longrightarrow} \pi(l) = \pi^{(1)}(k) \notin [\pi(m), \pi(n)]$, i.e. $\pi(l) > \pi(n)$ or $\pi(l) < \pi(m)$, but $\pi(l) < \pi(m) \overset{(a)}{\Longrightarrow} \pi(k) < \pi(m) \Longrightarrow$ (c), and $\pi(l) > \pi(n) \overset{(a)}{\Longrightarrow} \pi(n) < \pi(k) \Longrightarrow$ (c), which finishes the proof of the lemma. □

Example 1.8. We compute $h_3^{(3)} X_{25413}$. It is helpful to mark the position between places 3 and 4 by a vertical line, since this is the 'axis' around which the transposition of numbers takes place. In addition we underline the numbers $\pi^{(i)}(k_i)$, so that one can easily check the λ-condition (1.7).

step:	1	2	3
$254\vert 13$	$\underline{3}54\vert 12$	$3\underline{6}4\vert 125$	$3\underline{7}41256$
		$35\underline{6}\vert 124$	$35\underline{7}1246$
	$2\underline{6}4\vert 135$	$2\underline{7}4\vert 1356$	$2\underline{8}413567$
	$25\underline{6}\vert 134$	$25\underline{7}\vert 1346$	$25\underline{8}13467$

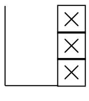
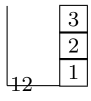
Of course the horizontal lines in the above table are to be understood as: 25413 in the first step gives rise to 35412, 264135, and 256134; and 35412 in the second step splits up into 364125 and 356124, etc. .

2. Observations for the Grassmannian case

Our overall assumption is that *the multiplication of an arbitrary Schubert polynomial, say $X_{\pi'}$, by another, say X_π, can be achieved through a sequence of $l(\pi)$ steps of the 'Monk form'* (0.4-5) *leading upwards in Bruhat order from π', where the admissible, i.e. correct, paths are selected by a rule depending essentially only on the diagram $D(\pi)$.*

More precisely: the summands $X_{\pi^{(N)}}$, respectively the permutations $\pi^{(N)}$, in the product $X_{\pi'}X_\pi$ are reached by executing $N = l(\pi)$ transpositions (k_i, l_i), $i = 1, \ldots, N$, with $\pi^{(0)} := \pi'$ and recursively $\pi^{(i)} := \pi^{(i-1)}(k_i, l_i)$, where for each i there is a natural number β_i such that $\pi^{(i-1)}(k_i) \overset{\beta_i}{\lhd} \pi^{(i-1)}(l_i)$. The number β_i will be called the *(transposition) axis* of the step i, and will in concrete calculations be represented as a vertical bar in $\pi^{(i-1)}$ between the places β_i and $\beta_i + 1$. The combinatorial rule, which says how to select an admissible transpositions (k_i, l_i) and an admissible axis β_i in each step (or admissible sequences (1.2) and (1.4)) will depend on the diagram $D(\pi)$ (with empty rows removed). The rule uses the notion of the *step tableaux* $S(\pi)$ for π, which is a certain filling of $D(\pi)$ with each of the numbers $1, \ldots, N$ occurring once, and the notion of the *weight tableaux* $W(\pi^{(N)})$ for each $\pi^{(N)}$, which is a filling of $D(\pi)$ with the *weights* $W(i) \equiv W(\pi^{(N)}, i) := \pi^{(i)}(k_i)$ on the places i determined by the step tableaux $S(\pi)$.

In the special case of Ex.1.5 one has $\lambda = 3$, $h_3^{(3)} = X_{\pi(3,3)} = X_{126345}$, $\beta_1 = \beta_2 = \beta_3 = 3$, and

$D(126345) =$  , $S(126345) =$  , $W(3741256) = W(3571246) =$

$\begin{array}{|c|}\hline 7 \\ \hline 6 \\ \hline 3 \\ \hline \end{array}$, and $W(28413567) = W(25813467) = \begin{array}{|c|}\hline 8 \\ \hline 7 \\ \hline 6 \\ \hline \end{array}$.

More generally one can rephrase Theorem 1.1 as follows:

> Let $\pi' \in S_\infty$, $\lambda \in \{N, 1^N\}$, $m \geq N$, and $\beta = (\beta_1, \ldots, \beta_N) = (m, \ldots, m) \equiv m^N$. Then the admissible permutations $\pi^{(N)}$ (or admissible paths leading to some $\pi^{(N)}$) for the product $X_{\pi'} X_{\pi(\lambda,m)}$ are the following:
> If $\lambda = N$, then the entries $1, \ldots, N$ are increasing from bottom to top in $S(\pi(\lambda, m))$, and so do the entries of each $W(\pi^{(N)})$.
> If $\lambda = 1^N$, then the entries $1, \ldots, N$ are increasing from right to left in $S(\pi(\lambda, m))$, and the entries of each $W(\pi^{(N)})$ are decreasing from right to left.

In other words: the question, which $\pi^{(N)}$ are admissible, is reduced to examine the tableaux $W(\pi^{(N)})$ in these cases, and the 'admissible tableaux' $W(\pi^{(N)})$ are described by the rule above. Below we describe an extensions of this notion of 'admissibility of a weight tableaux' to the cases of Grassmannian permutations, which gives the correct result in many cases, but fails for example for $X_{312645} X_{561234}$ – a simple counterexample found by N. Bergeron.

Observation 2.1. *Recall the definition* (0.7) *of the Grassmannian permutation* $\pi \equiv \pi(\lambda, m)$ *associated to a partition* $\lambda \equiv \lambda_1 \ldots \lambda_s$ $(\lambda_1 \geq \ldots \geq \lambda_s)$ *and a natural number* $m \geq l(\lambda) = s$, *and the fact that the number of boxes in column* i *of* $D(\pi)$ *is equal to the* i*-th entry of* $L(\pi)$. *(Therefore* $D(\pi)$ *is the Ferrer shape of* λ *with rows reflected into columns as indicated by* (0.7)*.) Then the product* $X_{\pi'} X_\pi$ *can be computed in many cases according to the following prescriptions:*

The step tableaux $S(\pi)$ *is given by numbering consecutively the columns from right to left and inside columns from bottom to top with the numbers* $1, \ldots, N = |\lambda|$.

$\beta_1 = \cdots = \beta_N := m$.

$W(\pi^{(N)})$ *is admissible iff the weights in* $W(\pi^{(N)})$ *increase in every column from bottom to top and for* $i = 1, \ldots, s-1$

$$b_j \leq a_j \quad \textit{for } j = 1, \ldots, \lambda_{i+1} \ , \tag{2.1}$$

where $a_1, \ldots, a_{\lambda_i}$ *and* $b_1, \ldots, b_{\lambda_{i+1}}$ *are the entries of columns* $(m-i+1)$ *and* $(m-i)$, *respectively, where the indices increase from top to bottom.*

Note that condition (2.1) can be rephrased by saying: if every column in a weight tableaux $W(\pi^{(N)})$ is flushed to a common top line, then the the weights increase in every column from bottom to top and decrease in every row from right to left. Furthermore we remark that different 'final' permutations $\pi^{(N)}$ may have the same weight tableaux, but equal permutations $\pi^{(N)}$ necessarily have different weight tableaux reflecting the fact that they are reached through different admissible path.

Example 2.2. We compute $X_{263514}\ X_{135624} = X_{263514}\ s^{(4)}_{221}(x)$ with $\beta_1 = \cdots = \beta_5 = 4$,

$$D(135624) = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \times & \times \\ \hline \times & \times & \times \\ \hline \end{array} \ , \text{ and } S(135624) = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 4 & 2 \\ \hline 5 & 3 & 1 \\ \hline \end{array} \ .$$

step:	1	2	3	4	5
$2635\vert 14$	$2\underline{7}35\vert 146$	$2\underline{8}35\vert 1467$	$28\underline{4}5\vert 1367$	$284\underline{6}\vert 1357$	$\underline{3}846\vert 1257$
					$28\underline{5}6\vert 1347$
			$283\underline{6}\vert 1457$	$283\underline{7}\vert 1456$	$28\underline{4}7\vert 1356$
	$26\underline{4}5\vert 13$	$264\underline{7}\vert 135$	$\underline{3}647\vert 125$	$36\underline{5}7\vert 124$	$\underline{4}657\vert 123$
		$2\underline{7}45\vert 136$	$\underline{3}745\vert 126$	$374\underline{6}\vert 125$	$37\underline{5}6\vert 124$
	$263\underline{7}\vert 145$	$263\underline{8}\vert 1457$	$26\underline{4}8\vert 1357$	$26\underline{5}8\vert 1347$	$\underline{3}658\vert 1247$
				$2\underline{7}48\vert 1356$	$\underline{3}748\vert 1256$
					$27\underline{5}8\vert 1346$

Note that e.g. $W(3756124) = \left\lfloor \begin{array}{ccc} & \boxed{6} & \boxed{7} \\ \boxed{5} & \boxed{3} & \boxed{4} \end{array} \right.$ and $W(37481256) = \left\lfloor \begin{array}{ccc} & \boxed{7} & \boxed{8} \\ \boxed{3} & \boxed{4} & \boxed{7} \end{array} \right.$.

Of course the Bergeron-Billey rule and the famous Littlewood-Richardson rule are contained as special cases in the above observation. Moreover the case of λ being a hook shape, proved as a simple consequence of the Bergeron-Billey rule in [So, Thm.8], is contained.

3. Observations for the L-unimodal case

In the introduction we mentioned already that L-unimodal permutations generalize both Grassmannian and dominant permutations. In order to formulate Observation 3.2 for the multiplication of a Schubert polynomial by an X_π with L-unimodal π we need the following simple

Lemma 3.1. *a) π is L-unimodal exactly when $D(\pi)$ has one component.*
b) If π is L-unimodal, then row j of $D(\pi)$ includes row $j+1$ for all j. If the rightmost box in row j stands in column $c_j \equiv c_j(\pi)$, then one has $c_1 \geq c_2 \geq \cdots \geq 0$.

Proof. Let $\pi \in S_n$ be L-unimodal with $l_{n-1} \leq \cdots \leq l_{n-\nu} \geq \cdots \geq l_0$ for some ν. Then a) and b) are simple consequences of the generation rules for $D(\pi)$, namely the removal of the 'hooks' of boxes $h_\pi(i) := \{[i', \pi(i)] \mid i' \geq i\} \cup \{[i, j'] \mid j' \geq \pi(i)\}$ from the $n \times n$-array of boxes. Observe that if $1 \leq i < \nu$, then the removal of the $h_\pi(i)$'s generates the initial step-backs of $D(\pi)$, and if $\nu < i \leq n$, then the removal of the $h_\pi(i)$'s generates proper inequalities in the sequence $c_1 \geq c_2 \geq \dots$. $\square$

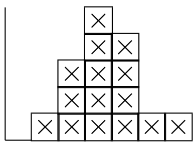
Observation 3.2. *Let $\pi \in S_n$ be L-unimodal as in the above lemma, and in particular let again c_j be the column, which contains the rightmost box in row j, and $r_1, r_2, \dots, r_q$ be the rows for which $c_{r_j} \gneq c_{r_j+1}$. Then exactly the $\pi^{(N)}$ occurring in the product $X_{\pi'} X_\pi$ are computed — but possibly with a greater multiplicity (!) — according to the following prescriptions:*

Subdivide $D(\pi)$ into q subdiagrams of Grassmannian shape, i.e. the q subdiagrams G_1, G_2 up to G_q consisting of rows $1, \dots, r_1$, rows $r_1+1, \dots, r_2$, up to rows $r_{q-1}+1, \dots, r_q$, respectively.

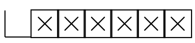
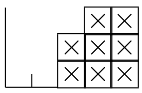
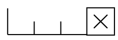
For $p = 1, \dots, q$ the single subdiagrams of Grassmannian shape G_p are treated now exactly as in Obs.2.1 or more correctly: "its yet to be found correct version", with m replaced by c_{r_p}, whereby one starts with G_1 and works up to G_q.

For $p = 1, \dots, q$ and $g_p := |G_1| + \dots + |G_p|$ we require that $\pi^{(g_p)}(k_{g_p})$ is not greater than any weight in the next higher row $r_p + 1$.

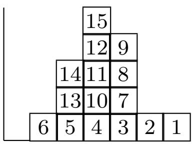
Example 3.3. For $\pi = 136984527 \in S_9$ one has the Lehmer code $L(\pi) = \overline{013541100}$, the diagram

$D(\pi) =$  ,

$c_1 = 7$, $c_2 = c_3 = c_4 = 5$, $c_5 = 4$, and $c_6 = 0$, whence $q = 3$, $r_1 = 1$, $r_2 = 4$, $r_3 = 5$. The subdiagrams of Grassmannian shape are

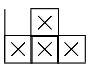
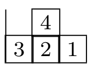
$G_1 =$  , $G_2 =$  , $G_3 =$  ,

and finally

$S(\pi) =$ 

with $g_1 = 6$, $g_2 = 14$, and $g_3 = 15$.

Example 3.4. We compute $X_{31542} X_{2431}$ with $\pi = 2431$, $L(\pi) = \overline{1210}$, $\beta_1 = 3$, $\beta_2 = 2$,

$D(\pi) =$  , and $S(\pi) =$ 

.

In the tableaux below we use the already established conventions, but add this time the 'dead ends' of the computation with the respective final steps in parenthesis.

step:	1	2	3	4
$315\|42$	$31\underline{6}\|425$	$\underline{4}16\|325$	$4\underline{3}\|6125$	$4\underline{6}3125$ $4\underline{5}6123$ $\underline{6}34125$ $\underline{5}36124$
			$4\underline{2}\|6315$	$4\underline{6}2315$ $\underline{6}24315$ $\underline{5}26314$ $4\underline{3}6215$
		$(3\underline{2}6\|415)$ $(3\underline{4}6\|125)$		
	$\underline{4}15\|32$	$(4\underline{3}5\|12)$ $(4\underline{2}5\|31)$		
	$(3\underline{4}5\|12)$			
	$(3\underline{2}5\|41)$			

If one tries a "column-wise" approach with step tableaux $S(\pi) = \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 4 & 2 & 1 \\ \hline \end{array}$ and $\beta_1 = 3$, then all reasonable combinations of β_2 and β_3 give false terms.

More elaborate examples like $X_{31524}X_{13542}$, $X_{136279485}X_{13542}$, $X_{31524}X_{13467582}$, and X_{31524} X_{136425} work equally well, but in the example $X_{31524}X_{146325}$ the multiplicities in some cases are to big. This shows that the requirement in Obs.3.2 for the admissible transitions between consecutive subshapes G_p, G_{p+1} is not yet exclusive enough.

4. Observations for the general case

This final section contains some observations, examples and remarks in view of a general rule how to multiply a Schubert polynomial by an X_π, when $D(\pi)$ has more than one component. We begin with a "rule", which works well in many cases, but has also severe defects in other cases to be discussed later on.

Observation 4.1. (First approximation of a general rule) *The following prescriptions give an overall and approximative "rule" for the multiplication of an arbitrary Schubert polynomial by an X_π in case $D(\pi)$ has $q > 1$ components $C_1, \ldots, C_q$:*

Every component is handled essentially as in Obs.3.2; the relative order of components is such that C_i precedes C_j, if the leftmost box of C_i is left to the leftmost box of C_j or, if the leftmost boxes of C_i and C_j are in the same columns, the lower component comes first. In this way the step tableaux $S(\pi)$ is completely determined.

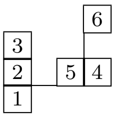
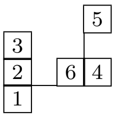
For the 'Monk step' i from $\pi^{(i-1)}$ to $\pi^{(i)}$ suppose that i is in column $c(i)$ and row $r(i)$ of $S(\pi)$. Then $\beta_i \geq c(i)$ is given by the requirement that in row $r(i)$ the next gap in $D(\pi)$ to the right of column $c(i)$ occurs at column $\beta_i + 1$. As usual $k_i \leq \beta_i < l_i$, where k_i can be chosen only from the set $\{1, \ldots, \beta_i\} \setminus F_\pi(i)$ with

$$F_\pi(i) := \{k < c(i) \mid [k, r(i)] \in D(\pi), \text{ and } \exists h : k < h < c(i),\ [h, r(i)] \notin D(\pi)\} \tag{4.1}$$

being called the π-fixed set for i. *In other words: $F_\pi(i)$ is the set of the numbers of those columns left to column $c(i)$, which contain boxes in row $r(i)$ and are separated by a gap from $[c(i), r(i)]$.*

Obs.4.1 already works well in many cases:

Example 4.2. We compute $X_{531624}X_{413625}$ with $\pi = 413625$, $L(\pi) = \overline{301200}$,

$S(\pi) =$  , *not* :  , and

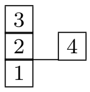
hence $\beta_1 = \beta_2 = \beta_3 = 1$, $\beta_4 = \beta_5 = \beta_6 = 4$, and $F_\pi(1) = F_\pi(2) = F_\pi(3) = \emptyset$, $F_\pi(4) = F_\pi(5) = \{1\}$, and $F_\pi(6) = \emptyset$. For the computation it is convenient to draw a box around the numbers on the places $F_\pi(i)$.

step:	... 3	4	5	6
5|31624	$\boxed{8}$ 315|2467	$\boxed{8}$ $\underline{4}$15|2367	84$\underline{2}$5|1367	$\underline{9}$42513678 84$\underline{3}$51267 842$\underline{6}$1357
		$\boxed{8}$ 31$\underline{6}$|2457	8$\underline{4}$16|2357	$\underline{9}$41623578 8$\underline{5}$162347 841$\underline{7}$2356
			83$\underline{2}$6|1457	$\underline{9}$32614578 8$\underline{4}$261357 83$\underline{4}$61257 832$\underline{7}$1456

Following simply the "rule" of Obs.4.1 can result in generating too many permutations, a fact which is partially taken into account by

Observation 4.3. *Assume that in the situation of Obs.4.1 i and $i+1$ are in different components, and that $\pi^{(i-1)}(k_i) \lhd \pi^{(i-1)}(l_i)$ for some $k_i \in F_\pi(i)$ and arbitrary $l_i > k_i$ (not necessarily $\beta_i < l_i$!). Then the permutation $\pi^{(i-1)}(k_i, l_i)$ is generated with a minus sign, i.e. it must be canceled in another place, where it occurs through "regular" generation in accordance to Obs.4.1 .*

Example 4.4. We compute $X_{13524}X_{4132}$ with $\pi = 4132$, $L(\pi) = \overline{3010}$, and

$S(\pi) =$  . and

step:	1	2	3	4
$1\|3524$	$\underline{3}\|1524$	$\underline{5}\|1324$	$\boxed{\underline{6}}\,13\|245$	$6\underline{2}3145$ $61\underline{4}235$
		$\underline{4}\|1523$	$\boxed{\underline{5}}\,14\|23$	$5\underline{2}413$ $51\underline{6}234$
	$\underline{2}\|3514$	$\underline{3}\|2514$	$\boxed{\underline{5}}\,23\|14$	$52\underline{4}13$
			$\boxed{\underline{4}}\,25\|13$	$4\underline{3}512$ $42\underline{6}135$ $-\,\underline{5}2413$ (!)

Observation 4.3 marks an important deviation from our general "philosophy" that the $\pi^{(N)}$ of the product $X_{\pi'}X_\pi$ are reached via admissible paths! But on the other hand it seems very artificial and arbitrary to exclude just one of the permutations 52413 in the above example from the final column.

In addition it is unclear, whether only the 'fixed places' $k_i \in F_\pi(i)$ with $k_i \le \beta_i$ should be excluded or all $k_i \le \max F_\pi(i)$. In the first case there are examples showing that too many permutations are generated and in the second case too less. So one can say that the 'interaction' of several components in $D(\pi)$ seems to be the cause, which makes it difficult to find a *simple* combinatorial rule for the multiplication of Schubert polynomials.

References

[BB] N. Bergeron, S. Billey, RC-Graphs and Schubert polynomials, *Experimental Mathematics* **2** (1993), 257 - 269.

[C] C. Chevalley, Sur le Dècomposition Cellulaires des Espaces G/P, Proceedings of Symposia in Pure Mathematics **56** (1994).

[Hi] H. Hiller, "Geometry of Coxeter groups", Pitman, Boston, 1982.

[LS1] A. Lascoux, M.-P. Schützenberger, Décompositions dans l'algébre des différences divisées, *discrete math.* **99** (1992), 165 - 179.

[LS2] A. Lascoux, M.-P. Schützenberger, Polynomes des Schubert, *C.R.Acad.Sci.Paris* **294** (1982), 447 - 450.

[M1] I.G. Macdonald, Schubert polynomials, *in*: A.D. Kendwell (ed.), "Surveys in Combinatorics", London Math. Soc. Lec. Notes Ser. **166**, 73 - 99, Cambridge University Press, Cambridge 1991.

[M2] I.G. Macdonald, "Notes on Schubert polynomials", Publications du L.A.C.I.M., vol. 6, Université du Québec, Montréal, 1991.

[M3] I.G. Macdonald, "Symmetric functions and Hall polynomials", Clarendon Press, Oxford 1979.

[Mo] D. Monk, The geometry of flag manifolds, *Proc. London Math. Soc.* **3** (1959), 253 - 286.

[Sa] B.E. Sagan, "The Symmetric Group", Wadsworth & Brooks/Cole, Pacific Grove, California (1991).

[So] F. Sottile, Pieri's formula for flag manifolds and Schubert polynomials, *Ann. Inst. Fourier, Grenoble***46** (1996), 89 - 110.

[W] R. Winkel, Recursive and combinatorial properties of Schubert polynomials, preprint (1995).

Institut für Reine und Angewandte Mathematik, RWTH Aachen, D-52056 Aachen, Germany, *papers/preprints*: http://www.iram.rwth-aachen.de/ ~winkel/

Current address: Dept. of Mathematics, MIT, 2-390, 77 Mass. Ave., Cambridge, MA 02139-4307, *current e-mail*: winkel@math.mit.edu

E-mail address: winkel@iram.rwth-aachen.de