ON THE EXPANSION OF SCHUR AND SCHUBERT POLYNOMIALS INTO STANDARD ELEMENTARY MONOMIALS

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ABSTRACT: Motivated by the recent discovery of a simple quantization procedure for Schubert polynomials we study the expansion of Schur and Schubert polynomials into standard elementary monomials (SEM). The SEM expansion of Schur polynomials can be described algebraically by a simple variant of the Jacobi-Trudi formula and combinatorially by a rule based on posets of staircase box diagrams. These posets are seen to be rank symmetric and order isomorphic to certain principal order ideals in the Bruhat order of symmetric groups ranging between the full symmetric group and the respective maximal Boolean sublattice. We prove and conjecture extensions of these results for general Schubert polynomials. The featured conjectures are: (1) an interpretation of SEM expansions as "alternating approximations" and (2) surprising properties of different numbers naturally associated to SEM expansions. This hints at as yet undiscovered deeper symmetry properties of the SEM expansion of Schubert polyonmials.

1. INTRODUCTION

The Gromov-Witten invariants of flag manifolds can be computed as structure constants of the ring of quantum Schubert polynomials (representing quantum cohomology classes), just the same way as the intersection coefficients of Schubert varieties can be computed as the structure constants of the ring of ordinary Schubert polynomials (representing ordinary cohomology classes). S. Fomin, S. Gelfand, and A. Postnikov recently [FGP] discovered that quantum Schubert polynomials can be computed through a simple *quantization* of the expansion of ordinary Schubert polynomials into standard elementary monomials (SEM): one has to substitute only the ordinary elementary symmetric polynomials occurring in the SEM by their quantized counterparts. In other words: the "trivialization of quantization" achieved by [FGP] reduces the understanding of quantum Schubert polynomials to the "really hard part" of understanding the SEM expansions of ordinary Schubert polynomials.

We therefore begin in the present paper with an investigation of these SEM expansions: in the case of Schur polynomials there exists a simple determinantal formula, which

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enables e.g. the derivation of a combinatorial rule. In the case of general Schubert polynomials we prove some generalizations of the Schur case results and formulate several conjectures, which hint at as yet undiscovered deeper symmetry properties of the SEM expansion of Schubert polyonmials.

In 1953 Borel proved that the ring of integral cohomology of (complete) flags over a complex vector space of dimension n is isomorphic to the ring $P_n := \mathbb{Z}[x_1, \ldots, x_n]$ factored by the ideal $\Lambda_n^+ := (e_1^{(n)}, \ldots, e_n^{(n)})$, where

$$e_i^{(n)} \equiv e_i(x_1, \ldots, x_n) := \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \ldots\ x_{j_i} \tag{1.1}$$

denotes the *elementary symmetric polynomial* of degree i in the variables $x_1, \ldots, x_n$ with $e_0^{(n)} := 1$ for all $n \ge 0$ and $e_i^{(n)} := 0$ unless $0 \le i \le n$. In signs:

$$H^*(Fl_n, \mathbb{Z}) \cong P_n/\Lambda_n^+ \ .$$

The free $\mathbb{Z}$-module H_n of residues of P_n/Λ_n^+ is spanned by the set of monomials

$$\{x^l := x_1^{l_{n-1}} \ldots x_{n-1}^{l_1} \mid l \in \mathbb{L}_n\} \ \text{ with} \tag{1.2}$$

$$\mathbb{L}_n := \{l = l_1 \ldots l_{n-1} \mid 0 \le l_\nu \le \nu,\ \nu = 1, \ldots, n-1\ \}\ , \tag{1.3}$$

that is

$$H_n = \langle x^l \mid l \in \mathbb{L}_n \rangle_{\mathbb{Z}}\ . \tag{1.4}$$

The partitioning of the manifold Fl_n of n-dimensional flags into Schubert varieties yields another distinguished basis of the module H_n under the Borel isomorphism, namely the basis $\{X_\pi \mid \pi \in S_n\}$ of Schubert polynomials associated to permutations π in the symmetric group S_n. The relation between the sets S_n and $\mathbb{L}_n$ is given by the following bijection known as the *(Lehmer) code* of a permutation: for $\pi \in S_n$ define the Lehmer code $L(\pi)$ by

$$L(\pi) = \overline{l_{n-1} \ldots l_1 l_0} \text{ with } l_{n-\nu}(\pi) := |\{\ j \mid \nu < j,\ \pi(\nu) > \pi(j)\}| \ \ (\nu = 1, \ldots, n)\ ,$$

e.g. $L(361542) = [240210]$ and $L(1257346) = [0023000]$. Subsequently we will often identify an element $l = l_1 \ldots\ l_{n-1} \in \mathbb{L}_n$ and a Lehmer code $\overline{l_{n-1} \ldots\ l_1 l_0}$. The permutation $\omega_n := n\ n-1 \ldots\ 1$ of maximal length in S_n and the sequence $\delta_n := 1 \ldots\ n-1 \in \mathbb{L}_n$ ("$\delta_n = L(\omega_n)$") are of special importance.

Bernstein, Gelfand, and Gelfand (1973), and Demazure (1973-74) introduced Schubert varieties into the Borel picture and describeded a calculus of divided differences for the corresponding Schubert classes, whereas A. Lascoux and M.-P. Schützenberger found the remarkable polynomial representatives of Schubert classes in type A, i.e. the Schubert polynomials for the symmetric groups, for which they developed (mainly 1982 - 1987) the fundamentals of the present algebraic-combinatorial theory.

They defined the Schubert polynomial associated to a permutation $\pi \in S_n$ by

$$X_\pi := \partial_{\pi^{-1}\omega_n} x^{\delta_n}\ , \tag{1.5}$$

where ∂_π for an arbitrary permutation π is a sequence of divided difference operators $\partial_k := (id - \sigma_k)/(x_k - x_{k+1})$: k is a natural number; the elementary transposition $\sigma_k = (k, k+1)$ acts as operators transposing the variables x_k and x_{k+1} of a polynomial $f \in \mathbb{Z}[x] := \mathbb{Z}[x_1, x_2, x_3, \ldots]$; and $\partial_\pi := \partial_{a_1} \ldots \partial_{a_p}$, where $a_1 \ldots a_p$ is any reduced sequence for π. For more information about the 'Schubert calculus' and the Lascoux-Schützenberger theory of Schubert polynomials see e.g. [Hi], [LS1,L] (and references therein), [M1, M2, W1].

In [LS2] Lascoux and Schützenberger have recognized also a third kind of basis for the space H_n via duality (cf. Sec.3), which seemed to be less interesting and has not been investigated until its recent appearance in [FGP], which revealed its importance in the context of quantum Schubert polynomials. This third kind of basis is the basis of *standard elementary monomials* (SEM):

$$\{e_l \mid l \in L_n\} \quad \text{with} \quad e_l = e_{l_1 l_2 \ldots\, l_{n-1}} := e_{l_1}^{(1)} e_{l_2}^{(2)} \ldots\, e_{l_{n-1}}^{(n-1)} \, . \tag{1.6}$$

Every polynomial in $\mathbb{Z}[x]$ can be expressed as the sum of products of elementary symmetric polynomials, because $x_m = e_1^{(m)} - e_1^{(m-1)}$. And any product of elementary symmetric polynomials can be 'straightened' to a $\mathbb{Z}$-linear combination of SEM by the repeated application of the following straightening formula ([FGP, (4.3)]):

$$e_i^{(k)} e_j^{(k)} = e_i^{(k+1)} e_j^{(k)} + \sum_{h \geq 1} e_{i-h}^{(k+1)} e_{j+h}^{(k)} - \sum_{h \geq 1} e_{i-h}^{(k)} e_{j+h}^{(k+1)} \quad (i, j, k \geq 1). \tag{1.7}$$

Since the upper index is increased by application of this formula one looks first for the minimal upper index occurring twice in a product of elementary symmetric polynomials. The divided difference operator ∂_k commutes with $e_i^{(n)}$ unless $i = k$, in which case one has $\partial_k e_k^{(n)} = e_{k-1}^{(n-1)}$ $(n \in \mathbb{N})$. Hence the SEM expansions of Schubert polynomials

$$X_\pi = \sum_{l \in \mathbb{L}_n} \alpha_l(\pi)\, e_l \quad \text{with } \alpha_l(\pi) \in \mathbb{Z} \tag{1.8}$$

can be computed in a simple manner from the definition (1.5), where $x^{\delta_n} = e_{12 \ldots n-1} = e_{\delta_n}$. By the results of [W1, Sec.3] this can furthermore be done recursively: in order to compute X_π for some $\pi \in S_{n+1}$ one easily determines a permutation $\bar{\pi} \in S_n$ and a natural number $k \leq n$, such that

$$X_\pi = \partial_k \partial_{k+1} \ldots \partial_n \, (e_n^{(n)} X_{\bar{\pi}}) \, ,$$

namely $k = \pi^{-1}(1)$ and $\bar{\pi}$ is the same as π with the 1 at place k removed and all remaining entries diminished by 1, e.g. for $\pi = 3517264$ one has $k = 3$ and $\bar{\pi} = 246153$.

If now the SEM expansion of an ordinary Schubert polynomial is known, then the 'quantisation' is achieved — this is the main result of [FGP] — by simply substituting every elementary symmetric polynomial $e_k^{(m)}$ appearing in a SEM expansion by its quantum counterpart $\widetilde{e}_k^{(m)}$. The latter polynomials can be computed algebraically as the coefficients of the characteristic polynomials of certain matrices or by the following combinatorial rule:

If the variable x_j is represented by the 'monomer' (or singleton) $\{j\}$, then all summands in $e_i^{(m)}$ can be understood as disjoint coverings of the integer nodes of the line segment $[1, m]$ with i monomers (or simply as i-element subsets of $\{1, \ldots, m\}$); if one adds now 'dimers' $\{j, j+1\}$ corresponding to variables q_j of weight 2, then all the summands in $\widetilde{e}_i^{(m)}$ can be understood as disjoint coverings of i nodes of $[1, m]$ with monomers and dimers. One of course wonders, whether the obvious generalization by adding 'trimers' and '4-mers' etc. has a meaning in terms of quantum cohomology.

Unfortunately the straightening approach for the computation of the SEM expansions described above does not give the slightest clue about the structure or properties of the result obtained.

In the next section we show that for *symmetric* Schubert polynomials, which are exactly the Schur polynomials, it is possible to say quite a lot about this structure: a simple variant of the Jacobi-Trudi determinantal formula for Schur polynomials (2.3) is used to show that the coefficients α_l appearing in the SEM expansions of $s_\lambda^{(m)}(x)$ for fixed partition λ are independent of m and either 0, 1, or -1 (Cor.2.2-3). These coefficients can also be derived by a combinatorial rule involving posets $\mathbb{D}(\lambda)$ of staircase box diagrams (SBD) (Thm.2.7). For all partitions of a given length there are only finitely many types of poset structures $\mathbb{D}(\lambda)$, which we show to be isomorphic to certain explicitly constructed intervals in Bruhat order on permutations (Cor.2.11). Moreover the posets $\mathbb{D}(\lambda)$ are rank symmetric (Prop.2.13).

The third section gives a coherent account of the mutual expansion formulas for the above introduced three kinds of bases: the monomial, Schubert polynomial, and SEM bases. All of these formulas appeared in [KM] in the quantized form, and most of them can be traced back via [M2] to the original papers of Lascoux and Schützenberger. The basic formula, which generalizes the determinantal formula (2.3) of the Schur case, is (3.11) for arbitrary Schubert polynomials. Unfortunately (3.11) is much less explicit than (2.3), and there remains a lot to be understood:

In Section 4 we show that the coefficients α_l appearing in the SEM expansion of an arbitrary Schubert polynomial are "essentially" independent of the number of variables involved (Prop.4.1), thus enabling the definition of 'basic SEM expansions' (Def.4.3). Prop.4.5 generalizes the following well known fact about the expansion of Schubert polynomials into (ordinary) monomials: let $\pi \in S_n$, then

$$X_\pi = x^{L(\pi)} + \sum \alpha_l \; x^l \; ,$$

where the sum is taken over all $l \in \mathbb{L}_n$, which are lexicographically smaller than the element $\bar{l} \in \mathbb{L}_n$ associated to $L(\pi)$. In case of the SEM expansions one can replace the lexicographical order '$<_{lex}$' by a stronger order relation '$\prec$' defined with the help of 'raising operators' R_{ij} (Def.4.4).

In Section 5 we conjecture a generalization of the poset structures $\mathbb{D}(\lambda)$ for arbitrary Schubert polynomials (Conj.5.2) and describe how these more general posets might be used to understand the SEM expansions of Schubert polynomials as 'alternating approximations' (Conj.5.4). For the multiplicities of the expansion coefficients α_l in (1.8) we conjecture a 'fast decay' (Conj.5.6) and a surprising symmetry property (Conj.5.7). Even more astonishing are the properties of the numbers l_i occurring in the standard elementary monomials of the SEM expansions (Thm.5.8 and Conjectures 5.9-10). Moreover, the proven and in particular the conjectured properties of SEM expansions should have a meaning in terms of quantum field theories.

2. The expansion of Schur polynomials into SEM

First of all we recall some results about the connection between Schur and Schubert polynomials, between partitions and Grassmannian permutations. A permutation π is called *Grassmannian* iff it has a unique descent at place m, i.e. $\pi(m) > \pi(m+1)$, or equivalently iff there is a partition $\lambda \equiv \lambda_1 \dots \ \lambda_n$ (with $\lambda_1 \geq \dots \ \geq \lambda_n > 0$) and a natural number $m \geq n$ such that

$$\pi \equiv \pi(\lambda, m) := L^{-1}\overline{(\underbrace{0 \dots \ 0\ \lambda_n \ \dots \ \lambda_1}_{m}\underbrace{0 \dots \ 0}_{\geq \lambda_1})} \ . \tag{2.1}$$

Then a result of fundamental importance is

$$X_{\pi(\lambda,m)} = s_\lambda^{(m)}(x) \ , \tag{2.2}$$

where $s_\lambda^{(m)}(x) = s_\lambda(x_1, \dots, x_m)$ is the Schur polynomial in the variables $x_1, \dots, x_m$. For example: $\pi = 1257346 = L^{-1}(0023000)$ is Grassmannian and $X_{1257346} = s_{32}^{(4)}(x)$. The following determinantal formula appeared in [K, Thm.1] in the quantized form with a quite complicated proof.

Theorem 2.1. *Let λ be a partition of length $l(\lambda) = n > 0$ and λ' its conjugate. Then for all $m \geq \lambda_1$:*

$$s_{\lambda'}^{(m)}(x) = \det\left(e_{\lambda_i - i + j}^{(m+j-1)}\right)_{1 \leq i,j \leq n} \ . \tag{2.3}$$

Proof. The number of variables of the entries in columns $1, \dots, n$ of the above determinant is $(m, m+1, \dots, m+n-1)$. Applications of the recursion

$$e_i^{(m)} = e_i^{(m-1)} + x_m e_{i-1}^{(m-1)}$$

to column n gives $(m, m+1, \dots, m+n-3, m+n-2, m+n-2)$, than to columns $n-1$ and n gives $(m, m+1, \dots, m+n-4, m+n-3, m+n-3, m+n-3)$, etc., and finally to columns $2, 3, \dots, n$ the well known Jacobi-Trudi formula (cf. [M3,Sa])

$$s_{\lambda'}^{(m)}(x) = \det\left(e_{\lambda_i - i + j}^{(m)}\right)_{1 \leq i,j \leq n} \ .$$

□

Subsequently we will often use the notation

$$(2.4)\qquad e_{(m),\,\dots} := e_{\underbrace{0\dots0}_{m},\,\dots}\ .$$

Corollary 2.2. *With the notations of Theorem 2.1,* (1.6) *and* (2.4) *one has*

$$s^{(m)}_{\lambda'}(x) = \sum_{\sigma\in S_n} sign(\sigma)\ e_{(m-1),\lambda_{\sigma(1)}-\sigma(1)+1\dots\lambda_{\sigma(n)}-\sigma(n)+n}\ .$$

Corollary 2.3. *With the notations of Theorem 2.1,* (1.6) *and* (2.4) *one has for all* $m \geq \lambda_1$*:*

$$(2.5)\qquad s^{(m)}_{\lambda'}(x) = \sum_{l\in\mathbb{L}_n} \alpha_l\ e_{(m-1),l_1\dots l_n}\quad \textit{with}\quad \alpha_l \in \{+1, 0, -1\}\ ,$$

i.e. the expansion coefficients α_l *do not depend on* m*.*

Proof. The numbers $\lambda_i - i$ for $i = 1, \dots, n$ are all different. □

We describe now a combinatorial rule, which generates the expansion (2.5).

Definition 2.4. *A* staircase box diagram (SBD) D *of degree* n *and weight* N *is a subset of cardinality* N *of the set*

$$\{(i,j) \mid 1 \leq j \leq i,\ i = 1, \dots, n\}$$

depicted by unit square boxes with center points in an $\mathbb{N}\times\mathbb{N}$ *grid. A* SBD D *is* admissible, *if*

$$(i,j) \in D \Longrightarrow \{(i,j') \mid j \leq j' \leq i\} \subset D\ .$$

For a partition λ *of length* s *and weight* $|\lambda| = N$ *the* SBD D_λ *is defined as the admissible* SBD *, which contains exactly* λ_i *boxes in row* $n - s + i$ $(i = 1, \dots, s)$*, such that rows* $1, \dots, n - s$ *remain empty.*

Definition 2.5. *An* admissible move *is a move, which transfers* $h \geq 1$ *boxes in their respective columns from the row* i *of an admissible* SBD D *to row* $i' > i$*, such that the resulting* SBD D' *is again admissible and the following condition is satisfied:*

if there is a ν *with* $i < \nu < i'$*, then all places in row* ν *of* D *traversed by the* h *boxes are either empty or occupied by other boxes; more precisely: if*

$$(2.6)\qquad a_i \equiv a_i(D) := |\{j \mid (i,j) \in D\}|$$

denote the row weights *of* D*, then the move transforming the admissible* SBD D *with row weights* $(\dots, a_i, \dots, a_{i'}, \dots)$ *into the admissible* SBD D' *with row weights* $(\dots, a_i - h, \dots, a_{i'} + h, \dots)$ *is admissible if and only if*

$$(2.7)\qquad i < \nu < i' \implies [\ a_\nu \geq a_i + \nu - i\quad \textit{or}\ a_\nu \leq a_i - h + \nu - i\ ]\ .$$

Definition 2.6. *For every partition* λ *of length* n *and weight* $|\lambda| = N$ *the* λ-derived poset $\mathbb{D}(\lambda)$ *is defined as the set of all* SBD *, which can be derived from* D_λ *by admissible moves, where* D_λ *is the bottom element and the covering relations are given by the admissible moves.* $\mathbb{D}(\lambda)$ *has the rank function 'rk' with* $rk(D_\lambda) = 0$*.*

Recall from [Hu] that the *Bruhat order* '$<_B$' on S_n is the transitive closure of the following covering relation: "$\bar{\pi}$ covers π" if and only if

(2.8)
$$\pi' = \pi \circ (j,j'),\ j < j',\ \pi(j) < \pi(j') \text{ and } |\{\nu \mid j < \nu < j', \pi(j) < \pi(\nu) < \pi(j')\}| = 0\ .$$

In other words: the transposition (j,j') increases the length of π by 1, because it introduces exactly one more inversion to the sequence of numbers of π.

Theorem 2.7. *Let λ be a partition of length $l(\lambda) = n > 0$ and λ' its conjugate. Using Definitions* 2.4-6 *above one has for all $m \geq s$:*

$$s^{(m)}_{\lambda'}(x) = \sum_{D \in \mathbb{D}(\lambda)} (-1)^{rk(D)}\ e_{(m-1),a_1(D)\dots a_n(D)}\ . \tag{2.9}$$

Moreover $\mathbb{D}(\lambda)$ is isomorphic to a principal order ideal in Bruhat order on S_n, i.e. to an interval $[id_n, \widehat{\pi}(\lambda)]$, where $\widehat{\pi}(\lambda)$ is constructed after formula (2.11) *below.*

Proof. Let $M(\lambda)$ be the matrix of degrees appearing in the determinant of (2.3), i.e.

$$M(\lambda) = (\lambda_i - i + j)_{1 \leq i,j \leq n}\ . \tag{2.10}$$

Let $a = a_1, \dots, a_n$ be a word, where every a_j is chosen from the column j of $M(\lambda)$. Since all entries of the column j are different, one gets a unique word $i(a) \equiv i_1(a) \dots i_n(a)$ by associating to every a_j it's row number $i_j(a) \equiv i(a_j)$. We call a word a λ-*admissible*, if it is of the form $a = (\lambda_{\sigma(1)} - \sigma(1) + 1)\ \dots\ (\lambda_{\sigma(n)} - \sigma(n) + n)$ for some $\sigma \in S_n$ and all entries $\lambda_{\sigma(j)} - \sigma(j) + j$ are non-negative. Clearly an $i(a)$ associated to a λ-admissible word is a permutation in S_n, which we denote by $\pi(a)$. The set of λ-*admissible permutations*

$$\Pi(\lambda) := \{\pi(a) \mid a \text{ is } \lambda\text{-admissable } \} \tag{2.11}$$

now forms an interval $[id_n, \widehat{\pi}(\lambda)]$ in Bruhat order of S_n, where $\pi(\lambda_1 \dots\ \lambda_n) = id_n$ is the bottom element, and where the top element $\widehat{\pi}(\lambda)$ is the permutation associated to the word a, which is constructed as follows: a_1 is the maximal non-negative number in column 1, a_2 is the maximal non-negative number in column 2, which is not in row $i(a_1)$, a_3 is the maximal non-negative number in column 3, which is not in rows $i(a_1), i(a_2)$, etc. . It remains to be shown that

$$\pi(a) \in \Pi(\lambda) \iff \pi(a) \leq_B \widehat{\pi}(\lambda)\ . \tag{2.12}$$

Assume that $\pi \in \Pi(\lambda)$, then by the construction of $\widehat{\pi}(\lambda)$ every λ-admissible π can be reached form $\widehat{\pi}(\lambda)$ trough a reduction of the number of inversions, i.e. $\pi \in \Pi(\lambda) \Longrightarrow \pi \leq_B \widehat{\pi}(\lambda)$ by (2.8). If on the other hand one goes down in Bruhat order from $\widehat{\pi}(\lambda)$, then one remains in $\Pi(\lambda)$, because the set

$$M^+(\lambda) := \{(i,j) \mid (M(\lambda))_{ij} = \lambda_i - i + j \geq 0\}$$

has the following obvious convexity property: $(i,j) \in M^+(\lambda) \Longrightarrow (i,j') \in M^+(\lambda)$ for all j' with $j \leq j' \leq n$. In other words: the set

$$M^-(\lambda) := \{(i,j) \mid 1 \leq i,j \leq n,\ (i,j) \notin M^+(\lambda)\} \tag{2.13}$$

has the shape of a partition

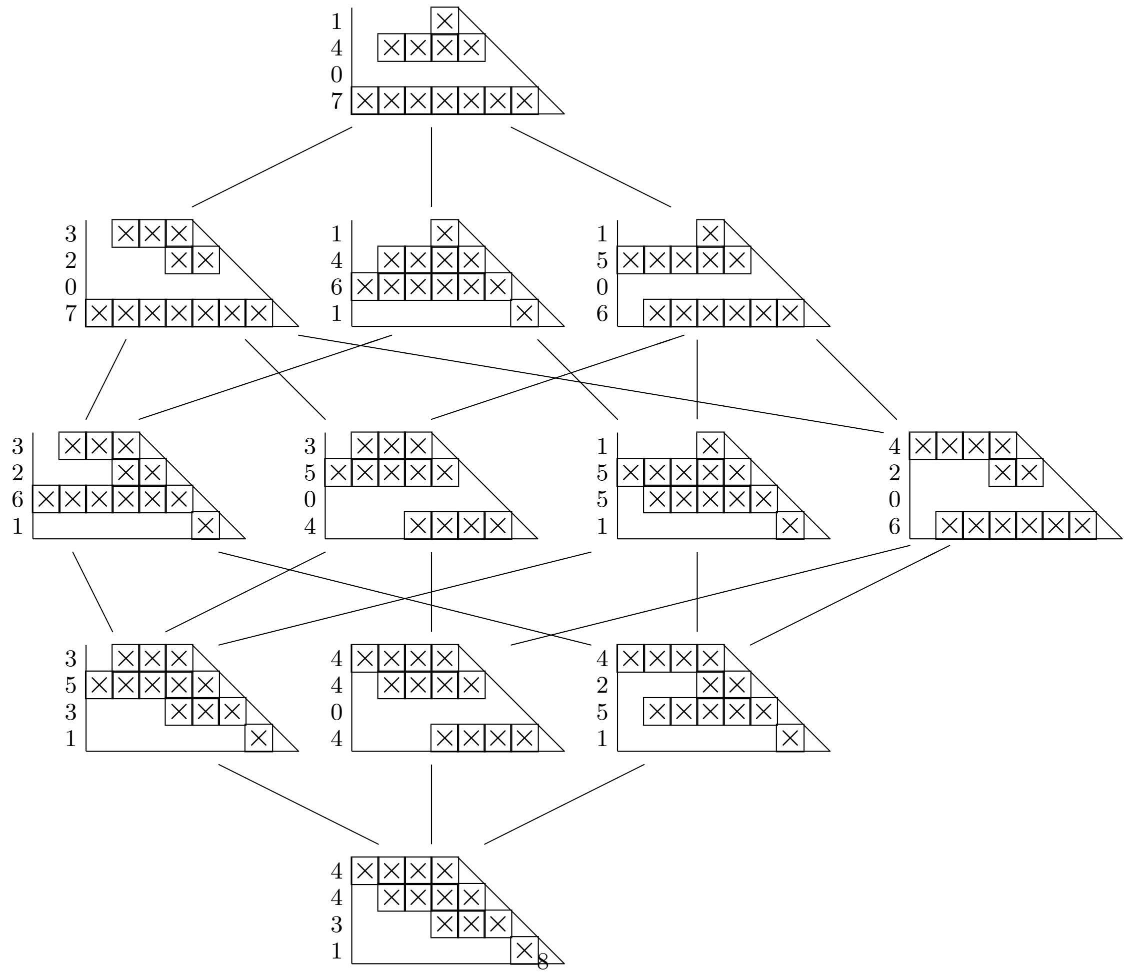
$$\rho \equiv \rho(\lambda) = \rho_1 \rho_2 \dots \qquad \text{with} \qquad \rho_j := \{ i \mid (i,j) \in M^-(\lambda) \} \ . \tag{2.14}$$

This establishes (2.12). — Now (2.9) can be proved by an easy induction: since D_λ and id_n obviously correspond, we have to show that an admissible move from D to D' in $\mathbb{D}(\lambda)$ corresponds to $\pi(a(D))$ being covered by $\pi(a(D'))$ (in Bruhat order on $\Pi(\lambda)$). This amounts to showing the equivalence of conditions (2.7) and (2.8). Using the notation of Def.2.5, the notation $i(a) \equiv i_1(a) \dots i_n(a)$ as explained above, and the structure of the matrix $M(\lambda)$, one sees that

$$\begin{aligned} a_\nu \geq a_j + \nu - j \quad &\Longleftrightarrow \quad i(a_j) > i(a_\nu) \quad \text{and} \\ a_\nu \leq a_j - h + \nu - j \quad &\Longleftrightarrow \quad i(a_{j'}) < i(a_\nu) \quad \text{, i.e.} \\ (2.7) \quad &\Longleftrightarrow \quad [i(a_j) > i(a_\nu) \ \text{ and } \ i(a_{j'}) < i(a_\nu)] \quad \Longleftrightarrow \quad (2.8)\ , \end{aligned}$$

which completes the proof of Thm.2.7 . $\square$

Example 2.8. Let $\lambda = 4^2 31$. Then $\mathbb{D}(\lambda)$ is:



Moreover $M(\lambda) = \begin{pmatrix} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ -2 & -1 & 0 & 1 \end{pmatrix}$ and $\rho(\lambda) = 1^2$.

$\widehat{\pi}(\lambda) = 3241$ is associated to the λ-admissible word 1407. The admissible move $3531 \longrightarrow 1551$ is of type '$a_\nu \geq a_i + \nu - i$', and the move $4251 \longrightarrow 3261$ of type '$a_\nu \leq a_i - h + \nu - i$', whereas the move $4431 \longrightarrow 1461$ is not admissible.

Note that the interval $[id_4, \pi(3207)]$ embedded into $\Pi(\lambda)$ is order-isomorphic to $\mathcal{B}(3)$, the Boolean lattice on 3 elements.

Remark 2.9. (a) "Nearly" admissible moves, i.e. moves failing to satisfy only condition (2.7), are non-covering transitions, which stay in $\mathbb{D}(\lambda)$.

(b) In accordance with part (a) the top element of $\mathbb{D}(\lambda)$ can be computed from D_λ by shifting as many boxes as possible to lower rows, such that the resulting diagram is admissible.

(c) The partition $\rho(\lambda)$ associated by (2.14) to any partition λ can be computed as follows: let $n = l(\lambda)$ and $\mu_i := \max(0, n - i - \lambda_{n-i+1})$ for $i = 1, \dots, n$; then ρ is the conjugate of μ.

(d) Part (c) above shows that in general there are infinitely many partitions λ, which have the same $\rho(\lambda)$.

(e) For every partition λ the permutation $\widehat{\pi} \equiv \widehat{\pi}(\lambda)$ of Thm.2.7 can be computed as follows: let $l(\lambda) = n$ and $l(\rho) = s \leq n - 2$; determine $\rho(\lambda)$ as in (c) above, and set $\widehat{\pi}(1) = n - \rho_1$; for $j = 2, \dots, s$ set $\widehat{\pi}(j) = \max(\{1, \dots, n\} \setminus \{\widehat{\pi}(1), \dots, \widehat{\pi}(j-1)\})$, if $\rho_j = \rho_{j-1}$, and $\widehat{\pi}(j) = n - \rho_j$ otherwise.

Definition 2.10. *Let $n \in \mathbb{N}$. The poset of n-*remainders *$R(n)$ is the set of all partitions $\rho \equiv \rho_1\rho_2\dots$, such that $\rho_j \leq n - 1 - j$ for $j = 1, 2, \dots$ and*

$$\rho_j \geq \rho_{j+1} \geq \rho_j - 1 \tag{2.15}$$

ordered by the inclusion of Ferrer shapes and the empty partition as minimal element.

*The poset of n-*maximal permutations *$\widehat{\Pi}(n)$ is the set*

$$\{\pi \in S_n \mid \pi = L^{-1}(\delta_n - \rho) \;\; \textit{for } \rho \in R(n)\}$$

ordered by Bruhat order.

Corollary 2.11. *For every $n \in \mathbb{N}$ there is an anti-isomorphism (order reversing bijection) between the posets $R(n)$ and $\widehat{\Pi}(n)$. The latter poset is isomorphic to the set of all poset structures $\{\ \mathbb{D}(\lambda) \mid l(\lambda) = n\ \}$ ordered by embeddings, which fix the bottom elements.*

The bottom element of $R(n)$ is the empty partition $\emptyset_n$, which corresponds to $[id_n, \omega_n] \cong S_n$, i.e. the full Bruhat ordered symmetric group of degree n.

The top element of $R(n)$ is the partition $\delta_{n-2} := n-2\ n-3\ \dots\ 1$, which corresponds to $[id_n, 2\ 3 \dots\ n\ 1] \cong \mathcal{B}(n-1)$, i.e. the Boolean lattice on $n-1$ elements.

The set of all poset structures $\{\ \mathbb{D}(\lambda) \mid l(\lambda) = n,\ n \in \mathbb{N}\ \}$ can be ordered by embeddings, which fix the bottom elements, where one uses the natural embeddings of the symmetric groups: $S_n \hookrightarrow S_{n+1},\ \pi \mapsto \pi(1) \dots \pi(n)\ (n+1)$.

Proof. Condition (2.15) is necessary in order to avoid equal rows in the matrix $M(\lambda)$. (2.17) below shows that $\widehat{\Pi}(n)$ is in fact the set of maximal elements $\widehat{\pi}$ associated to partitions λ of length n. The rest is immediate from Thm.2.7 and Rem.2.9 except for the fact that $[id_n, 2\ 3 \dots\ n\ 1] \cong \mathcal{B}(n-1)$. To see the latter observe that $2\ 3 \dots\ n\ 1$ is uniquely represented by the reduced word $\sigma_1\sigma_2 \dots\ \sigma_{n-1}$, where $\sigma_j = (j, j+1)$. By the 'subword property' of Bruhat order (see [Hi, Cor.I.6.5-6] or [Hu, Thm.5.10]) there exists a bijection between all permutations in the interval $[id_n, 2\ 3 \dots\ n\ 1]$ and the subwords of $\sigma_1\sigma_2 \dots\ \sigma_{n-1}$ respective the subsets of $\{1, \dots, n-1\}$. $\square$

Remark 2.12. (1) $R(n)$ is empty for $n = 1, 2$.
(2) $R(3) = \{1, \emptyset\}$ and $R(4) = \{21, 2, 1, \emptyset\}$, both totally ordered.
(3) By (2.15) the Hasse diagrams of $R(2m-1)$ and $R(2m)$ contain isomorphic copies of $\mathcal{B}(m)$, but not of $\mathcal{B}(m+1)$.

We conjecture that all poset structures $\mathbb{D}(\lambda)$ respective all intervals $[id, \widehat{\pi}(\lambda)]$ are self dual (anti-isomorphic to itself), and prove the following weaker statement:

Proposition 2.13. *The posets $\mathbb{D}(\lambda)$ are rank symmetric for all partitions λ.*

Proof. Let $n = l(\lambda)$, $s = l(\rho)$, $\widehat{\pi} \equiv \widehat{\pi}(\lambda)$, and $r = l(\widehat{\pi}(\lambda))$, i.e. r is the rank of $\mathbb{D}(\lambda)$. Let furthermore w_ν be the number of elements in $\mathbb{D}(\lambda)$ of rank ν: $w_\nu := |\{a \in \mathbb{D}(\lambda) \mid rk(a) = \nu\}|$. Then the assertion is

$$w_\nu = w_{r-\nu} \quad \text{for } \nu = 0, 1, \dots, r. \tag{2.16}$$

We show first that

$$L(\widehat{\pi}) = \delta_n - \rho\ . \tag{2.17}$$

Recall Rem.2.9(e) and (2.15). Let $1 = j_1 < j_2 < \dots < j_{\rho_1} < n$ be the sequence of numbers such that $\rho_{j_\nu} = \rho_{j_{\nu-1}} - 1$ for $\nu = 2, \dots, \rho_1$ and $1 < j'_1 < j'_2 < \dots < j'_{n-\rho_1} = n$ the complementary sequence. Then

$$\widehat{\pi}(j_1) < \widehat{\pi}(j_2) < \dots < \widehat{\pi}(j_{\rho_1}) = n \ \text{ and } \ \widehat{\pi}(1) > \widehat{\pi}(j'_1) > \widehat{\pi}(j'_2) > \dots > \widehat{\pi}(j'_{n-\rho_1}) = 1\ .$$

Since on the other hand by Rem.2.9(e) $\widehat{\pi}(j_\nu) = n - \rho_{j_\nu}$, it follows from the definition of the Lehmer code that

$$l_{j_\nu}(\widehat{\pi}) = |\{j'_t \mid j'_t > j_\nu\}| = n - j_\nu - |\{j_t \mid j_t > j_\nu\}| = n - j_\nu - \rho_\nu$$

and similarly

$$l_{j'_\nu}(\widehat{\pi}) = |\{j'_t \mid j'_t > j'_\nu\}| = n - j'_\nu - \rho_\nu\ ,$$

which completes the proof of (2.17). We generalize now (2.17) by defining for every $\pi \in \mathbb{D}(\lambda)$ the dual π^* as

$$\pi^* = \psi_\rho(\pi) := L^{-1}(L(\pi\omega_n) - \rho) \ . \tag{2.18}$$

(Note $(id_n)^* = \widehat{\pi}$.) In order to prove the assertion (2.16) it remains to show that for π of rank $0, \dots, \lfloor r/2 \rfloor$ one has

$$\pi \in \mathbb{D}(\lambda) \iff \pi^* \in \mathbb{D}(\lambda) \ \text{ and} \tag{2.19}$$

$$\pi \text{ covers } \pi' \iff (\pi')^* \text{ covers } \pi^* \ . \tag{2.20}$$

It is convenient to set $\rho_{s+1} = \cdots = \rho_n = 0$. For (2.19) observe now that $\pi^* \in \mathbb{D}(\lambda)$ iff for all j: $\pi^*(j) \le n - \rho_j$. Using the j_ν and j'_ν as introduced above this is equivalent to $l_j(\pi^*) \le n - j - \rho_j \iff l_j(\pi\omega_n) \le n - j$, which is true for all π. For (2.20) note that π covers π' in Bruhat order iff $\pi'\omega_n$ covers $\pi\omega_n$ in Bruhat order, and that the 'shift' by ρ does not change the covering relation.

We have actually proved that the "lower half" and "upper half" of $\mathbb{D}(\lambda)$ are anti-isomorphic, but difficulties with the "middle part" prevent us from showing the conjectured self duality of $\mathbb{D}(\lambda)$. □

3. Orthogonality and expansion formulas in H_n

In the last section the whole development was based on the determinantal formula (2.3), which itself is a simple variant of the Jacobi-Trudi formula. For arbitrary Schubert polynomials the basic formula is now (3.11) below, which appeared in [KM, Thm.6] along with many other (known) expansion formulas in quantized form. For the convenience of the reader and in view of the importance of (3.11) we give in this section a concise treatment of expansion formulas as conceived already by Lascoux and Schützenberger. 'Orthogonality' in the Schubert case is very similar to orthogonality in the Schur case (see [M3, Sec.I.4]).

The *Cauchy formula* (cf. [M3, (5.10)]) for Schubert polynomials is given by

$$\sum_{\pi \in S_n} X_\pi(x)\ X_{\pi\omega_n}(y) = \sum_{1 < i+j \le n} (x_i + y_j) =: \Delta_n(x,y) \ . \tag{3.1}$$

Lemma 3.1.

$$\Delta_n(x,y) = \sum_{l \in \mathbb{L}_n} e_l(x)\ y^{\delta_n - l} = \sum_{l \in \mathbb{L}_n} e_{\delta_n - l}(x)\ y^l \quad \textit{for all } n \in \mathbb{N}. \tag{3.2}$$

Proof. Expand the left side of (3.2) into monomials $y^{\delta_n - l}$ with coefficients $\beta_l \in P_n := \mathbb{Z}[x_1, \dots, x_n]$. To every factor $y_{n-\nu}^{\nu - l_\nu}$ ($\nu \in \{1, \dots, n-1\}$) of $y^{\delta_n - l}$ there corresponds in every monomial of β_l (written with coefficients 0 or $+1$) a product of l_ν different x_i's with $1 \le i \le \nu$, i.e. every summand of $e_{l_\nu}^{(\nu)}$ occurs for this $y_{n-\nu}^{\nu - l_\nu}$. Multiplying together all factors for $\nu = 1, \ldots, n-1$ gives $\beta_l = e_l$. This proves the first equality; the second is immediate from the involution $l \mapsto \delta_n - l$ of $\mathbb{L}_n$. $\square$

The identification $\overline{l_{n-1} \ldots\ l_1 l_0} \mapsto l_1 \ldots\ l_{n-1}$ together with the bijection $S_n \longrightarrow \mathbb{L}_n,\ \pi \mapsto L(\pi)$ shows, that (3.2) is of the same type as (3.1):

$$\Delta_n(x,y) = \sum_{\pi \in S_n} e_{L(\pi)}(x)\ y^{L(\omega_n \pi)} \ ,$$

where we have used

$$L(\omega_n \pi) = \delta_n - L(\pi) = L(\omega_n) - L(\pi) \ . \tag{3.3}$$

We moreover need the *scalar product* (symmetric bilinear form) (cf. [M2, (5.2)])

$$\langle\ ,\ \rangle : P_n \times P_n \longrightarrow \Lambda_n := P_n^{S_n},\ (f,g) \mapsto \langle f,g \rangle := \partial_{\omega_n}(fg) \ ,$$

which has the property (cf. [M2, (5.3)(i)])

$$\langle \partial_\pi f, g \rangle = \langle f, \partial_{\pi^{-1}} g \rangle \ \text{ for } \pi \in S_n,\ f,g \in P_n \ , \tag{3.4}$$

and the 'orthogonality' of Schubert polynomials (cf. [M2, (5.4)])

$$\langle X_u, X_v \rangle = \begin{cases} 1 & \text{, if } u = \omega_n v \\ 0 & \text{, otherwise.} \end{cases} \tag{3.5}$$

Since $\{X_\pi \mid \pi \in S_n\}$ is a $\mathbb{Z}$-basis of H_n and H_n is the space of residues of P_n/Λ_n^+, one sees immediately that $\{X_\pi \mid \pi \in S_n\}$ is a Λ_n-basis of P_n, i.e. every $f \in P_n$ has an expansion

(3.6)

$$f = \sum_{\pi \in S_n} \gamma_\pi X_\pi \ \text{ with } \gamma_\pi = \langle f, X_\pi \rangle \in \Lambda_n, \text{ where } (\gamma_\pi \in \mathbb{Z} \text{ for all } \pi) \iff (f \in H_n) \ .$$

Let $\eta : \mathbb{Z}[x] \longrightarrow \mathbb{Z}$ be the $\mathbb{Z}$-algebra homomorphism, which maps $f \in \mathbb{Z}[x]$ onto its constant term.

Lemma 3.2.

$$\langle f, X_\pi \rangle = \eta(\partial_{\omega_n \pi}\ f) \ \text{ for all } f \in H_n,\ \pi \in S_n. \tag{3.7}$$

Proof. We compute directly from the definition (1.5) of Schubert polynomials and (3.4) that

$$\langle f, X_\pi \rangle = \langle f, \partial_{\pi^{-1}\omega_n}\ X_{\omega_n} \rangle = \langle \partial_{\omega_n \pi}\ f, X_{\omega_n} \rangle \equiv \langle g, X_{\omega_n} \rangle \ .$$

But by (3.6) one has

$$\langle g, X_{\omega_n} \rangle = \sum_{\pi \in S_n} \gamma_\pi \langle X_\pi, X_{\omega_n} \rangle = \gamma_{id} = \eta(g) = \eta(\partial_{\omega_n \pi}\ f) \ .$$

$\square$

The following expansion formulas are now immediate:

$$e_l = \sum_{\pi \in S_n} \eta(\partial_{\omega_n \pi}\ e_l)\ X_\pi \ , \tag{3.8}$$

$$x^l = \sum_{\pi \in S_n} \eta(\partial_{\omega_n \pi}\ x^l)\ X_\pi \ . \tag{3.9}$$

The expansion formula (3.11) below connects in a remarkable way all the distinguished bases of H_n considered so far: the Schubert, monomial, and SEM basis.

Theorem 3.3 (KM)**.** *For all $n \in \mathbb{N}$ and $\pi \in S_n$ one has the following expansion formula for the Schubert polynomial X_π into* SEM *(let $|l| := l_1 + \ldots + l_{n-1}$):*

$$X_\pi = \sum_{l \in \mathbb{L}_n} \alpha_l\ e_l \quad \textit{with } \alpha_l = \eta(\partial_{\pi\omega_n}\ x^{\delta_n - l}) \tag{3.10}$$

or without η:

$$X_\pi = \sum_{\substack{l \in \mathbb{L}_n \\ |l| = l(\pi)}} (\partial_{\pi\omega_n}\ x^{\delta_n - l})\ e_l \ . \tag{3.11}$$

Proof. Combination of (3.1) and (3.2) gives

$$\sum_{\pi \in S_n} X_\pi(x)\ X_{\pi\omega_n}(y) = \sum_{l \in \mathbb{L}_n} e_l(x)\ y^{\delta_n - l} \ . \tag{3.12}$$

Changing π to $\bar{\pi}$, multiplying with $X_\pi(y)$, and applying $\partial^{(y)}_{\omega_n}$ (the upper index indicates that the divided difference operators act on the variables y) gives

$$\sum_{\bar{\pi} \in S_n} X_{\bar{\pi}}(x)\ \langle X_{\bar{\pi}\omega_n}(y), X_\pi(y)\rangle = \sum_{l \in \mathbb{L}_n} e_l(x)\ \langle y^{\delta_n - l}, X_\pi(y)\rangle \ .$$

By the orthogonality (3.5) the left hand side of the above equation is seen to be $X_{\omega_n\pi\omega_n}$. For the scalar product of the right hand side one uses (3.7). Equation (3.10) is now immediate by the involution $\pi \mapsto \omega_n\pi\omega_n$ of S_n.

For (3.11) observe that every application of a divided difference operator ∂_k to a homogeneous polynomial in $\mathbb{Z}[x]$ either lowers the degree by exactly 1 or gives zero. This means that the result in case of $|l| > l(\pi)$ is always zero and that η has some effect only for $l \in \mathbb{L}_n$ if $|l| \leq l(\pi)$. Hence the requirement $|l| = l(\pi)$ makes η superfluous. $\square$

The following variants did also appear in [KM] in quantized form:

$$X_\pi = \sum_{l \in \mathbb{L}_n} \eta(\partial_{\pi\omega_n}\ x^l)\ e_{\delta_n - l} \tag{3.13}$$

originates from the second equality of (3.2). The observation that $\Delta_n(x,y) = \Delta_n(y,x)$ together with interchanging x and y variables in (3.2) gives the two formulas

$$X_\pi = \sum_{l \in \mathbb{L}_n} \eta(\partial_{\pi\omega_n}\ e_l)\ x^{\delta_n - l}\ , \tag{3.14}$$

$$X_\pi = \sum_{l \in \mathbb{L}_n} \eta(\partial_{\pi\omega_n}\ e_{\delta_n - l})\ x^{l}\ . \tag{3.15}$$

4. The expansion of Schubert polynomials into sem : results

We first prove a generalization of Cor.2.3, which needs some notational preparation.

A permutation $\pi \in S_n$ is called *unembedded*, if $\pi(1) \neq 1$ and $\pi(n) \neq n$. The reason for this definition is that Schubert polynomials are invariant under the (left) embedding of permutations (as described at the end of Cor.2.10), and that right embedding, i.e.

$$S_n \hookrightarrow S_{n+m},\ \pi^{(m)} \equiv id_m \times \pi := 1 \dots m\ (\pi(1)+m) \dots\ (\pi(n)+m)$$

with $\pi^{(0)} := \pi$, leads to an increase of the number of variables involved without changing the "essential structure" of the polynomial. For example: $\pi \equiv \pi(\lambda, l(\lambda))$ (cf. (2.1)) is unembedded, $\pi^{(m)} = \pi(\lambda, l(\lambda)+m)$, and $X_{\pi^{(m)}} = s_\lambda^{(l(\lambda)+m)}(x)$ (as stated already in (2.2) in a slightly different notation). Recall that the length $l(\pi)$ of an arbitrary permutation $\pi \in S_n$ can be computed as the sum of its Lehmer code entries: $l(\pi) = |L(\pi)|$, and using (1.8) set

$$\beta(\pi) := |\{l \in \mathbb{L}_n \mid \alpha_l(\pi) \neq 0\}|\ .$$

Proposition 4.1. *Let $\pi \in S_n$ be an unembedded permutation. Then there exists an integer $N \equiv N(\pi)$ with $0 \leq N \leq l(\pi)$ such that for all $m \geq N$:*

$$X_{\pi^{(m)}} = \sum_{l \in \mathbb{L}_{n+N}} \alpha_l\ e_{(m-N),l_1 \dots l_{n+N}} \quad \textit{with} \quad \alpha_l \in \mathbb{Z} \quad \textit{and} \tag{4.1}$$

$$\alpha_l \neq 0 \implies l_1 = \dots = l_N = 0\ . \tag{4.2}$$

Moreover the coefficients $\alpha_l^{(m)}$ appearing in the SEM *expansions of $X_{\pi^{(m)}}$ for $0 \leq m < N$ can be recovered from the α_l with $l = l_1 \dots l_{n+N} \in \mathbb{L}_{n+N}$ by setting*

$$\alpha_l^{(m)} = 0, \quad \textit{if } l_{1+N-m} \dots l_{n+N} \notin \mathbb{L}_{n+m}\ ,$$

and otherwise $\alpha_l^{(m)} = \alpha_l$, i.e.

$$\beta(\pi) < \beta(\pi^{(1)}) < \dots < \beta(\pi^{(N)}) = \beta(\pi^{(N+1)}) = \dots \tag{4.3}$$

Moreover for all $\pi \in S_n$ one has

$$\alpha_l = 1\ , \ \textit{if} \ \ l := L(\omega_n \pi \omega_n)\ . \tag{4.4}$$

Proof. Let first $\pi \in S_n$ be arbitrary and $l := L(\omega_n \pi \omega_n) \in \mathbb{L}_n$ (with the usual identification). Then

$$\eta(\partial_{\pi\omega_n}\ x^{\delta_n - l}) = \eta(\partial_{\pi\omega_n}\ x^{L(\pi\omega_n)})$$

by (3.3), and because $\pi \mapsto \pi\omega_n$ is an involution of S_n it is enough to show that

$$\partial_\pi\ x^{L(\pi)} = 1 \quad \text{for all } \pi \in S_n. \tag{4.5}$$

It is known (cf.[W1, Cor.2.11]) that for $\pi \in S_n$ with *Lehmer code* $l = l_{n-1} \dots\ l_1$ a canonical reduced sequence for π can be computed as

$$l_{n-1} \dots 1\ \ l_{n-2} \dots 2\ \ l_{n-3} \dots 3 \quad \dots \quad l_1 \dots n-1, \tag{4.6}$$

where for $\nu = 1, \dots, n-1$ the notation $l_{n-\nu} \dots \nu$ means the sequence of natural numbers descending from $l_{n-\nu}$ to ν, if $l_{n-\nu} \geq 1$, and the empty sequence, if $l_{n-\nu} = 0$, e.g. for $\pi = 4216735$ one has $L(\pi) = \overline{3102200}$ and the reduced word 32125465. Hence equation (4.5) follows from

$$\partial_{i+k-1} \dots \partial_i\ x_i^k = 1 \quad \text{for } i, k \in \mathbb{N}.$$

To see the latter equation note that

$$\partial_i\ x_i^k = x_i^{k-1} x_{i+1}^0 + x_i^{k-2} x_{i+1}^1 + \dots\ + x_i^0 x_{i+1}^{k-1}\ ,$$

and that only the last term may not cancel due to the subsequent application of $\partial_{i+k-1} \dots \partial_{i+1}$. Therefore we have

$$\partial_{i+k-1} \dots \partial_i\ x_i^k = \partial_{i+k-1} \dots \partial_{i+1}\ x_{i+1}^{k-1} = \dots\ = \partial_{i+k-1}\ x_{i+k-1}^1 = 1\ .$$

This proves (4.5) and therefore (4.4).

Assume now that $\pi \in S_n$ is unembedded and set (with $((m), l) := \underbrace{0 \dots 0}_{m}\ l$)

$$\gamma_m := \eta(\partial_{\pi^{(m)}\omega_{n+m}}\ x^{\delta_{n+m} - ((m),l)}) \ \text{ for fixed } \pi, n \text{ and } l \in \mathbb{L}_n. \tag{4.7}$$

We show next that

$$\gamma_1 = \gamma_0\ ,\ \text{ if } l \in \mathbb{L}_n \text{ and } ((1), l) \in \mathbb{L}_{n+1}\ . \tag{4.8}$$

Observe first that

$$x^{\delta_{n+1} - ((1),l)} = x^{\delta_n - l} \cdot (x_1 \dots x_n)$$

and that for $L(\pi) = \overline{l_{n-1} \dots l_0}$ one has

$$L(\pi^{(1)}\omega_{n+1}) = 1_+ L(\pi\omega_n) := \overline{(l_{n-1}+1) \dots (l_0+1)\ 0}\ .$$

The computation of the canonical reduced sequence for $\pi^{(1)}\omega_{n+1}$ according to (4.6) then yields

$$\begin{aligned} &(l_{n-1}+1) \dots 1\ \ (l_{n-2}+1) \dots 2\ \dots\ (l_1+1) \dots\ n = \\ &\qquad (l_{n-1}+1) \dots 2\ \ (l_{n-2}+1) \dots 3\ \dots\ (l_1+1) \dots (n+1)\ 1\ 2\ \dots\ n\ , \end{aligned}$$

whence

$$\partial_{\pi^{(1)}\omega_{n+1}} = 1_+(\partial_{\pi\omega_n})\ \partial_1 \partial_2 \dots\ \partial_n \equiv \partial_+\ \partial_1 \partial_2 \dots\ \partial_n\ ,$$

where of course $\partial_+ = 1_+(\partial_{\pi\omega_n})$ means that every number occurring in the reduced sequence of $\pi\omega_n$ is increased by one. By the product rule for divided differences

$$\partial_k(fg) = (\partial_k f)g + \sigma_k(f)(\partial_k g)$$

we can therefore compute with $f(x_1, \ldots, x_{n-1}) := x^{\delta_n - l}$ that

$$\begin{aligned}
&\eta\,(\partial_+\ \partial_1\partial_2 \ldots\ \partial_n\ ((f(x_1, \ldots, x_{n-1})(x_1 \ldots x_n))) = \\
&\qquad\qquad \eta\,(\partial_+\ \partial_1\partial_2 \ldots\ \partial_{n-1}\ ((f(x_1, \ldots, x_{n-1})(x_1 \ldots x_{n-1}))) = \\
&\eta\,(\partial_+\ \partial_1\partial_2 \ldots\ \partial_{n-2}\ [(\partial_{n-1} f(x_1, \ldots, x_{n-1}))(x_1 \ldots x_{n-1}) + f(x_1, \ldots, x_{n-2}, x_n)(x_1 \ldots x_{n-2})])\ .
\end{aligned}$$

Now the first summand equals

$$\begin{aligned}
&\eta\,(\partial_+\ \partial_1\partial_2 \ldots\ \partial_{n-2}\ (x_1 \ldots x_{n-1})(\partial_{n-1} f(x_1, \ldots, x_{n-1})) = \\
&\qquad\qquad \eta\,(\partial_+\ (x_1 \ldots x_{n-1})\partial_1\partial_2 \ldots\ \partial_{n-2}\ \partial_{n-1} f(x_1, \ldots, x_{n-1})\ ,
\end{aligned}$$

but since η — ∂_+ does not contain ∂_1 — is applied to a polynomial containing x_1, the expression necessarily vanishes. Therefore

$$\begin{aligned}
&\eta\,(\partial_+\ \partial_1 \ldots\ \partial_n\ ((f(x_1, \ldots, x_{n-1})(x_1 \ldots x_n))) = \\
&\qquad \eta\,(\partial_+\ \partial_1 \ldots\ \partial_{n-1}\ ((f(x_1, \ldots, x_{n-2}, x_n)(x_1 \ldots x_{n-2}))) = \ldots\ = \\
&\eta\,(\partial_+\ f(x_2, \ldots, x_n)) = \eta\,(1_+(\partial_{\pi\omega_n})\ f(x_2, \ldots, x_n)) = \eta\,(\partial_{\pi\omega_n}\ f(x_1, \ldots, x_{n-1}))\ .
\end{aligned}$$

Since $l \in \mathbb{L}_n \implies ((1), l) \in \mathbb{L}_{n+1}$ but not necessarily reverse, the equality (4.8) is true only if $l \in \mathbb{L}_n$ *and* $((1), l) \in \mathbb{L}_{n+1}$. Similarly one shows that

$$\gamma_{m+1} = \gamma_m \quad \text{for } m \in \mathbb{N}, \text{ if } l \in \mathbb{L}_{n+m} \text{ and } ((1), l) \in \mathbb{L}_{n+m+1}\ .$$

If $m \geq l(\pi)$ then by (4.1-2) every $l \in \mathbb{L}_{n+m}$ with $\alpha_l \neq 0$ has shown up, which explains the upper bound for N. $\square$

Example 4.2. $\pi = 53421 \in S_5$ is unembedded with $l(\pi) = |\ \overline{42200}\ | = 8$, $\pi^{(0)} = 53421$, $\pi^{(1)} = 164532$, $\pi^{(2)} = 1275643$,$\pi^{(3)} = 12386754$, and $N = 3$. One computes:

$$\begin{aligned}
X_{\pi^{(0)}} =& e_{1134} - e_{0234} \\
X_{\pi^{(1)}} =& e_{01134} - e_{00234} - e_{01044} - e_{01125} + e_{00225} \\
X_{\pi^{(2)}} =& e_{001134} - e_{000234} - e_{001044} - e_{001125} + e_{000225} + e_{001026} + e_{000054} \\
X_{\pi^{(3)}} =& e_{0001134} - e_{0000234} - e_{0001044} - e_{0001125} + e_{0000225} + e_{0001026} + e_{0000054} - e_{0000027}\ .
\end{aligned}$$

For $m > 3$ the expression for $X_{\pi^{(m)}}$ changes only due to prefixing additional $m - 3$ zeroes to the sequences of indices.

Since every permutation is contained in some sequence $(\pi^{(m)})_{m=0,1,2,\ldots}$ for some unembedded $\pi \in S_n$, and since by Prop.4.1 every coefficient in the SEM expansion of $X_{\pi^{(m)}}$ can be easily computed from those of $X_{\pi^{(N)}}$, we make the following

Definition 4.3. *A permutation $\pi^{(N)}$ (in the notation of Prop.*4.1*), a Schubert polynomial $X_{\pi^{(N)}}$, or its* SEM *expansion is called a* basic *permutation, Schubert polynomial, or* SEM *expansion, respectively.*

Definition 4.4. *Let $l \in \mathbb{L}_n$ with $l_i > 0$ and $l_j < j$ for $1 \le i < j \le n-1$. Then there is a well defined operator*

$$R_{ij}(l) = R_{ij}(\dots l_i \dots\ l_j \dots) := \dots (l_i - 1) \dots\ (l_j + 1) \dots \quad ,$$

and we have the order relation $l' \prec l :\iff$ there is a sequence of operators $R_{i_\nu j_\nu}$, such that $l = R_{i_s j_s} \dots\ R_{i_1 j_1}(l')$. The set

$$Cone(l') := \{l \in \mathbb{L}_n \mid l' \preccurlyeq l\} \ ,$$

is called the cone *of $l' \in \mathbb{L}_n$, and the order relation '$\preccurlyeq$' the* cone order.

We denote by $\underline{Cone}(l')$ the ranked poset built on $Cone(l')$ with the partial order induced by the covering relation: "l covers h" iff $l = R_{i,i+1}h$ for some $i < n-1$.

The above poset $\underline{Cone}(l')$ is well defined, because every well defined operator R_{ij} on some $l \in Cone(l')$ can be written as a sequence: $R_{ij} = R_{i,i+1}R_{i+1,i+2} \dots\ R_{j-1,j}$. (Note that writing R_{ij} as $R_{j-1,j} \dots\ R_{i+1,i+2}R_{i,i+1}$ may lead outside of $\mathbb{L}_n$.)

If we write "$l <_{lex} l'$" for "l is lexicographically smaller than l'" (with respect to the the lexicographic order induced by $0 < 1 < 2 < \dots$), then obviously

$$l' \prec l \quad \Longrightarrow \quad l <_{lex} l' \ ,$$

but the converse is in general not true, e.g. $0203 <_{lex} 1013$, but $1013 \not\prec 0203$.

The next proposition generalizes the following well known fact about the expansion of Schubert polynomials into (ordinary) monomials: let $\pi \in S_n$, then

$$X_\pi = x^{L(\pi)} + \sum \alpha_l\ x^l \ ,$$

where the sum is taken over all $l \in \mathbb{L}_n$ with $|l| = l(\pi)$, $l <_{lex} \bar{l} \in \mathbb{L}_n$, $\bar{l} = L(\pi)$, and l componentwise smaller than $1\ 2 \dots n-1$.

Proposition 4.5. *Let $\pi \in S_n$ and $\bar{l} := L(\omega_n \pi \omega_n) \in \mathbb{L}_n$. Then the* SEM *expansion of X_π has the form*

$$X_\pi = e_{\bar{l}} + \sum_{\bar{l} \prec l} e_l \ . \tag{4.9}$$

Proof. By (4.4) and (3.11) above it is enough to show that the $\prec$-order is preserved by the divided difference operators ∂_k ($k = 1, \dots, n-1$), whose application generates all Schubert polynomials from X_{ω_n} (cf. (1.5)), i.e. we prove the '$\prec$' part of (4.9) by (downward) induction over the length of π. For $\pi = \omega_n$ the assertion is clear:

$$X_{\omega_n} = e_{1,\dots,n-1} \ .$$

Assume now that $l(\pi) < l(\omega_n)$. Then there is a k with $\pi(k) < \pi(k+1)$ and it is well known that $X_\pi = \partial_k X_{\pi'}$ with $\pi' := \pi\sigma_k$. Hence it is enough to show that

$$(4.10) \qquad (e_{l'} \prec e_l \ , \ l = R_{ij}(l') \ ) \quad \Longrightarrow \quad \mathrm{cmin}\ \partial_k(e_{l'}) \preccurlyeq \mathrm{cmin}\ \partial_k(e_l) \ ,$$

where we used the induced order on the SEM and the notation 'cmin' for the $\prec$-minimum of a subset of $\mathbb{L}_n$. To show (4.10) observe first that by the straightening rule (1.7) one has for all $a, b \in \mathbb{N}$

$$\mathrm{cmin}\ \partial_k(e_a^{(k-1)} e_{b+1}^{(k-1)}) = \mathrm{cmin}\ (e_a^{(k-1)} e_b^{(k-1)}) = e_{a+b}^{(k)} \ .$$

(In case of $b+1 = 0$ the result is zero.) This implies (4.10), because it is not hard to see that for the $\prec$-minimum of the terms involved one has:
'$R_{ij}\partial_k = \partial_k R_{ij}$' for $k \neq i, j$,
'$R_{ik}\partial_k = \partial_k R_{i(k-1)}$' and '$R_{kj}\partial_k = \partial_k R_{(k-1)j}$' for $i+1 < j$, and that
equality occurs on the r.h.s. of (4.10) for $i+1 = j$. □

Remark 4.6. In fact the terms appearing in the expansion of Schubert polynomials into ordinary monomials lay in the 'cone' of $L(\pi)$ as well, instead of being just lex-greater. This follows immediately from Kohnert's rule for the generation of Schubert polynomials [W2].

The last proposition enables the improvement of the upper bound for the numbers $N(\pi)$ appearing in Prop.4.1:

Corollary 4.7. *Let $l = l_1 \dots\ l_{n-1} := L(\omega_n \pi \omega_n)$, and $l^{(\nu)} := 0 \dots\ 0\ (l_1 + \dots + l_\nu) \dots\ l_{n-1}$ for $\nu = 1, \dots, n-1$. Then*

$$(4.11) \qquad N(\pi) \leq \max_\nu\ \min\ \{ m \mid ((m), l^{(\nu)}) \in \mathbb{L}_{n+m} \} \ .$$

For example (continuing Ex.4.2) let $\pi = 53421$ with $N = 3$ and $l^{(1)} = l = 1134 \in \mathbb{L}_5$. Then $l^{(2)} = 0234 \in \mathbb{L}_5$, $l^{(3)} = 000054 \in \mathbb{L}_7$, and $l^{(4)} = 000000009 \in \mathbb{L}_{10}$. Therefore one gets the bound $N \leq \max\{0, 2, 5\} = 5$ instead of $8 = l(\pi)$.

Remark 4.8. Formulas (4.9) and (3.11) together yield a straightening free way to compute the SEM expansions: while proceeding from one level of $\underline{Cone}(\bar{l})$ to the next (in the notations of Def.4.3 and Prop.4.4) one adds simultaneously terms to the SEM expansion. Since $\underline{Cone}(\bar{l})$ is quite big, this algorithm works much slower than the 'straightening approach', but for large n this may be out-weighted by a lesser amount of memory necessary.

5. The expansion of Schubert polynomials into SEM : conjectures

At present we do not know a generalization of Thm.2.7, but it seems that an analog of the poset $\mathbb{D}(\lambda)$ is still available (Conj.5.2), which indicates how the SEM expansion of an arbitrary Schubert polynomial can be understood (Conj.5.3).

Definition 5.1. *Let $l \in \mathbb{L}_n$, $1 \le i < i' \le n-1$, and $h \le l_i$. Then the transformation $l = \dots, l_i, \dots, l_{i'}, \dots) \mapsto l' = \dots, l_i - h, \dots, l_{i'} + h, \dots$ is called* admissible *if and only if*

$$\text{(1): } l' \in \mathbb{L}_n \quad \text{and} \quad \text{(2): } i < \nu < i' \implies [\ a_\nu \ge a_i + \nu - i \quad \text{or } a_\nu \le a_i - h + \nu - i\ ]\ .$$

The definition says especially that an admissible move is of the form: $l' = (R_{ii'})^h(l)$, i.e. $l' \in Cone(l)$.

It is known that for $\pi \in S_n$ the involution $\pi \mapsto \omega_n \pi \omega_n$ generalizes the conjugation of partitions: $\pi(\lambda', l(\lambda')) = \omega_n \pi(\lambda, l(\lambda))\omega_n$ (cf. [W1, Prop.4.9]).

Conjecture 5.2. *For basic* SEM *expansions (and therefore for all* SEM *expansions) it is possible to define a poset $\mathbb{D}(\pi^{(N)})$ for the previously computed set $\{l \in \mathbb{L}_{n+N} \mid \alpha_l \ne 0\}$ in the following way:*

the bottom element is $l_\pi := ((N), l)$ with $l = L(\omega_n \pi \omega_n)$ (a top element in general does not exist);

l' covers l'' in $\mathbb{D}(\pi)$, if $l', l'' \in \mathbb{L}_{n+N}$ and l' is the result of an admissible transformation applied to l''.

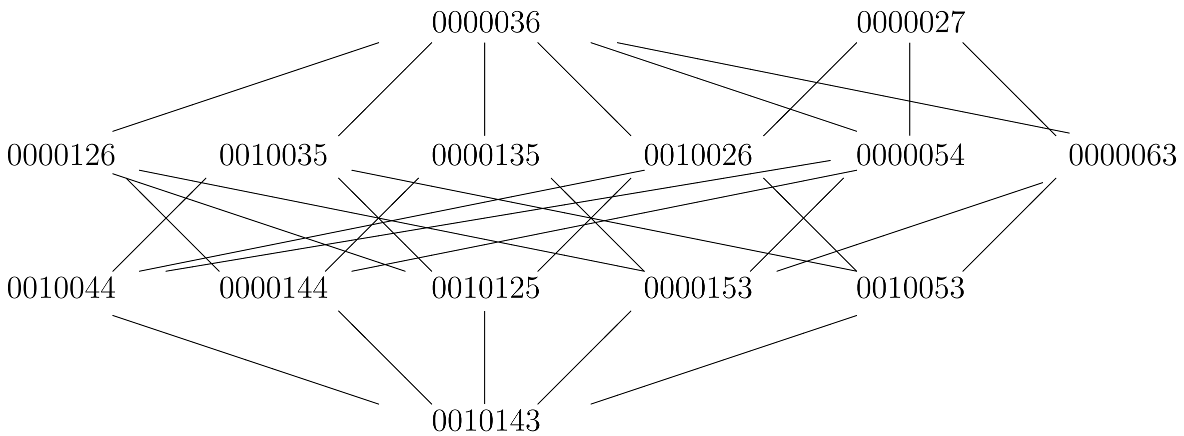
Then the poset $\mathbb{D}(\pi)$ is ranked with $rk(l_\pi) = 0$, where $rk(l)$ is even, if $\alpha_l > 0$, and $rk(l)$ is odd, if $\alpha_l < 0$.

Note that Thm.2.7 in case of a Grassmannian π implies the validity of Conj.5.2. Note further that $\mathbb{D}(\pi)$ is different from $\underline{Cone}(l_\pi)$ even if one restricts the latter to l with $\alpha_l \ne 0$.

Example 5.3. For the unembedded $\pi = 426513$ one has $N(\pi) = 2$ and the basic SEM expansion is

$$\begin{aligned} X_{12648735} = {} & e_{0010143} - e_{0010044} - e_{0000144} - e_{0010125} - e_{0000153} - e_{0010053} + e_{0000126} + e_{0010035} \\ & + e_{0000135} + e_{0010026} + 2e_{0000054} + e_{0000063} - 2e_{0000036} - e_{0000027}\ . \end{aligned}$$

The poset $\mathbb{D}(12648735)$ has the following Hasse diagram:



The significance of the poset $\mathbb{D}(\lambda)$ is clarified by the following

Conjecture 5.4. *Using the notations of Prop.*4.1 *and Conj.*5.2 *above it is possible to define for all permutations* π *the polynomials*

$$\Sigma_{\leq\nu}(\pi) := \sum_{l\in\mathbb{D}(\pi),\ rk(l)\leq\nu} (-1)^{rk(l)}\ |\alpha_l|\ e_l \quad \textit{and}\ \Sigma'_{\leq\nu}(\pi) := \Sigma_{\leq\nu}(\pi) - X_\pi\ .$$

Then all monomials in $(-1)^\nu\ \Sigma'_{\leq\nu}(\pi)$ *have non-negative integer coefficients. Equivalently let*

$$\Sigma_{\geq\nu}(\pi) := \sum_{l\in\mathbb{D}(\pi),\ rk(l)\geq\nu} (-1)^{rk(l)}\ |\alpha_l|\ e_l \quad \textit{and}\ r = rk\ \mathbb{D}(\pi)\ ,$$

then all monomials in $(-1)^{r-\nu}\ \Sigma_{\geq\nu}(\pi)$ *have non-negative integer coefficients.*

Conj.5.3 can be rephrased by saying: $\Sigma_0(\pi) = e_{l_\pi}$ is the "positive SEM closure of X_π", and the introduction of successively higher levels of $l \in \mathbb{D}(\pi)$ in the $\Sigma_\nu(\pi)$ leads to an alternating approximation of X_π. In general the number of terms in the $\Sigma'_\nu(\pi)$ first increases with increasing ν and then decreases to zero.

Remark 5.5. Every e_l with $l \in \mathbb{L}_n$ can be viewed as a collection of SBD D originating from the unique admissible SBD D_l with l_i boxes in row i by moving the boxes in their rows freely to the left. The monomial associated to such a SBD D has as the exponent of x_j the number of boxes in column j of D.

In contrast to this *unrestricted* moves departing from an admissible SBD , Kohnert's rule for the combinatorial generation of Schubert polynomials says (using the conventions of [W2]), that X_π originates from the collection of SBD , which are derived from the Rothe diagram $D(\pi)$ by moving the boxes to the left in their rows and *observing certain restrictions*. ($D(\pi)$ is a SBD of the right degree and weight, but is in general not admissible.) This hitherto unexplored relationship between the alternating approximation of X_π by the $\Sigma'_{\leq\nu}(\pi)$ and Kohnert's rule applied to $D(\pi)$ might give a clue how to prove the above (and below) conjectures or even to extend them to a combinatorial rule for the SEM expansion of Schubert polynomials. We emphasize that an *algebraic* interpretation (and proof) of Kohnert's rule is still unknown.

For every SEM expansion (1.8) of a Schubert polynomial one can define the numbers

$$A_k(\pi) := |\{l \mid \alpha_l(\pi) = k\}| \quad \text{for } k \in \mathbb{Z}\setminus\{0\}\ . \tag{5.1}$$

By Cor.2.3 the following conjecture is obviously true for Grassmannian π.

Conjecture 5.6. *For fixed* π *one has*

$$A_{\pm1}(\pi) \gg A_{\pm2}(\pi) \gg A_{\pm3}(\pi) \gg \dots\ ,$$

where '$a \gg b$*' means "*a *is much greater then* b *or* $b = 0$*".*

For example: $\pi = (11)\ 4\ 8\ 5\ 2\ 6\ 9\ (10)\ 1\ 3\ 7$ has the numbers of coefficients $A_1(\pi) = A_{-1}(\pi) = 3699$, $A_2(\pi) = A_{-2}(\pi) = 198$, $A_3(\pi) = A_{-3}(\pi) = 12$, and $A_{\pm k}(\pi) = 0$ for $k > 3$.

The distribution of the numbers $A_k(\pi)$ for single permutations or all permutations of some symmetric group, seems to be an interesting subject, which is not easily studied with the help of a computer: let

$$(5.2) \qquad B_k := \min \{n \mid A_k(\pi) > 0 \text{ for some } \pi \in S_n\} \quad (k \in \mathbb{Z} \setminus \{0\}) \ ,$$

then $B_{\pm 1} = 1$, $B_{\pm 2} = 7$, and $B_{\pm k} \geq 10$ for $k \geq 3$ are the only results known so far.

That *the* SEM *expansion of Schubert polynomials (or quantum cohomology itself ?) may have deep symmetry properties, which are not at all understood*, is indicated by the subsequent conjectures:

Conjecture 5.7. *For all basic expansions — with the exception of the elementary symmetric polynomials itself — one has*

$$A_k = A_{-k} \quad \textit{for all } k \in \mathbb{N}.$$

The conjecture is true in the Schur case: since all unembedded Grassmannian permutations are basic by Cor.2.3, one has in this case $A_1 = A_{-1}$ by (2.16). The conjecture has been established by computer calculations up to S_6, i.e. for every $\pi \in S_6$ the associated basic permutation $\pi^{(N)}$, which may be in S_{10} or S_{11} (!), has been determined with the help of (4.3) and checked.

We introduce the following numbers for $\pi \in S_n$ with SEM expansion (1.8):

$$\xi_\pi(i) := \sum_{l \in \mathbb{L}_n} \alpha_l \ l_i \ , \quad \xi_\pi := (\xi_\pi(1), \ldots, \xi_\pi(n-1)) \ ,$$
$$\zeta_\pi := \sum_{i=1}^{n-1} \xi_\pi(i) \ , \quad \bar{\zeta}_\pi := \sum_{i=1}^{n-1} |\xi_\pi(i)| \ ,$$

and for all $n \in \mathbb{N}$:

$$C_n := |\{\pi \in S_n \mid \zeta_\pi = 0\}| \ , \quad \bar{C}_n := |\{\pi \in S_n \mid \bar{\zeta}_\pi = 0\}| \ .$$

Since clearly $[\ \bar{\zeta}_\pi = 0 \Longrightarrow \zeta_\pi = 0\ ]$, $\bar{\zeta}_{id} = 0$, and $\zeta_{\omega_n} > 0$, it follows

$$1 \leq \bar{C}_n \leq C_n < n! \ .$$

But more can be said:

Theorem 5.8. *Let $\pi = \pi(\lambda, m)$ be a Grassmannian permutation associated to the partition λ. Then*

$$\bar{\zeta}_\pi = 0 \iff 33 \subset \lambda \ ,$$

where '$33 \subset \lambda$' means that the shape of λ includes the shape 33.

Proof. Let $\lambda \equiv \lambda_1 \dots\ \lambda_n$ a partition of length n and $\pi \equiv \pi(\lambda', \lambda_1)$ the unembedded Grassmannian permutation associated to the conjugate partition of λ, i.e. we have to show

$$\bar{\zeta}_\pi = 0 \iff 222 \subset \lambda \ .$$

Since $\bar{\zeta}_\pi = 0 \iff \forall j : \xi_\pi(j) = 0$, it follows from Cor.2.2 that

$$\bar{\zeta}_\pi = 0 \iff \forall j : \sum_{\sigma \in S_n(\rho)} sign(\sigma)\ (\lambda_{\sigma(j)} - \sigma(j) + j) = 0 \ ,$$

where we used the notation '$S_n(\rho)$' to indicate that the shape $\rho(\lambda)$ (see (2.14)) may restrict S_n to some subgroup. For an arbitrary $a = (a_i) \in \mathbb{Z}^n$ let $\bar{a} := \sum_{\sigma \in S_n} sign(\sigma)\ a_{\sigma(1)}$. Clearly $\bar{a} = a_1$ for $n = 1$, $\bar{a} = a_1 - a_2$ for $n = 2$; and $\bar{a} = 0$ for $n \geq 3$, because every number a_i occurs with an equal number $(n-1)!/2$ of $+$ and $-$ signs. But this implies the 'if' direction of the assertion: in case of $222 \subset \lambda$ all permutations of at least S_3 are applied to any column vector $(\lambda_i - i + j)_{i=1,\dots,n}$.

It remains to investigate the 'only if' direction: if $n \leq 2$, then the obvious direct calculation shows $\bar{\zeta}_\pi \neq 0$. Assume therefore $n \geq 3$ and $222 \not\subset \lambda$, i.e. λ is of the form $1\dots1$, $21\dots1$, or $221\dots1$. In all these cases the associated $\rho(\lambda)$ equals δ_{n-2}. Since this is all that matters, we take for every $n \geq 3$ one of the associated Grassmannian permutations and denote it by $\pi(n)$. Then the validity of

$$\xi_{\pi(n)}(n) = (-1)^n (a_2 - a_1) \ , \tag{5.3}$$

where $a = (a_i)$ is the vector of the last column of $M(\lambda)$ (see (2.10)), implies $\bar{\zeta}_\pi \neq 0$. A simple direct calculation shows (5.3) in case of $n = 3$. Assume (5.3) to be true for some $n \geq 3$. Then we have to investigate the two minors $M_{1,1}$ and $M_{2,1}$ of the $(n+1) \times (n+1)$ matrix $M(\lambda)$. Since the $\rho(\lambda)$ of both minors is of the form δ_{n-2}, we see that $\xi_{\pi(n+1)}(n+1) = (-1)^n(a_3 - a_2) - (-1)^n(a_3 - a_1) = (-1)^{n+1}(a_2 - a_1)$. □

The property '$\bar{\zeta}_\pi = 0$' is not restricted to Grassmannian permutations π — at least a very high percentage of "arbitrary" permutations, for which a simple characterization has yet to be found, has this property, too:

Conjecture 5.9. *The sequences $C_n/n!$ and $\bar{C}_n/n!$ are strictly increasing for $n \geq 4$, and*

$$\lim_{n \longrightarrow \infty} C_n/n! = \lim_{n \longrightarrow \infty} \bar{C}_n/n! = 1 \ .$$

Evidence for this conjecture is given by the following table:

n	C_n	$C_n/n!$	$\bar{C}_n$	$\bar{C}_n/n!$	$C_n/\bar{C}_n$
4	1	0.04166...	11	0.45833...	...
5	4	0.033...	75	0.625	0.0533...
6	71	0.09611...	542	0.75277...	0.1309...
7	1017	0.20...	4221	0.8375	0.24...
8	12566	0.3116...	35962	0.8919...	0.349...

From Prop.4.1 it is immediate that $\xi_{\pi^{(m)}} = (0,\ldots,0)$ implies $\xi_{\pi^{(m+1)}} = (0,\ldots,0)$ for all $m \geq N(\pi)$. This is in general not true for $m < N(\pi)$, even if $\xi_{\pi^{(N)}} = (0,\ldots,0)$: for example $N(\pi) = 3$ for $\pi = 3751462$ has $\xi_{\pi^{(m)}} = (0,\ldots,0)$ for $m = 1,3,4,\ldots$, but $\xi_\pi = (0,0,-1,0,1,0)$ and $\xi_{\pi^{(2)}} = (0,0,0,0,-2,2,0,0)$.

Since the number of non-zero terms $\beta(\pi)$ occurring in the SEM expansion of some $\pi \in S_n$ can be arbitrarily large compared to n, the entries of the vector ξ_π could be arbitrary large. In fact the conjecture above shows that allmost allways the vector ξ_π is the zero vector or at least 'balanced' in the sense: $\zeta_\pi = 0$. (It would be interesting to investigate the distribution of different kinds of vectors ξ_π: for example the zero vector and vectors ξ_π with exactly one $+1$ and one -1 and zeros otherwise occur most often.) But even more seems to be true: the following final conjecture has been established by computer calculations up to S_7, i.e. for every $\pi \in S_7$ the condition (5.4) below has been checked, and in case of π being 'exceptional' (see below) the associated basic permutation $\pi^{(N)}$ has been determined from (4.3) and checked.

Conjecture 5.10. *For all basic permutations π the numbers $\xi_\pi(i)$ obey*

$$(-i) \leq \xi_\pi(i) \leq i \quad \textit{for all } i\ , \tag{5.4}$$

which can be expressed in the short form: $|\xi_\pi| \leq \delta$.

Permutations π which do not obey (5.4) are called *δ-exceptional.* The first δ-exceptional permutation is $\pi = 31542 \in S_5$ with $\xi_\pi = (0,0,0,-5)$, whereas S_6 and S_7 contain 10 and 85 δ-exceptional permutations, respectively. Most of them "miss" δ by only ± 1 in one entry, but there occur also higher 'defects' (the first time) in S_7:
$\xi_{2714365} = (0,0,0,-1,1,-8)$, $\xi_{3154762} = (-1,0,-1,0,-1,-13)$, and $\xi_{7531246} = (0,0,-1,-4,-8,-11)$.

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