SCHUBERT POLYNOMIALS OF TYPE A – D

RUDOLF WINKEL*

Abstract. Schubert polynomials of type B, C, and D have been described first by S. Billey and M. Haiman [BH] using a combinatorial method. In this paper we give a unified algebraic treatment of Schubert polynomials of types A – D in the style of the Lascoux-Schützenberger theory in type A, i.e. Schubert polynomials are generated by the application of sequences of divided difference operators to 'top polynomials'. The use of the creation operators for Q-Schur and P-Schur functions allows us to give: (1) simple and natural forms of the 'top polynomials', (2) formulas for the easy computation with all divided differences, (3) recursive structures, and (4) simplified derivations of basic properties.

1. Introduction

Schubert polynomials made their first appearence in algebraic geometry in connection with the intersection calculus of flag manifolds, but because of their beautiful algebraic and combinatorial properties they have now superseded Schur polynomials and tableaux in computer algebra packages designed for the representation theory, invariant theory, and combinatorics of the symmetric groups (cf. [KKL,V]). In the next three paragraphs we briefly review the cohomological origin of Schubert polynomials (a fuller account may be found in [BH] and [F]).

Let G be one of the classical groups $SL(n,\mathbb{C}), SO(2n+1,\mathbb{C}), Sp(2n,\mathbb{C})$, $SO(2n,\mathbb{C})$ with Weyl groups $W_n = A_{n-1}, B_n, C_n, D_n$, respectively, and let B be the Borel subgroup of the respective upper triangular matrices. In 1953 Borel [B] has proved that the ring of integral cohomology $H^*(Fl_n,\mathbb{Z})$ of the flag manifolds $Fl_n = G/B$ is isomorphic to the ring $P_n = \mathbb{Q}[z_1,\dots,z_n]$ factored by the ideal I_n, which is generated by W_n-invariant polynomials without constant term. In type A this ideal is $I_n := (e_1^{(n)},\dots,e_n^{(n)})$, where

$$e_k^{(n)} \equiv e_k(z_1,\dots,z_n) := \sum_{1\le j_1<\dots<j_k\le n} z_{j_1}\dots\ z_{j_k}$$

are the elementary symmetric polynomials of degree k in the variables $z_1,\dots,z_n$. In types B and C the ideal is $I_n = (p_2^{(n)}, p_4^{(n)},\dots)$, where $p_k^{(n)} := z_1^k + \dots + z_n^k$ are the

Date: July 2, 1998.

1991 *Mathematics Subject Classification.* 05E15, 14M15.

Key words and phrases. Schubert polynomials, classical groups, Q-Schur and P-Schur functions, divided differences.

* Supported by Deutsche Forschungsgemeinschaft (DFG).

power sum symmetric polynomial, and in type D an additional polynomial $z_1 \dots z_n$ is included into I_n.

The Bruhat decomposition $G = \sum_{w \in W_n} BwB$ of G induces a partition of flag manifold G/B into Schubert cells, whose closures are the Schubert varieties. Combinatorially the closure of a Schubert cell associated to some W can be described as the union over all Schubert cells with w' less or equal to w in Bruhat order. The polynomial representatives of the cohomology classes of Schubert varieties under the Borel isomorphism are now the *Schubert polynomials* X_w^A, X_w^B, X_w^C, and X_w^D, respectively. Since the Schubert polynomials are invariant under the natural embeddings of the classical groups resp. Weyl groups $W_n \hookrightarrow W_{n+1}$, it is possible to view them as elements of the rings $\mathbb{Z}[z] = \mathbb{Z}[z_1, z_2, z_3, \dots]$ in type A, and $\mathbb{Q}[z; p(Z)] = \mathbb{Q}[z_1, z_2, z_3, \dots; p_1, p_3, \dots]$ in types B, C, D, where $p_k := z_1^k + z_2^k + \dots$ is the power sum symmetric function. The λ-ring substitution $p_k(X) = x_1^k + x_2^k + \dots = -p_k(Z)/2$ yields an isomorphic ring $\mathbb{Q}[z; p(X)]$ with nicer formulas: the Schubert polynomials X_w^B and X_w^D are now elements of the ring $\mathbb{Z}[z; P_\mu(X)] \cong \mathbb{Z}[P_\mu(X)]\ [z]$, and the Schubert polynomials X_w^C of $\mathbb{Z}[z; Q_\mu(X)] \cong \mathbb{Z}[Q_\mu(X)]\ [z]$. Here Q_μ is the Q-Schur function (see Section 2 below) associated to the partition μ with distinct parts, i.e.,

$$\mu \in \mathcal{D} := \{\mu \equiv \mu_1 \dots \mu_l \mid \mu_1 > \dots > \mu_l > 0\}$$

($l = l(\mu)$ being the length of μ); and P_μ is the P-Schur function defined by

$$P_\mu = 2^{-l(\mu)}\ Q_\mu\ . \tag{1.1}$$

Bernstein, Gelfand, and Gelfand ([BGG], 1973), and Demazure (1973-74) introduced the Schubert varieties into the Borel picture and discovered that Schubert classes can be computed by applying a w-dependent sequence of divided difference operators to the cohomology class of highest codimension, i.e. the cohomology class of a point (see also [Hi]). In the years 1982 - 1987 Lascoux and Schützenberger in a series of papers described the type A polynomial representatives of these classes of highest codimension, henceforth called *top polynomials*, thereby giving rise to the beautiful algebro-combinatorial theory of the Schubert polynomials X_w^A. Again we recall here only some basics, a fuller account of which may be found e.g. in [LS1, L] (and references therein) and [M1, M2].

The Weyl group W_n of type A can be identified with the Coxeter group A_{n-1} on generators $s_1, \dots, s_{n-1}$ with relations $(s_i s_j)^{m(s_i,s_j)} = id$, where $m(s_i, s_i) = 1$ for all i, and $m(s_i, s_j) \geq 2$ for $i \neq j$. The numbers $m(s_i, s_j)$ are comfortably encoded into the Coxeter graph 

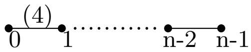
, where each generator s_i is represented by a vertex with label i, and where: in case of $m(s_i, s_j) = 2$, i.e., $s_i s_j = s_j s_i$, no edge is drawn between vertices i and j; in case of $m(s_i, s_j) = 3$, i.e. $s_i s_j s_i = s_j s_i s_j$, an edge is drawn between vertices i and j; and in case of $m(s_i, s_j) \geq 4$ an edge is drawn between vertices i and j with label $(m(s_i, s_j))$.

It is well known that the abstract Coxeter group A_{n-1} can be identified with the symmetric group S_n of permutations, where the generators s_i are interpreted as the elementary transpositions $\sigma_i = (i, i+1)$. Then every permutation $\pi \in S_n$ can be written as a product $\pi = \sigma_{a_1} \dots \sigma_{a_l}$ of minimal length $l = l(\pi)$, where $a \equiv a_1 \dots a_l$ is called a *reduced sequence* for π. Let $\omega_n = n\ n-1 \dots\ 1$ be the permutation of maximal length in S_n, set $\delta_n = n \dots\ 1$, and $z^{\delta_{n-1}} := z_1^{n-1} \dots z_{n-1}^1$. Let furthermore $\partial_i := (id - \sigma_i)/(z_k - z_{k+1})$ be the divided difference operator, where $\sigma_i = (i, i+1)$ acts by transposing the variables z_i and z_{i+1}. Set finally $\partial_\pi := \partial_{a_1} \dots\ \partial_{a_p}$, where $a_1 \dots\ a_p$ is any reduced sequence for π. Then the Schubert polynomial associated to a permutation $\pi \in S_n$ is defined by

$$X_\pi \equiv X_\pi^A := \partial_{\pi^{-1}\omega_n}\ z^{\delta_{n-1}}\ , \tag{1.2}$$

i.e., $X_{\omega_n} = z^{\delta_{n-1}}$ is the top polynomial in P_n/I_n as identified by Lascoux and Schützenberger.

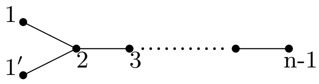
The Weyl groups W_n of type B and C — differing only by the length of their roots (n-cubes and n-octahedra !) — can be identified as abstract groups with the Coxeter group B_n on generators $s_0, \dots, s_{n-1}$ with relations given by the Coxeter graph



.

The Coxeter group B_n in turn can be identified with the group $\bar{S}_n$ of signed permutations, where the additional generator s_0 is interpreted as the operator σ_0 changing the sign at place 1. The longest element of $\bar{S}_n$ is now $\bar{\omega}_n = \bar{1} \dots \bar{n}$.

The Weyl group W_n of type D can be identified similarly with the Coxeter group D_n on generators $s_{1'}, s_1, \dots, s_{n-1}$ with relations given by the Coxeter graph



, or the group $\bar{\bar{S}}_n$ of signed permutations with an even number of bars, where the additional generator $s_{1'}$ is interpreted as the operator $\sigma_{1'}$ changing simultaneously the numbers at places 1 and 2 and their signs, i.e., "$\sigma_{1'} = \sigma_0\sigma_1\sigma_0$". The longest element of $\bar{\bar{S}}_n$ is now $\bar{\bar{\omega}}_n = \bar{1} \dots \bar{n}$, if n is even, and $\bar{\bar{\omega}}_n = 1\ \bar{2} \dots \bar{n}$, if n is odd.

The divided difference operators, which need to be added in types B, C, and D, are:

$$\partial_0^C \equiv \partial_0 := \frac{id - \sigma_0}{-2z_1}\ , \quad \partial_0^B := \frac{id - \sigma_0}{z_1}\ , \quad \partial_{1'} := \frac{id - \sigma_{1'}}{-z_1 - z_2}\ ,$$

where σ_0 acts as the operator, which replaces z_1 by $-z_1$, and $\sigma_{1'}$ acts as the operator, which replaces simultaneously z_1 by $-z_2$ and z_2 by $-z_1$, i.e., "$\sigma_{1'} = \sigma_0\sigma_1\sigma_0$" again. Note that for $i \geq 1$ the σ_i leave the p_k's invariant, whereas $\sigma_0 p_k = p_k - 2z_1^k$.

One of the great achievements of [BH] is the description of the top polynomials in types B, C, and D. For partitions λ, μ of length at most k and $\lambda \subset \mu$ the skew multi-Schur function is defined by

$$S_{\mu/\lambda}(Z_1, Z_2, \dots, Z_k) = \det(h_{\mu_i - \lambda_j + j - i}(z_1, \dots, z_i))_{i,j=1,\dots,k}\ , \tag{1.3}$$

where Z_i is $z_1 + \cdots + z_i$ in λ-ring notation, and $h_\nu^{(i)} = h_\nu(z_1, \ldots, z_i)$ is the complete symmetric polynomial of degree ν with variables $z_1, \ldots, z_i$. Writing $\delta_n + \lambda$ for the componentwise sum of δ_n and λ, $\delta_n = n \ldots 1$, and λ' for the conjugate of λ, the top polynomial of type C is given as [BH, (4.59)]

$$X^C_{\bar{\omega}_n} = \sum_{\lambda \subset \delta_{n-1}} Q_{\delta_n+\lambda}(X)\ S_{\delta_{n-1}/\lambda'}(Z_1, Z_2, \ldots, Z_{n-1}) \ , \tag{1.4}$$

and the top polynomial of type D is given as [BH, (4.60)]

$$X^D_{\bar{\bar{\omega}}_n} = \sum_{\lambda \subset \delta_{n-1}} P_{\delta_{n-1}+\lambda}(X)\ S_{\delta_{n-1}/\lambda'}(Z_1, Z_2, \ldots, Z_{n-1}) \ . \tag{1.5}$$

Rescaling of (1.4) in accordance to (1.1) gives the formula for $X^B_{\bar{\omega}_n}$ in terms of P-Schur functions. In Theorem 3.1 below we provide simple formulas for the top polynomials $X^C_{\bar{\omega}_n}$ and $X^D_{\bar{\bar{\omega}}_n}$, which are the natural generalizations of the top polynomials $x^{\delta_{n-1}}$ in type A.

Now the Schubert polynomials of types B, C, and D can be defined for all $\bar{\pi} \in \bar{S}_n$ and $\bar{\bar{\pi}} \in \bar{\bar{S}}_n$ by

$$X^B_{\bar{\pi}} := \partial_{\bar{\pi}^{-1}\bar{\omega}_n}\ X^B_{\bar{\omega}_n} \ , \tag{1.6}$$

$$X^C_{\bar{\pi}} \equiv \bar{X}_{\bar{\pi}} := \partial_{\bar{\pi}^{-1}\bar{\omega}_n}\ X^C_{\bar{\omega}_n} \ , \tag{1.7}$$

$$X^D_{\bar{\bar{\pi}}} \equiv \bar{\bar{X}}_{\bar{\bar{\pi}}} := \partial_{\bar{\bar{\pi}}^{-1}\bar{\bar{\omega}}_n}\ X^D_{\bar{\bar{\omega}}_n} \ . \tag{1.8}$$

By a well known theorem of by J. Tits (1968) (cf. [B, Thm.II.3]) the above sequences of divided difference operators are independent of the reduced sequence used to represent the (signed) permutations, whence the left hand sides of (1.2) and (1.6-8) are well defined.

Let '$\diamond$' stand for one of the letters A, B, C, D, then the top polynomials can be written as $X^\diamond_{\omega_n^\diamond}$, and the definition of Schubert polynomials as

$$X^\diamond_\pi := \partial_{\pi^{-1}\omega_n^\diamond}\ X^\diamond_{\omega_n^\diamond} \quad \text{for every } \pi \in S_n^\diamond \ , \tag{1.9}$$

where of course: $X^\diamond = X^A, X^B, X^C, X^D$, $\omega_n^\diamond = \omega_n, \bar{\omega}_n, \bar{\omega}_n, \bar{\bar{\omega}}_n$, and $S_n^\diamond = S_n, \bar{S}_n, \bar{S}_n, \bar{\bar{S}}_n$ for $\diamond =$ A, B, C, D, respectively.

Since from the definitions and (1.1) it is not hard to see that

$$X^B_{\bar{\pi}} = 2^{-s(\bar{\pi})}\ X^C_{\bar{\pi}} \ , \tag{1.10}$$

where $s(\bar{\pi})$ is the number of bars in $\bar{\pi}$, we subsequently in this paper consider only the types A, C, and D.

The approach of Billey and Haiman in [BH] to Schubert polynomials of type B, C, D, is based on a generalization of the combinatorial Edelman-Greene correspondence [EG] in type A. The central Theorems 3 and 4 of [BH] describe expansions of the Schubert polynomials $X^C_{\bar{\pi}}$ and $X^D_{\bar{\bar{\pi}}}$ as linear combinations of type A Schubert polynomials with coefficients in the rings $\mathbb{Z}[Q_\mu(X)]$ and $\mathbb{Z}[P_\mu(X)]$, respectively, where the coefficients are given by the generalizations of the Edelman-Greene correspondence. But because these correspondences are not suitable for actual computations, and also the proofs in this setting tend to be very cumbersome, the simplifications in the present paper seem to meet a demand both in theory and computation.

The paper [FK] of Fomin and Kirillov, which has been written in cross-fertilization to [BH] features also B_n-analogues of Schubert polynomials in the ring $\mathbb{Q}[z]$. Especially the B_n-analogues of the "first kind" seem to be valuable as tools for the computation of multiplication constants for the X_π^C and the Q-Schur functions; those of "second kind" are specializations of the Billey-Haiman Schubert polynomials upon setting $z_{n+1} = z_{n+2} = \cdots = 0$. All the B_n-analogues do not have the property of invariance under the natural embeddings of the Weyl groups (compare Prop.3.5 below).

Since the complicated form of the top polynomials in types B, C, D, and the application of the divided differences ∂_0 and $\partial_{1'}$ to Q-Schur and P-Schur functions, respectively, causes computational difficulties — the resulting skew forms (cf. (2.6) below) must be expanded into ordinary forms —, S. Billey in her recent papers [B1,B2] has developed methods for the computation of Schubert polynomials avoiding divided differences: in [B1] recursive formulas for Schubert polynomials of type A – D based on the 'transition equations' introduced by Lascoux and Schützenberger in type A are given, and in [B2] an approach based on Kostant polynomials includes in addition the exceptional root systems.

In the present paper we follow closely the originial approach of Lascoux and Schützenberger in type A and present formulas, which make theory and calculation in types C and D almost as simple as in type A: in Section 2 the generating operators for Q-Schur and P-Schur functions are described, and their interactions with divided differences are investigated in a sequence of technical lemmas. In particular we deduce commutator formulas for the generating operators and the divided differences ∂_0 (Lemma 2.2) and $\partial_{1'}$ (Lemma 2.4). In Section 3 we show that the complicated formulas (1.4) and (1.5) for the top polynomials in type C and D can be substituted by simple expressions, which in a natural way generalize the factorization $(z_1 \dots z_{n-1}) \cdot \dots \cdot (z_1 z_2) \cdot z_1 = e_{n-1}^{(n-1)} \dots e_1^{(1)}$ of the top polynomial $X_{\omega_n} = z^{\delta_{n-1}}$ in type A.

For the computation of all the $X_\pi^\diamond$ one can proceed either from the definition (1.9) using the top polynomials as given in Section 3 or by using the recursive method described in Section 4, which generalizes one of the recursive structures for type A given in [W1]. Basic properties of Schubert polynomials — e.g. stability (Prop.3.5) — are derived, whenever the necessary tools are available. Due to the use of generating operators (Section 2) and the recursive structure (Section 4) these derivations are often much more simple than the corresponding proofs in [BH] (compare especially the five line proof of Proposition 5.1).

Moreover, we expect the formalism described in the present paper to be important for the development of a theory of "quantum Schubert polynomials of type B, C, D " in analogy to [FGP] and [W3].

2. The generating operators of Q-Schur functions and their interaction with divided differences

First of all we recall the construction of Q-Schur functions and their operator representation, for which full details and proofs may be found in the book [HH] of Hoffman and Humphreys. Let $\Delta_{\mathbb{Q}} := \mathbb{Q}[p_1, p_3, p_5, \dots]$ be the polynomial ring over the rationals generated by the power sum symmetric functions of odd degree. Set

$$z_\lambda = \prod_{i>0} i^{m_i(\lambda)}\ (m_i(\lambda))!\ ,$$

where λ is a partition and $m_i(\lambda)$ are the number of parts of size i occurring in λ. Then set [HH, (7.9)]

$$q_m := \sum_{\substack{\lambda \in \mathcal{D} \\ |\lambda| = m}} 2^{l(\lambda)}\ z_\lambda^{-1}\ p_\lambda$$

with $p_\lambda = p_{\lambda_1} \dots p_{\lambda_l}$ for $\lambda = \lambda_1 \dots \lambda_l$. Furthermore set $q_\emptyset = q_0 = 1$, $q_\lambda = 0$, if some $\lambda_i < 0$, and $q_\lambda = q_{\lambda_1} \dots q_{\lambda_l}$ for any $\lambda \in \mathcal{D}$. Then the Q-Schur functions can be defined recursively by the Schur formulas [HH, Thm. 9.14]

$$\begin{aligned}
Q_a &= q_a\ ,\\
Q_{ab} &= q_a q_b + 2\sum_{n>0} (-1)^n q_{a+n} q_{b-n} \quad \text{for } a > b\ ,\\
Q_{\lambda_1 \dots \lambda_{2k+1}} &= \sum_{i=1}^{2k+1} (-1)^{i+1}\ q_{\lambda_i}\ Q_{\lambda_1 \dots \widehat{\lambda_i} \dots \lambda_{2k+1}} \quad \text{for } k \geq 0,\ l(\lambda) \text{ odd}\ ,\\
Q_{\lambda_1 \dots \lambda_{2k}} &= \sum_{i=2}^{2k} (-1)^{i}\ Q_{\lambda_1 \lambda_i}\ Q_{\lambda_2 \dots \widehat{\lambda_i} \dots \lambda_{2k+1}} \quad \text{for } k > 0,\ l(\lambda) \text{ even}\ ,
\end{aligned}$$

where the notation $\widehat{\lambda_i}$ means the removal of λ_i. Moreover one can deduce that $\Delta_{\mathbb{Q}} = \mathbb{Q}[q_1, q_2, q_3, \dots]/I = \mathbb{Q}[Q_\mu]$, where I is the ideal generated by the relations for the q_i.

The above formulas have been found by Schur already in 1911 as a result of his investigation of projective representations of the symmetric groups. Much more recent are the operator representations of Q-Schur functions, which are specializations in the 'projective case' $t = -1$ of the vertex operator representations of Hall-Littlewood functions $Q_\lambda(z;t)$ (cf. [M3]) given by Nihuang Jing [J], and which are fore-shadowed by formulas of J.N. Bernstein for Schur functions (cf.[Z, p.69]).

For all natural numbers i the $\mathbb{Q}$-linear mappings $q_i^\perp :\ \Delta_{\mathbb{Q}} \longrightarrow \Delta_{\mathbb{Q}}$ are defined on the q_n's by [HH, (7.20)]

$$q_i^\perp(q_n) = 2q_{n-i} \tag{2.1}$$

and extend uniquely (by the "product rule" [HH, (7.19)]) to all of $\Delta_{\mathbb{Q}}$. The basic operators

$$\mathcal{B}_n : \ \Delta_{\mathbb{Q}} \longrightarrow \Delta_{\mathbb{Q}}, \ f \mapsto \mathcal{B}_n(f) := \sum_{i\geq 0} (-1)^i \ q_{n+i} \ q_i^{\perp}(f) \text{ for } n \geq 0 \tag{2.2}$$

with $\mathcal{B}_0(1) = 1$ increase the degree of f by n and 'create' for any given $\mu \in \mathcal{D}$ the Q-Schur function according to:

$$Q_\mu = \mathcal{B}_{\mu_1} \circ \cdots \circ \mathcal{B}_{\mu_l}(1) \ . \tag{2.3}$$

Moreover the operators $\mathcal{B}_n$ obey the following relations [HH, Thm.9.1]:

$$\begin{aligned} \mathcal{B}_n \circ \mathcal{B}_n =& 0 \ \text{ for all } n \geq 1 \\ \mathcal{B}_i \circ \mathcal{B}_j =& - \mathcal{B}_j \circ \mathcal{B}_i \ \text{ for } i \neq j \ . \end{aligned} \tag{2.4}$$

We investigate next the interaction between the operators $q_i^{\perp}$, $\mathcal{B}_n$, and the divided differences. Note first that by the 'product rule'

$$\partial_i(fg) = (\partial_i f)g + \sigma_i(f)(\partial_i g) \ , \tag{2.5}$$

which is valid for all $i \in \mathbb{N} \cup \{0, 1'\}$ and all functions f, g in the z-variables, the operators ∂_k for $k \geq 1$ commute with any function, which is symmetric in the variables x_k, x_{k+1} or z_k, z_{k+1}; in particular these ∂_k commute with all the p_i, q_i, and Q_μ. The operators $q_i^{\perp}$ act on the symmetric *functions* in $\Delta_{\mathbb{Q}}$, but we assume that they commute with all *polynomials* in the x and z variables, as do the $\mathcal{B}_n$'s. The proof of the formula

$$\partial_0 \ Q_\mu(X) = \sum_{0<k\leq\mu_1} Q_{\mu/(k)}(X) \ z_1^{j-1} \tag{2.6}$$

relays on the combinatorial definition of Q-Schur functions via tableaux of shifted shape and can be found in [BH, Cor.4.5].

Lemma 2.1.

$$\partial_0 \ q_m = \sum_{j>0} q_{m-j} \ z_1^{j-1} \tag{2.7}$$

$$\partial_0 \ q_{m+1} = q_m + z_1(\partial_0 \ q_m) \tag{2.8}$$

$$\sigma_0 f = f + 2z_1(\partial_0 f) \tag{2.9}$$

$$\partial_k \ q_m^{\perp} = q_m^{\perp} \ \partial_k \ \ \textit{for all } k, m \geq 1 \tag{2.10}$$

$$\partial_0 \ q_m^{\perp} = q_m^{\perp} \ \partial_0 \ \ \textit{for all } m \geq 0 \tag{2.11}$$

$$\partial_0 \ (q_{m+i} \ q_i^{\perp}) = (\partial_0 \ q_{m+i}) \ q_i^{\perp} + 2z_1(\partial_0 \ q_{m+i}) \ q_i^{\perp} \ \partial_0 + q_{m+i} \ q_i^{\perp} \ \partial_0 \tag{2.12}$$

$$\begin{aligned} &\partial_0 \ (q_{m+i+1} \ q_i^{\perp}) - \ z_1 \ \partial_0 \ (q_{m+i} \ q_i^{\perp}) - q_{m+i} \ q_i^{\perp} = \\ &q_{m+i+1} \ q_i^{\perp} \ \partial_0 \ + \ z_1 \ q_{m+i} \ q_i^{\perp} \ \partial_0 \ \ \textit{for } m \geq 1, \ i \geq 0 \end{aligned} \tag{2.13}$$

Proof. For $\mu = (m)$ formula (2.6) specializes to formula (2.7). (2.8) follows from (2.7) and (2.9) directly from the definition of ∂_0. For (2.10) one applies both sides to some q_i and computes

$$\partial_k \; q_m^\perp \; (q_i) = \partial_k \; 2 \; q_{i-m} = 2q_{i-m} \; \partial_k = q_m^\perp \; (q_i) \; \partial_k = q_m^\perp \; \partial_k \; (q_i) \; .$$

The proof of (2.11) is similar to that of (2.10):

$$\partial_0 \; q_m^\perp \; (q_i) = 2\partial_0 \; q_{i-m} = \sum_{j>0} 2q_{i-m-j} \; z_1^{j-1} = q_m^\perp \; (\sum_{j>0} q_{i-j} \; z_1^{j-1}) = q_m^\perp \; \partial_0 \; (q_i) \; .$$

For (2.12) one computes with (2.7), the product rule for ∂_0, (2.9-11), and the shorthand $q_i^0 := \partial_0 q_i$

$$\begin{aligned}\frac{1}{2}\partial_0 \; (q_{m+i} \; q_i^\perp(q_n)) = \partial_0 \; (q_{m+i} \; q_{n-i}) = q_{m+i}^0 \; (\sigma_0 \; q_{n-i}) + q_{m+i} \; q_{n-i}^0 = \\ q_{m+i}^0 \; q_{n-i} + 2z_1 \; q_{m+i}^0 \; q_{n-i}^0 + q_{m+i} \; q_{n-i}^0 = \\ \frac{1}{2}\left(q_{m+i}^0 \; q_i^\perp(q_n) + 2z_1 q_{m+i}^0 \; q_i^\perp \; \partial_0(q_n) + q_{m+i} \; q_i^\perp \; \partial_0(q_n)\right) \; .\end{aligned}$$

The left hand side of (2.13) equals by (2.12)

$$\begin{aligned}q_{m+i+1}^0 \; q_i^\perp + 2z_1 \; q_{m+i+1}^0 \; q_i^\perp \; \partial_0 + q_{m+i+1}^0 \; q_i^\perp \; \partial_0 - q_{m+i} \; q_i^\perp \\ - z_1 \; q_{m+i} \; q_i^\perp - 2z_1^2 \; q_{m+i}^0 \; q_i^\perp \; \partial_0 - z_1 \; q_{m+i} \; q_i^\perp \; \partial_0 \; ,\end{aligned}$$

which by (2.8) is the right hand side of (2.13). □

Lemma 2.2.

(2.14) $\mathcal{B}_0 \; Q_\mu = (-1)^{l(\mu)} Q_\mu$ *for all* $\mu \in \mathcal{D}$

(2.15) $\partial_\nu \; \mathcal{B}_n = \mathcal{B}_n \; \partial_\nu$ *for all* $n, \nu \geq 1$

(2.16) $\partial_0 \; \mathcal{B}_n = \mathcal{B}_n \; \partial_0 + \mathcal{B}_{n-1} + z_1 \; \partial_0 \; \mathcal{B}_{n-1} + z_1 \; \mathcal{B}_{n-1} \; \partial_0$ *for all* $n > 0$

(2.17) $\partial_0 \; \mathcal{B}_n = \sum_{k=1}^{n} z_1^{k-1} \; \mathcal{B}_{n-k} + (\mathcal{B}_n + 2\sum_{k=1}^{n-1} z_1^k \; \mathcal{B}_{n-k} + z_1^n \; \mathcal{B}_0) \; \partial_0 + z_1^n \; \partial_0 \mathcal{B}_0$

Proof. (2.14) follows from (2.4), (2.15) from (2.2) and (2.10), (2.16) from (2.2) and (2.13), (2.17) from (2.16). □

Lemma 2.3. *With $p_m := q_m/2$ $(m \geq 0)$ one has for $m, i \geq 0$:*

$$\partial_{1'} = \sigma_0 \partial_1 \sigma_0 \tag{2.18}$$

$$\sigma_{1'} f = f + (z_1 + z_2)(\partial_{1'} f) \tag{2.19}$$

$$\sigma_0\ p_m = p_m + 2 \sum_{k=1}^{m} p_{m-k}\ z_1^k \tag{2.20}$$

$$\sigma_0\ p_{m+1} = p_{m+1} + z_1\ p_m + z_1(\sigma_0\ p_m) \tag{2.21}$$

$$\partial_{1'}\ q_m^\perp = q_m^\perp\ \partial_{1'} \tag{2.22}$$

$$\partial_{1'}\ p_m = 2 \sum_{k=1}^{m} p_{m-k}\ h_{k-1}^{(2)}\ , \quad \text{where } h_k^{(2)} = \partial_1\ z_1^{k+1} \tag{2.23}$$

$$\partial_{1'}\ p_{m+1} = p_m + z_2(\partial_{1'}\ p_m) + \sigma_0\ p_m \tag{2.24}$$

$$\partial_{1'}\ (p_{m+i}\ q_i^\perp) = (\partial_{1'}\ p_{m+i})\ q_i^\perp + (z_1 + z_2)(\partial_{1'}\ p_{m+i})\ q_i^\perp\ \partial_{1'} + p_{m+i}\ q_i^\perp\ \partial_{1'} \tag{2.25}$$

$$\begin{aligned} &\partial_{1'}\ (p_{m+2+i}\ q_i^\perp) - 2p_{m+1+i}\ q_i^\perp - (z_1 + z_2)\ \partial_{1'}\ (p_{m+1+i}\ q_i^\perp) + z_1 z_2 \partial_{1'}\ (p_{m+i}\ q_i^\perp) \\ &= p_{m+2+i}\ q_i^\perp\ \partial_{1'} + (z_1 + z_2)\ p_{m+1+i}\ q_i^\perp\ \partial_{1'} + z_1 z_2 p_{m+i}\ q_i^\perp\ \partial_{1'} \end{aligned} \tag{2.26}$$

$$\partial_{1'}\ (p_{1+i}\ q_i^\perp) = p_{1+i}\ q_i^\perp\ \partial_{1'} + 2p_i\ q_i^\perp + (z_1 + z_2)\ \left[\partial_{1'} p_i\ q_i^\perp + p_i\ q_i^\perp \partial_{1'}\right] \tag{2.27}$$

Proof. (2.18) and (2.19) are immediate from the definitions. (2.20) follows from [BH, (4.15)] and the proof of [BH, Cor.4.5]. (2.21) follows from (2.20) and the proof of (2.22) is analogous to that of (2.11). For (2.23) one computes first

$$\partial_1\ \sigma_0\ p_m \overset{(2.20)}{=} 2 \sum_{k=1}^{m}\ p_{m-k}\ h_{k-1}^{(2)}\ .$$

This equation surprisingly turns out to be invariant under the left action of σ_0, i.e.,

$$\partial_{1'}\ p_m \overset{(2.18)(2.20)}{=} 2 \sum_{k=1}^{m} \left(p_{m-k} + 2 \sum_{j=1}^{m-k} p_{m-k-j}\ z_1^j \right)\ (\sigma_0 h_{k-1}^{(2)}) \overset{(!)}{=} 2 \sum_{k=1}^{m}\ p_{m-k}\ h_{k-1}^{(2)}\ .$$

Comparing the coefficients of the $p_{m-\nu}$ $(\nu = 1, \ldots, m)$ on both sides shows that (!) is equivalent to $\sigma_0 h_0^{(2)} = 1 = h_0^{(2)}$ for $\nu = 1$ and

$$\sigma_0 h_{\nu-1}^{(2)} + 2 \sum_{k=1}^{\nu-1} z_1^{\nu-k} (\sigma_0 h_{k-1}^{(2)}) = h_{\nu-1}^{(2)} \quad \text{for } \nu = 2, \ldots, m,$$

which is readily proven by induction:

$$\begin{aligned} \sigma_0 h_\nu^{(2)} + 2 \sum_{k=1}^{\nu} z_1^{\nu+1-k} (\sigma_0 h_{k-1}^{(2)}) &= z_1 \left[h_{\nu-1}^{(2)} + (\sigma_0 h_{\nu-1}^{(2)}) \right] + (\sigma_0 h_\nu^{(2)}) = \\ & z_2^\nu + (2 z_1 z_2^{\nu-1} - z_1 z_2^{\nu-1}) + z_1^2 z_2^{\nu-2} + (2 z_1^3 z_2^{\nu-3} - z_1^3 z_2^{\nu-3}) + \cdots = h_\nu^{(2)}\ . \end{aligned}$$

(2.24) follows from (2.23), and the proof of (2.25) is analogous to that of (2.12).

To verify (2.26) first expand the left hand side of (2.26) with the help of (2.25). The resulting equation is

$$\partial_{1'} p_{m+2+i} + z_1 z_2 \partial_{1'} p_{m+i} = 2p_{m+1+i} + (z_1+z_2)\partial_{1'} p_{m+1+i}$$

multiplied by $q_i^\perp + (z_1+z_2) q_i^\perp \partial_{1'}$ on the right. But the latter equation by application of (2.24) reduces to (2.21). (2.27) is seen similarly. □

Lemma 2.4. *Set $\bar{\mathcal{B}}_n := \mathcal{B}_n/2$ for $n \geq 1$ and $\bar{\mathcal{B}}_0 := \mathcal{B}_0$. Then for $n \geq 3$:*

$$(2.28)\quad \begin{aligned} \partial_{1'}\ \bar{\mathcal{B}}_n = \bar{\mathcal{B}}_n\ \partial_{1'} + 2\bar{\mathcal{B}}_{n-1} + (z_1+z_2)\ \ (\partial_{1'}\ \bar{\mathcal{B}}_{n-1} + \bar{\mathcal{B}}_{n-1}\ \partial_{1'}) \\ + z_1 z_2\ \ (\bar{\mathcal{B}}_{n-2}\ \partial_{1'} - \partial_{1'}\ \bar{\mathcal{B}}_{n-2}) \end{aligned}$$

$$(2.29)\quad \partial_{1'}\ \bar{\mathcal{B}}_2 = \bar{\mathcal{B}}_2\ \partial_{1'} + 2\bar{\mathcal{B}}_1 + (z_1+z_2)\ \ (\partial_{1'}\ \bar{\mathcal{B}}_1 + \bar{\mathcal{B}}_1\ \partial_{1'}) + \frac{1}{2} z_1 z_2\ \ (\bar{\mathcal{B}}_0\ \partial_{1'} - \partial_{1'}\ \bar{\mathcal{B}}_0)$$

$$(2.30)\quad \partial_{1'}\ \bar{\mathcal{B}}_1 = \bar{\mathcal{B}}_1\ \partial_{1'} + 2\bar{\mathcal{B}}_1 + \frac{1}{2}(z_1+z_2)\ \ (\partial_{1'}\ \bar{\mathcal{B}}_0 + \bar{\mathcal{B}}_0\ \partial_{1'})$$

Proof. Set $b_n := \sum_{i\geq 0} (-1)^i\ p_{n+i}\ q_i^\perp = \mathcal{B}_n/2$ for $m \geq 0$, i.e., $\bar{\mathcal{B}}_n = b_n$ for $n \geq 1$ and $\bar{\mathcal{B}}_0 = 2b_0$ by (2.2). Then (2.28-29) are immediate consequences of (2.26), and (2.30) follows from (2.27). □

Remark 2.5. In spite of the appearance of the fraction 1/2 in (2.29-30) the application of $\partial_{1'}$ to some P_μ with $\mu \in \mathcal{D}$ does not result in rational coefficients:

since $\mu_1 > \cdots > \mu_l > 0$ for $\mu \equiv \mu_1 \ldots \mu_l \in \mathcal{D}$, there exist only three "situations", where (2.29-30) apply:

$$\partial_{1'}\ \bar{\mathcal{B}}_1(1) = \bar{\mathcal{B}}_0(1) = 1\ ,$$

$$\partial_{1'}\ \bar{\mathcal{B}}_2(1) = 2\bar{\mathcal{B}}_1(1) + (z_1+z_2)\ \partial_{1'}\ \bar{\mathcal{B}}_1(1) = 2P_1 + z_1 + z_2\ ,$$

and

$$\partial_{1'}\ \bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) = \bar{\mathcal{B}}_2(1) + (z_1+z_2)\ \bar{\mathcal{B}}_1(1) + \frac{1}{2}z_1z_2 - \frac{1}{2}z_1z_2\partial_{1'}\ \bar{\mathcal{B}}_0\bar{\mathcal{B}}_1(1)\ .$$

But

$$-\frac{1}{2}z_1z_2\partial_{1'}\ \bar{\mathcal{B}}_0\bar{\mathcal{B}}_1(1) = \frac{1}{2}z_1z_2\partial_{1'}\ \bar{\mathcal{B}}_1\bar{\mathcal{B}}_0(1) = \frac{1}{2}z_1z_2\partial_{1'}\ \bar{\mathcal{B}}_1(1) = \frac{1}{2}z_1z_2\ ,$$

whence

$$\partial_{1'}\ P_{21} = P_2 + (z_1+z_2)P_1 + z_1z_2\ .$$

3. The top polynomials

Obviously the type A top polynomial $X_{\omega_n} = z^{\delta_{n-1}}$ equals $(z_1 \ldots z_{n-1}) \ldots (z_1 z_2) \cdot z_1 = e_{n-1}^{(n-1)} \ldots e_1^{(1)}$. This simple kind of decomposition generalizes for the top polynomials $X^C_{\bar{\omega}_n}$ and $X^D_{\bar{\bar{\omega}}_n}$ as follows:

Theorem 3.1. *Recall that $e_k^{(n)}$ is the elementary symmetric polynomial of degree k in $z_1, \ldots, z_n$ and that it commutes with the operators $\mathcal{B}_n$. Set*

$$\mathcal{B}_{0,n+1} := \sum_{k=0}^{n} e_{n-k}^{(n)} \; \mathcal{B}_{n+k+1} \tag{3.1}$$

for all integers $n \geq 0$, i.e., the $e_{n-k}^{(n)}$ act as multiplication operators, and: $\mathcal{B}_{0,1} = \mathcal{B}_1$, $\mathcal{B}_{0,2} = \mathcal{B}_3 + z_1\mathcal{B}_2$, $\mathcal{B}_{0,3} = \mathcal{B}_5 + (z_1 + z_2)\mathcal{B}_4 + z_1z_2\mathcal{B}_3$, etc. . Then for all $n \in \mathbb{N}$

$$X^C_{\bar{\omega}_n} = \mathcal{B}_{0,n} \circ \mathcal{B}_{0,n-1} \circ \cdots \circ \mathcal{B}_{0,1}(1) \ . \tag{3.2}$$

Set $\bar{\mathcal{B}}_0 := \mathcal{B}_0$, $\bar{\mathcal{B}}_n := \mathcal{B}_n/2$ for $n \geq 1$ in accordance with (1.1) and (2.3), and

$$\bar{\mathcal{B}}_{0,n+1} := \sum_{k=0}^{n} e_{n-k}^{(n)} \; \bar{\mathcal{B}}_{n+k} \tag{3.3}$$

i.e., $\bar{\mathcal{B}}_{0,1} = 1$, $\bar{\mathcal{B}}_{0,2} = \bar{\mathcal{B}}_2 + z_1\bar{\mathcal{B}}_1$, $\bar{\mathcal{B}}_{0,3} = \bar{\mathcal{B}}_4 + (z_1 + z_2)\bar{\mathcal{B}}_3 + z_1z_2\bar{\mathcal{B}}_2$, etc. . Then for all $n \in \mathbb{N}$

$$X^D_{\bar{\bar{\omega}}_n} = \bar{\mathcal{B}}_{0,n} \circ \bar{\mathcal{B}}_{0,n-1} \circ \cdots \circ \bar{\mathcal{B}}_{0,1}(1) \ . \tag{3.4}$$

Proof. Clearly $X^C_{\bar{\omega}_1} = X^C_{(\bar{1})} = Q_1 = \mathcal{B}_{0,1}(1)$. In order to prove (3.2) we have therefore to show that

$$X^C_{\bar{\omega}_{n+1}} = \mathcal{B}_{0,n+1} \; X^C_{\bar{\omega}_n} \tag{3.5}$$

for all $n \geq 1$. From the definition of skew multi-Schur functions (1.3) one has

$$S_{\delta_n/\lambda'}(Z_1, Z_2, \ldots, Z_n) = \det(\ h^{(i)}_{(n+1-i)-\lambda'_j+j-i}\ )_{i,j=1,\ldots,n} = \det(\ h^{(i)}_{n+1+j-\lambda'_j-2i}\ )_{n\times n}$$

for any $\lambda \subset \delta_n$, which by [M2, (3.8″)] equals

$$\det(\ e^{(n+1-i)}_{n+1+j-\lambda_j-2i}\ )_{n\times n} =: E(\lambda) \ .$$

Expansion of the determinant with respect to the first row gives

$$\sum_{j=1}^{n} (-1)^{j-1} \ e^{(n)}_{n-1+j-\lambda_j} \ \det(E(\lambda)^{[j]}) \ ,$$

where $E(\lambda)^{[j]}$ is the cofactor of the element $e^{(n)}_{n-1+j-\lambda_j}$. Note that $e^{(n)}_{n-1+j-\lambda_j} \neq 0 \iff [\ j \leq \lambda_j + 1 \ \wedge \ n \geq \lambda_j + 1 - j\ ] \iff 0 \leq k \leq n$, where $k \equiv k(j, \lambda) := \lambda_j + 1 - j$.

For $\lambda = \lambda_1 \ldots \lambda_n$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $1 \leq j \leq n$ set

$$\lambda[j] := \lambda_1 + 1 \ \ldots \ \lambda_{j-1} + 1 \quad \lambda_{j+1} \ \ldots \ \lambda_n \ .$$

Then

$$\mu_1 \ldots \mu_n = \delta_n + \lambda \iff \mu_1 \ldots \widehat{\mu_j} \ldots \mu_n = \delta_{n-1} + \lambda[j] \ .$$

With Lemma 3.2, the notation $\mu(\widehat{j}) = \mu_1 \dots \widehat{\mu_j} \dots \mu_n$, (1.4), and the observation that two elements of $\{\lambda, \lambda[j], j\}$ determine uniquely the third, one calculates:

$$
\begin{aligned}
X^C_{\bar\omega_{n+1}} &= \sum_{\lambda \subset \delta_n} Q_{\delta_{n+1}+\lambda} \ \det E(\lambda) \\
&= \sum_{\substack{j=1 \\ j \le \lambda_j + 1 \\ \lambda[j] \subset \delta_{n-1}}}^{n} (-1)^{j-1} \, Q_{\delta_{n+1}+\lambda} \, e^{(n)}_{n-k} \ \det(E(\lambda)^{[j]}) \\
&= \sum_{\substack{k=0 \\ \lambda[j] \subset \delta_{n-1}}}^{n} e^{(n)}_{n-k} \, \mathcal{B}_{n+k+1} \, Q_{(\delta_{n+1}+\lambda)(\widehat{j})} \ \det(E(\lambda)^{[j]}) \\
&= \sum_{k=0}^{n} e^{(n)}_{n-k} \, \mathcal{B}_{n+k+1} \sum_{\lambda[j] \subset \delta_{n-1}} Q_{\delta_n + \lambda[j]} \ \det E(\lambda[j]) \\
&= \sum_{k=0}^{n} e^{(n)}_{n-k} \, \mathcal{B}_{n+k+1} \sum_{\lambda \subset \delta_{n-1}} Q_{\delta_n + \lambda} \, S_{\delta_{n-1}/\lambda'} \\
&= \mathcal{B}_{0,n+1} \, X^C_{\bar\omega_n} \ ,
\end{aligned}
$$

which proves (3.2). The proof of formula (3.4) is analogous. □

Lemma 3.2. *With the notations for λ, $E(\lambda)$, $E(\lambda)^{[j]}$, and $\lambda[j]$ of the proof of Theorem* 3.1 *one has*

$$\det(E(\lambda)^{[j]}) = \det E(\lambda[j])$$

for any partition $\lambda \subset \delta_n$. Moreover $\det E(\lambda) \neq 0 \Longleftrightarrow \lambda \subset \delta_n$.

Proof. Actually one has $E(\lambda)^{[j]} = E(\lambda[j])$, which is obvious for the upper indices of the entries. For the lower indices observe that they are completely determined by the lower indices in any row. It is therefore enough to show that the lower indices in the first row of $E(\lambda)^{[j]}$ are equal to the lower indices in the second row of $E(\lambda)$ without the j-th entry, which is

$$
\begin{aligned}
&(n-3)+1-\lambda_1 \ \dots \ (n-3)+(j-1)-\lambda_{j-1} \\
&\qquad\qquad (n-3)+(j+1)-\lambda_{j+1} \ \dots \ (n-3)+n-\lambda_n \\
&= \ (n-2)+1-(\lambda_1+1) \ \dots \ (n-2)+(j-1)-(\lambda_{j-1}+1) \\
&\qquad\qquad (n-2)+j-\lambda_{j+1} \ \dots \ (n-2)+(n-1)-\lambda_n \ .
\end{aligned}
$$

But these are the lower indices in the first row of $E(\lambda[j])$.

For the second assertion observe that more generally by [M2, (3.2)]:
$S_{\mu/\lambda}(Z_1, \dots, Z_n) \neq 0 \Longleftrightarrow \lambda \subset \mu$. □

Let $\mathcal{B}_n^\diamond = e_{n-1}^{(n-1)}, \mathcal{B}_n, \bar{\mathcal{B}}_n$ and $\mathcal{B}_{0,n}^\diamond = e_{n-1}^{(n-1)}, \mathcal{B}_{0,n}, \bar{\mathcal{B}}_{0,n}$ for $\diamond =$ A, C, D, respectively. Then for all $n \in \mathbb{N}$:

$$X_{\omega_n^\diamond}^\diamond = \mathcal{B}_{0,n}^\diamond \circ \mathcal{B}_{0,n-1}^\diamond \circ \cdots \circ \mathcal{B}_{0,1}^\diamond(1)\ . \tag{3.6}$$

It is now possible to draw some first conclusions from the definition (1.9) of Schubert polynomials of type A, C, D and the operator formula (3.6) for the top polynomials. Let $S_\infty^\diamond$ be the direct limit of the groups $S_n^\diamond$, where $\pi \in S_n^\diamond$ is identified with $\pi(1)\dots\pi(n)\ (n+1) \in S_{n+1}^\diamond$ for all n.

Proposition 3.3. *Let $\pi, \rho \in S_\infty^\diamond$. Then*
a) $\partial_\pi \partial_\rho = \partial_{\pi\rho}$, if $l(\pi\rho) = l(\pi) + l(\rho)$, and $= 0$ otherwise.
b) $\partial_\pi X_\rho^\diamond = X_{\rho\pi^{-1}}^\diamond$, if $l(\rho\pi^{-1}) = l(\rho) - l(\pi)$, and $= 0$ otherwise.
c) $\partial_i X_\pi^\diamond = X_{\pi\sigma_i}^\diamond$, if $l(\pi\sigma_i) = l(\pi) - 1$, and $= 0$ otherwise, for $i \in \mathbb{N} \cup \{0, 1'\}$.
d) $X_\pi^\diamond$ is a homogeneous function of degree $l(\pi)$.

Proof. a) follows from the fact that $\partial_\pi \neq 0$ if and only if the sequence of generators representing π is reduced. b) is immediate from a) and the definition (1.9). c) is a specialization of b).

For d) note first that the degree of some $z^d \in \mathbb{Z}[z]$ is the usual polynomial degree $|d|$ and the degree of Q_μ or P_μ is $|\mu| = \mu_1 + \cdots + \mu_l$, since every term has (polynomial) degree $|\mu|$. By (1.2) and Theorem 3.1 the top polynomials are homogeneous of degrees:
$1 + 2 + \cdots + (n-1) = n(n-1)/2 = l(\omega_n)$ in type A,
$1 + 3 + \cdots + (2n-1) = n^2 = l(\bar{\omega}_n)$ in type C, and
$2 + 4 + \cdots + (2n-2) = n(n-1) = l(\bar{\bar{\omega}}_n)$ in type D.
It is well known from type A theory (and in fact not hard to prove) that the the application of a ∂_i with $i \in \mathbb{N} \cup \{1'\}$ to a homogeneous polynomial either lowers the degree by 1 or produces the zero function, whence by c) these divided differences pose no problem. But the same is true for ∂_0 by the product rule (2.5) and (2.6). ☐

It is not hard to see that

$$l(\pi\sigma_i) = l(\pi) - 1 \quad \Longleftrightarrow \quad \pi(i) > \pi(i+1) \tag{3.7}$$

for every $\pi \in S_\infty$ and $i \geq 1$; otherwise $l(\pi\sigma_i) = l(\pi) + 1$. Let $(*, *)$ denote the sign pattern on places i and $i+1$; then for every $\pi \in \bar{S}_\infty$ or $\in \bar{\bar{S}}_\infty$ and $i \in \mathbb{N}$:

$$l(\pi\sigma_i) = l(\pi) - 1 \quad \Longleftrightarrow \quad \begin{cases} \pi(i) > \pi(i+1) & \text{for } (+,+) \\ \text{always} & \text{for } (+,-) \\ \text{never} & \text{for } (-,+) \\ \pi(i) < \pi(i+1) & \text{for } (-,-) \end{cases} \tag{3.8}$$

In case of $\pi \in \bar{S}_\infty$ one has $l(\pi\sigma_0) = l(\pi) - 1 \iff \pi(1)$ has a bar. In case of $\pi \in \bar{\bar{S}}_\infty$ and $i = 1'$ one has:

$$(3.9) \qquad l(\pi\sigma_{1'}) = l(\pi) - 1 \quad \iff \quad \begin{cases} \text{never} & \text{for } (+,+) \\ \pi(1) < \pi(2) & \text{for } (+,-) \\ \pi(1) > \pi(2) & \text{for } (-,+) \\ \text{always} & \text{for } (-,-) \end{cases}$$

Subsequently the following (signed) permutations are especially important:

$$(3.10) \qquad \begin{aligned} \omega^A_{0,n+1} &\equiv \omega_{0,n+1} := \sigma_1 \dots \sigma_n = 2 \dots n\ (n+1)\ 1 \in S_{n+1}\ , \\ \omega^C_{0,n+1} &\equiv \bar{\omega}_{0,n+1} := \sigma_n \dots \sigma_1 \sigma_0 \sigma_1 \dots \sigma_n = 1 \dots n\ \overline{n+1} \in \bar{S}_{n+1}\ , \\ \omega^D_{0,n+1} &\equiv \bar{\bar{\omega}}_{0,n+1} := \sigma_n \dots \sigma_1 \sigma_{1'} \dots \sigma_n = \bar{1}\ 2 \dots n\ \overline{n+1} \in \bar{\bar{S}}_{n+1}\ . \end{aligned}$$

Therefore

$$\begin{aligned} \partial_{(\omega^A_{0,n+1})^{-1}} &= \partial_n \dots \partial_1 \equiv \partial_{n\ \dots\ 1}\ , \\ \partial_{(\omega^C_{0,n+1})^{-1}} &= \partial_{\omega^C_{0,n+1}} = \partial_{n\ \dots\ 1\ 0\ 1\ \dots\ n}\ , \\ \partial_{(\omega^D_{0,n+1})^{-1}} &= \partial_{\omega^D_{0,n+1}} = \partial_{n\ \dots\ 1'\ 1\ \dots\ n}\ . \end{aligned}$$

Moreover for later use we note that by symmetry $\partial_\nu\ e^{(n)}_k = 0$ unless $\nu = n$, in which case it follows from the recursion

$$(3.11) \qquad e^{(n)}_k = e^{(n-1)}_k + z_n\ e^{(n-1)}_{k-1}$$

that

$$(3.12) \qquad \partial_n\ e^{(n)}_k = e^{(n-1)}_{k-1}\ .$$

Using the notation $e^{[n]}_k := e_k(z_2, \dots, z_n)$ with $e^{[n]}_0 := 1$ and $e^{[n]}_k = 0$ unless $0 \le k \le n-1$ one sees similarly from $e^{(n)}_k = e^{[n]}_k + z_1\ e^{[n]}_{k-1}$ that

$$(3.13) \qquad \partial_0 e^{(n)}_k = -e^{[n]}_{k-1}\ .$$

For $e^{[[n]]}_k := e_k(z_3, \dots, z_n)$ with $e^{[[n]]}_0 := 1$ and $e^{[[n]]}_k = 0$ unless $0 \le k \le n-2$ one sees from

$$e^{(n)}_k = e^{[[n]]}_k + (z_1 + z_2)\ e^{[[n]]}_{k-1} + z_1 z_2\ e^{[[n]]}_{k-2}$$

that

$$(3.14) \qquad \partial_{1'} e^{(n)}_k = -2 e^{[[n]]}_{k-1} \quad \text{except for } \partial_{1'} e^{(1)}_1 = \partial_{1'} z_1 = -1.$$

Proposition 3.4. *With the notations of* (1.9) *and* (3.10) *one has for* $\diamond = A,\ C,\ D$:

$$(3.15) \qquad \omega^\diamond_n = \omega^\diamond_{0,n} \dots \omega^\diamond_{0,1}$$

$$(3.16) \qquad \partial_{(\omega^\diamond_{0,n+1})^{-1}}\ X^\diamond_{\omega^\diamond_{n+1}} = X^\diamond_{\omega^\diamond_n}$$

and $X^\diamond_{id} = 1$.

Proof. For arbitrary $\pi \in S_n^\diamond$ the permutation $\omega_{0,n+1}^\diamond \pi \in S_{n+1}^\diamond$ is given by $1_+(\pi)\ 1 := (\pi(1)+1)\dots(\pi(n)+1)\ 1$ in type A, by $\pi\ \overline{n+1} := \pi(1)\dots\pi(n)\ \overline{n+1}$ in type C, and with an additional sign change for the number 1 in type D. Then (3.15) is immediate from the fact that the left action of $S_n^\diamond$ on itself is an action on the 'numbers' (whereas the right action is an action on the 'places').

In type A there are several possible proofs of (3.16). The one that is closest in structure to our proof below in types C and D is the following:

$$\partial_{n\ \dots\ 1}\ e_n^{(n)} \dots e_2^{(2)}\ e_1^{(1)} = (\partial_n\ e_n^{(n)}) \dots\ (\partial_2\ e_2^{(2)})\ (\partial_1\ e_1^{(1)}) \overset{(3.12)}{=} e_{n-1}^{(n-1)} \dots\ e_1^{(1)} \cdot 1\ .$$

In type C the following notation will be useful:

$$\mathcal{B}_{0,n+1}^{(i,j)} := \sum_{k \in \mathbb{Z}} e_{n-k-i}^{(n)}\ \mathcal{B}_{n+k+j} \tag{3.17}$$

It includes as special cases $\mathcal{B}_{0,n+1}^{(0,1)} = \mathcal{B}_{0,n+1}$ and

$$\sum_k (\partial_n\ e_{n-k}^{(n)})\ \mathcal{B}_{n+k+1} = \sum_k e_{n-k-1}^{(n-1)}\ \mathcal{B}_{n+k+1} = \mathcal{B}_{0,n}^{(0,2)}\ ,$$

the latter expression appearing in

$$\partial_n\ \bar{X}_{\bar{\omega}_{n+1}} \overset{(3.2)}{=} \partial_n\ \mathcal{B}_{0,n+1} \circ \bar{X}_{\bar{\omega}_n} = \mathcal{B}_{0,n}^{(0,2)} \circ \bar{X}_{\bar{\omega}_n}\ .$$

We calculate next with the product rule (2.5) and (3.2) that

$$\begin{aligned}
\partial_{n-1}\partial_n \bar{X}_{\bar{\omega}_{n+1}} &= \partial_{n-1}\ \mathcal{B}_{0,n}^{(0,2)} \circ \bar{X}_{\bar{\omega}_n} = \partial_{n-1}\ \left[\sum_k e_{n-k-1}^{(n-1)}\ \mathcal{B}_{n+k+1} \circ \bar{X}_{\bar{\omega}_n}\right] \\
&= \sum_k e_{n-k-2}^{(n-2)}\ \mathcal{B}_{n+k+1} \circ \bar{X}_{\bar{\omega}_n} + \sum_k (\sigma_{n-1} e_{n-k-1}^{(n-1)})\ \mathcal{B}_{n+k+1} \circ \partial_{n-1}\ \bar{X}_{\bar{\omega}_n} \\
&= \left[\sum_k e_{n-k-2}^{(n-2)}\ \mathcal{B}_{n+k+1} \circ \mathcal{B}_{0,n} + \sum_k (\sigma_{n-1} e_{n-k-1}^{(n-1)})\ \mathcal{B}_{n+k+1} \circ \mathcal{B}_{0,n-1}^{(0,2)}\right] \circ \bar{X}_{\bar{\omega}_{n-1}} \\
&\overset{(!)}{=} \mathcal{B}_{0,n+1}^{(0,0)} \circ \mathcal{B}_{0,n-1}^{(0,2)} \circ \bar{X}_{\bar{\omega}_{n-1}}\ ,
\end{aligned}$$

where the equality (!) follows from the equality

$$\begin{aligned}
\sum_{k,l} e_{n-k-2}^{(n-2)}\ e_{n-l-1}^{(n-1)}\ \mathcal{B}_{n+k+1} \circ \mathcal{B}_{n+l} + \sum_{k,l} (\sigma_{n-1} e_{n-k-1}^{(n-1)})\ e_{n-l-2}^{(n-2)}\ \mathcal{B}_{n+k+1} \circ \mathcal{B}_{n+l} \\
= \sum_{k,l} e_{n-k-1}^{(n)}\ e_{n-l-2}^{(n-2)}\ \mathcal{B}_{n+k+1} \circ \mathcal{B}_{n+l}
\end{aligned}$$

for $k+1 \neq l$. Collecting terms with

$$\mathcal{B}_{n+k+1} \circ \mathcal{B}_{n+l} = \mathcal{B}_{n-1+r} \circ \mathcal{B}_{n-1+s}\ \text{ for } r > s$$

shows that the latter equation reduces to

$$e^{(n-2)}_{n-r} e^{(n-1)}_{n-s} - e^{(n-2)}_{n-s} e^{(n-1)}_{n-r} + (\sigma_{n-1} e^{(n-1)}_{n-r+1})\ e^{(n-2)}_{n-s-1} - (\sigma_{n-1} e^{(n-1)}_{n-s+1})\ e^{(n-2)}_{n-r-1} = e^{(n)}_{n-r+1} e^{(n-2)}_{n-s-1} - e^{(n)}_{n-s+1} e^{(n-2)}_{n-r-1}\ ,$$

which can be verified by an elementary but tedious calculation using (3.11). Repeated application of

$$\partial_{n-1}\ \mathcal{B}^{(0,2)}_{0,n} \circ \bar{X}_{\bar{\omega}_n} = \mathcal{B}^{(0,0)}_{0,n+1} \circ \mathcal{B}^{(0,2)}_{0,n-1} \circ \bar{X}_{\bar{\omega}_{n-1}}$$

now shows that

$$\partial_0\ \partial_1\ \ldots\ \partial_n\ \bar{X}_{\bar{\omega}_{n+1}} = \partial_0\ \mathcal{B}^{(0,0)}_{0,n+1} \circ \cdots \circ \mathcal{B}^{(0,0)}_{0,2} \circ \mathcal{B}_{0,1}\ (1)\ .$$

But an examination of the proof of (4.2) below shows that (4.2) is valid, too, if one replaces $\mathcal{B}^{(0,1)}_{0,n+1} = \mathcal{B}_{0,n+1}$ by $\mathcal{B}^{(0,0)}_{0,n+1}$. Therefore

$$\partial_0\ \mathcal{B}^{(0,0)}_{0,n+1} \circ \cdots \circ \mathcal{B}^{(0,0)}_{0,2} \circ \mathcal{B}_{0,1}\ (1) = \mathcal{B}^{(0,0)}_{0,n+1} \circ \cdots \circ \mathcal{B}^{(0,0)}_{0,2} \circ \partial_0\ \mathcal{B}_{0,1}\ (1) = \mathcal{B}^{(0,0)}_{0,n+1} \circ \cdots \circ \mathcal{B}^{(0,0)}_{0,2}\ (1)\ ,$$

whence the computation

$$\partial_{n\ \ldots\ 1}\ \mathcal{B}''_{0,n+1} \circ \cdots \circ \mathcal{B}''_{0,2}\ (1) = (\partial_n\ \mathcal{B}''_{0,n+1}) \ldots\ (\partial_1\ \mathcal{B}''_{0,2})\ (1) = \mathcal{B}_{0,n} \circ \cdots \circ \mathcal{B}_{0,1}\ (1)$$

in analogy to type A above finishes the proof of (3.16) for type C. The proof of (3.16) for type D is analogous, and the last assertion of the lemma is now immediate from (3.15-16). □

Proposition 3.5. (Stability) *The Schubert polynomials $X^\diamond_\pi$ of type $\diamond = A, C, D$ are invariant under the natural embedding of π, i.e., $X^\diamond_{\pi'} = X^\diamond_\pi$ for $n < m$, $\pi \in S^\diamond_n$, and $\pi' = \pi(1) \ldots \pi(n)\ (n+1) \ldots m \in S^\diamond_m$.*

Proof. Clearly it is enough to show the assertion for $m = n+1$. Since the set of reduced sequences for $(\pi')^{-1}\omega^\diamond_{n+1} = (\omega^\diamond_{n+1}\ \pi')^{-1}$ equals the set of reduced sequences for $(\omega^\diamond_{n+1}\ \pi)^{-1} = (\omega^\diamond_{0,n+1}\ \omega^\diamond_n\ \pi)^{-1} = (\pi)^{-1}\ \omega^\diamond_n\ (\omega^\diamond_{0,n+1})^{-1}$, one calculates

$$X^\diamond_{\pi'} \overset{(1.9)}{=} \partial_{(\pi')^{-1}\omega^\diamond_{n+1}}\ X^\diamond_{\omega^\diamond_{n+1}} \overset{(3.10)}{=} \partial_{\pi^{-1}\omega^\diamond_n}\ \partial_{(\omega^\diamond_{0,n+1})^{-1}}\ X^\diamond_{\omega^\diamond_{n+1}} \overset{(3.16)}{=} \partial_{\pi^{-1}\omega^\diamond_n}\ X^\diamond_{\omega^\diamond_n} \overset{(1.9)}{=} X^\diamond_\pi\ .$$

□

4. The recursive structures

In [W1] two natural recursive structures for Schubert polynomials of type A have been established, which describe how a $X_{\pi'}$ with $\pi' \in S_{n+1}$ can be computed from a X_π with $\pi \in S_n$. These two recursive structures correspond to the two natural embeddings of the Coxeter graph of S_n as a connected subgraph into the Coxeter graph of S_{n+1}. Since the Coxeter graphs of type B and D allow by their asymmetry only one kind of natural embedding, we will focus here on that recursive structure in type A (called the 'up case' in [W1]), whose analogs exist in types B, C, D.

Lemma 4.1. *Let* $\mathcal{B}^\diamond_n = e^{(n-1)}_{n-1},\ \mathcal{B}_{0,n},\ \bar{\mathcal{B}}_{0,n}$ *for* $\diamond = A,\ C,\ D$*, respectively. Then*

$$\partial_\nu\ \mathcal{B}^\diamond_{0,n+1} = \mathcal{B}^\diamond_{0,n+1}\ \partial_\nu \quad \text{for all } n \in \mathbb{N},\ \nu \neq n \quad \text{and } \diamond = A,\ C,\ D. \tag{4.1}$$

$$\partial_0\ \mathcal{B}_{0,n+1} = \mathcal{B}_{0,n+1}\ \partial_0 \quad \text{for all } n \in \mathbb{N} \tag{4.2}$$

$$\partial_{1'}\ \bar{\mathcal{B}}_{0,n+1} = \bar{\mathcal{B}}_{0,n+1}\ \partial_{1'} \quad \text{for } n \geq 2 \tag{4.3}$$

Proof. (4.1) is a simple consequence of the product rule (2.5) in type A and of the definitions and (2.15) in types C, D. For (4.2) one computes

$$\begin{aligned}
\partial_0\ \mathcal{B}_{0,n+1}(g) &= \sum_{k=0}^{n}\ \partial_0(e^{(n)}_{n-k}\ \mathcal{B}_{n+k+1}(g)) \\
&= \partial_0\mathcal{B}_{2n+1}(g) + \sum_{k=0}^{n-1}\Big[(e^{[n]}_{n-k} - z_1 e^{[n]}_{n-k-1})(\partial_0\mathcal{B}_{n+k+1}(g)) - e^{[n]}_{n-k-1}\mathcal{B}_{n+k+1}(g)\Big] \\
&= \sum_{k=1}^{n} e^{[n]}_{n-k}(\partial_0\mathcal{B}_{n+k+1}(g)) - \sum_{k=0}^{n-1} e^{[n]}_{n-k-1}\,[\mathcal{B}_{n+k+1}(g) + z_1(\partial_0\mathcal{B}_{n+k+1}(g))] \\
&= \sum_{k=1}^{n} e^{[n]}_{n-k}\,[\ \partial_0\mathcal{B}_{n+k+1}(g) - \mathcal{B}_{n+k}(g) - z_1(\partial_0\mathcal{B}_{n+k}(g))\ ] \\
&\overset{(!)}{=} \sum_{k=0}^{n} e^{(n)}_{n-k}\mathcal{B}_{n+k+1}(\partial_0 g) = \mathcal{B}_{0,n+1}\ (\partial_0(g))
\end{aligned}$$

But since

$$\sum_{k=0}^{n} e^{(n)}_{n-k}\ \mathcal{B}_{n+k+1}(\partial_0 g) = \sum_{k=1}^{n} e^{[n]}_{n-k}\ \mathcal{B}_{n+k+1}(\partial_0 g) + z_1\sum_{k=1}^{n} e^{[n]}_{n-k}\ \mathcal{B}_{n+k}(\partial_0 g)\ ,$$

the equality (!) follows from

$$\partial_0\ \mathcal{B}_{m+1} - \mathcal{B}_m - z_1\ \partial_0\ \mathcal{B}_m = \mathcal{B}_{m+1}\ \partial_0 + z_1\ \mathcal{B}_m\ \partial_0 \quad \text{for all } m \geq 0,$$

which is (2.16).
For (4.3) one computes similarly

$$\begin{aligned}
\partial_{1'}\ \bar{\mathcal{B}}_{0,n+1}(g) &= \sum_{k=2}^{n}\ e^{[[n]]}_{n-k}\left[\partial_{1'}\bar{\mathcal{B}}_{n+k} - 2\bar{\mathcal{B}}_{n+k-1} - (z_1+z_2)\partial_{1'}\bar{\mathcal{B}}_{n+k-1} + z_1 z_2\partial_{1'}\bar{\mathcal{B}}_{n+k-2}\right](g) \\
&\overset{(!)}{=} \sum_{k=2}^{n}\ e^{[[n]]}_{n-k}\left[\bar{\mathcal{B}}_{n+k}\partial_{1'} + (z_1+z_2)\bar{\mathcal{B}}_{n+k-1}\partial_{1'} + z_1 z_2\bar{\mathcal{B}}_{n+k-2}\partial_{1'}\right](g)\ ,
\end{aligned}$$

which follows from (2.28). □

Proposition 4.2.

$$X^\diamond_{\omega^\diamond_{0,n}} = \mathcal{B}^\diamond_{0,n}\ (1) \qquad \text{for } \diamond = A,\ C,\ D \text{ and } n \in \mathbb{N}, \tag{4.4}$$

and more generally

$$X^\diamond_{\omega^\diamond_{0,n}\cdots\ \omega^\diamond_{0,k}} = \mathcal{B}^\diamond_{0,n} \circ \cdots \circ \mathcal{B}^\diamond_{0,k}\ (1) \quad \text{for } \diamond = A,\ C,\ D \text{ and } 1 \le k \le n, \tag{4.5}$$

which includes (3.6) *and* (4.4) *as special cases.*

Proof. Using the fact that $\omega^\diamond_n = (\omega^\diamond_n)^{-1}$ one computes

$$\begin{aligned} X^\diamond_{\omega^\diamond_{0,n}} &\overset{(1.9)}{=} \partial_{(\omega^\diamond_{0,n})^{-1}\ \omega^\diamond_n} X^\diamond_{\omega^\diamond_n} \overset{(3.15)}{=} \partial_{\omega^\diamond_{n-1}} X^\diamond_{\omega^\diamond_n} \overset{(3.6)}{=} \partial_{\omega^\diamond_{n-1}} \mathcal{B}^\diamond_{0,n} \circ X^\diamond_{\omega^\diamond_{n-1}} \\ &\overset{(4.1-3)}{=} \mathcal{B}^\diamond_{0,n}\ \partial_{\omega^\diamond_{n-1}} X^\diamond_{\omega^\diamond_{n-1}} = \mathcal{B}^\diamond_{0,n}\ \partial_{(\omega^\diamond_{n-1})^{-1}} X^\diamond_{\omega^\diamond_{n-1}} \overset{(3.15-16)}{=} \mathcal{B}^\diamond_{0,n}\ (1)\ . \end{aligned}$$

The proof of (4.5) is analogous. □

Theorem 4.3. *With the notation of* (3.6) *one has for every* $\pi \in S^\diamond_n$ *and* $\diamond = A,\ C,\ D$:

$$X^\diamond_{\omega^\diamond_{0,n+1}\pi} = \mathcal{B}^\diamond_{0,n+1}\ X^\diamond_\pi\ . \tag{4.6}$$

Proof. One calculates

$$\begin{aligned} X^\diamond_{\omega^\diamond_{0,n+1}\pi} &\overset{(1.9)}{=} \partial_{(\omega^\diamond_{0,n+1}\pi)^{-1}\omega^\diamond_{n+1}}\ X^\diamond_{\omega^\diamond_{n+1}} \\ &\overset{(3.6)}{=} \partial_{(\omega^\diamond_{0,n+1}\pi)^{-1}\omega^\diamond_{n+1}}\ \mathcal{B}^\diamond_{0,n+1}\ X^\diamond_{\omega^\diamond_n} \\ &\overset{(3.15)}{=} \partial_{\pi^{-1}\omega^\diamond_n}\ \mathcal{B}^\diamond_{0,n+1}\ X^\diamond_{\omega^\diamond_n} \\ &\overset{(4.1-3)}{=} \mathcal{B}^\diamond_{0,n+1}\ \partial_{\pi^{-1}\omega^\diamond_n}\ X^\diamond_{\omega^\diamond_n} \\ &\overset{(1.9)}{=} \mathcal{B}^\diamond_{0,n+1}\ X^\diamond_\pi\ . \end{aligned}$$

□

For the next result we need some notations (recall that ∂_1 and $\partial_{1'}$ in type D commute):

$$\begin{aligned} \mathcal{B}^\diamond_{k,n+1} &:= \partial_{n+1-k} \ldots \partial_n\ \mathcal{B}^\diamond_{0,n+1} \quad \text{for } 1 \le k \le n \text{ and } \diamond = \text{A, C, D} \\ \mathcal{B}_{n+1,n+1} &:= \partial_0\ \mathcal{B}_{n,n+1} \\ \mathcal{B}_{n+1+k,n+1} &:= \partial_k \ldots \partial_1\ \mathcal{B}_{n+1,n+1} \quad \text{for } 1 \le k \le n \\ \bar{\mathcal{B}}_{n',n+1} &:= \partial_{1'}\ \bar{\mathcal{B}}_{n-1,n+1} \\ \bar{\mathcal{B}}_{n+1,n+1} &:= \partial_{1'}\ \bar{\mathcal{B}}_{n,n+1} = \partial_1\ \bar{\mathcal{B}}_{n',n+1} \\ \bar{\mathcal{B}}_{n+1+k,n+1} &:= \partial_k \ldots \partial_2\ \bar{\mathcal{B}}_{n+1,n+1} \quad \text{for } 2 \le k \le n \end{aligned} \tag{4.7}$$

Note that k in $\mathcal{B}^\diamond_{k,n+1}$ takes values between zero and n, $2n+1$, $2n$ for $\diamond$ = A, C, D, respectively. If we therefore set

$$\begin{aligned} \mathbb{L}^A_{n+1} &:= \{ l^A \equiv l = [l_n\ \ldots\ l_0] \mid 0 \le l_\nu \le \nu,\ \nu = 0, \ldots, n\ \} \\ \mathbb{L}^C_{n+1} &:= \{ l^C \equiv \bar{l} = \overline{l_n\ \ldots\ l_0} \mid 0 \le l_\nu \le 2\nu+1,\ \nu = 0, \ldots, n\ \} \\ \mathbb{L}^D_{n+1} &:= \{ l^D \equiv \bar{\bar{l}} = \overline{\overline{l_n\ \ldots\ l_0}} \mid 0 \le l_\nu \le 2\nu,\ \nu = 0, \ldots, n, \text{ or } l_\nu = \nu'\ \} \end{aligned} \tag{4.8}$$

then all the operators

$$\mathcal{B}^{\diamond}_{l^{\diamond},n+1} := \mathcal{B}^{\diamond}_{l_n,n+1} \circ \cdots \circ \mathcal{B}^{\diamond}_{l_0,1} \quad \text{for } l^{\diamond} \in \mathbb{L}^{\diamond}_{n+1} \text{ and } \diamond = \text{A, C, D} \tag{4.9}$$

are well defined. Since the sets $S_n^{\diamond}$ and $\mathbb{L}_n^{\diamond}$ have equal cardinalities for all $n \in \mathbb{N}$ and $\diamond =$ A, C, D, there are many possible bijections between two corresponding sets. Some of them are especially useful for the study of Schubert polynomials (see [W1, Sec.4] and [W2, Sec.2] for a systematic exploration): the maps $\mathcal{L}^{\diamond}_{n+1} : S^{\diamond}_{n+1} \longrightarrow \mathbb{L}^{\diamond}_{n+1}$ are defined by

$$l^A_{n+1-i}(\pi) \equiv l_{n+1-i} := \sharp\{\ j \mid j > \pi^{-1}(i),\ \pi(j) > i\ \} \tag{4.10}$$

$$l^C_{i-1}(\pi) \equiv \bar{l}_{i-1} := \begin{cases} \sharp\{\ j \mid j > \pi^{-1}(i),\ \pi(j) < i\ \} & \text{if } u(\pi^{-1}(i)) = 1 \\ i + \sharp\{\ j \mid j < \pi^{-1}(i),\ \pi(j) < i\ \} & \text{if } u(\pi^{-1}(i)) = 0 \end{cases} \tag{4.11}$$

(4.12)

$$l^D_{i-1}(\pi) \equiv \bar{\bar{l}}_{i-1} := \begin{cases} \sharp\{\ j \mid j > \pi^{-1}(i),\ \pi(j) < i\ \} & \text{if } i \notin E(\pi),\ u(\pi^{-1}(i)) = 1 \\ i - 1 + \sharp\{\ j \mid j < \pi^{-1}(i),\ \pi(j) < i\ \} & \text{if } i \notin E(\pi),\ u(\pi^{-1}(i)) = 0 \\ (i-1) & \text{if } i \in E(\pi),\ D(i) \text{ even} \\ (i-1)' & \text{if } i \in E(\pi),\ D(i) \text{ odd} \end{cases}$$

where $i = 1, \dots, n+1$, $u(i) = 1$, if there is a bar on the letter i, and $= 0$ otherwise, $E(\pi) = \{i \mid \neg\exists j :\ j < \pi^{-1}(i),\ \pi(j) < i\}$, $D(i) = \sum_{\substack{j \geq i \\ u(j)=0}} 1$, and $0' := 0$. Examples are: $\mathcal{L}^A(362154) = [223000]$, $\mathcal{L}^C(\bar{3}\bar{5}1\bar{2}4) = \overline{37201}$ and $\mathcal{L}^C(35\bar{1}2\bar{4}) = \overline{60330}$, and $\mathcal{L}^D(35\bar{1}2\bar{4}) = \overline{\overline{50220'}}$ with $E(\pi) = \{1,3\}$ and $D(1) = 3$, $D(3) = 2$, and $\mathcal{L}^D(6\bar{7}\bar{5}4\bar{2}\bar{3}1) = \overline{\overline{55'4'3110'}}$ with $E(\pi) = \{6,5,4,2,1\}$.

Theorem 4.4. *With the notations of* (4.7-12) *above one has for every* $\pi \in S^{\diamond}_{n+1}$ *(*$n \geq 0$*) and* $\diamond =$ *A, C, D:*

$$X^{\diamond}_{\pi} = \mathcal{B}^{\diamond}_{\mathcal{L}^{\diamond}(\pi),n+1}\ (1)\ . \tag{4.13}$$

Proof. Let $\pi \in S^{\diamond}_{n+1}$. Then in type A determine k such that $\pi(n+1-k) = 1$. If $k \geq 1$ then it follows from Prop.3.3 c) and (3.7) that

$$X_\pi = \partial_{n+1-k} \dots \partial_n\ X_{\omega_{0,n+1}\pi'} \overset{(4.6)}{=} \partial_{n+1-k} \dots \partial_n\ \mathcal{B}^A_{0,n+1}\ X_{\pi'} \overset{(4.7)}{=} \mathcal{B}^A_{k,n+1}\ X_{\pi'}\ ,$$

where $\pi' \in S_n$ is the same as π with the 1 at place $n+1-k$ removed and all remaining entries diminished by 1, e.g. for $\pi = 3517264$ one has $k = 5$ and $\pi' = 246153$. If $k = 0$ then of course Theorem 4.3 directly gives $X_\pi = \mathcal{B}^A_{0,n+1}\ X_{\pi'}$. Repetition of this procedure gives an operator representation of X_π, which is of the form (4.8). But instead of reducing π to π' one can simply count for $l^A_{n+1-i}(\pi)$ the cardinality of numbers greater than i on all places right to the place of i, which is $l^A_{n+1-i}(\pi)$ as given by (4.9).

In types C and D the argument is analogous, but this time on starts with the position of the number $n+1$ instead of 1. In case $n+1$ has a bar in π, one simply moves $\overline{n+1}$

from its place $n+1$ in $\omega_{0,n+1}^{\diamond}\pi'$ by elementary transpositions to the left to its place in π. If $n+1$ has no bar in π, one first moves $\overline{n+1}$ from place $n+1$ in $\omega_{0,n+1}^{\diamond}\pi'$ to place 1 in type C [1 or 2 in type D], than changes the sign [applies $\sigma_{1'}$ in type D], and finally moves it to the right to its place in π. The number [and "kind" in type D] of transpositions necessary are recorded by the $l_{n+1-i}^{\diamond}(\pi)$ of (4.11-12). Note that in type D the set $E(\pi) = \{i_s, \ldots, i_1\}$ with $i_s > \cdots > i_1 = 1$ and $\pi^{-1}(i_s) = 1$ is exactly the set of those numbers, where a decision between $(i-1)$ or $(i-1)'$ has to be made. □

Using (2.4), (2.16), and (2.28-30) it is now easy to compute — or teach a computer how to compute — Schubert polynomials of type A – D directly form the definition (1.9) or recursively with (4.13).

Example 4.5. In type A let $n = 4$ and $\pi = 41253$. Then $\mathcal{L}^A(\pi) = [32010]$ and

$$\begin{aligned}\mathcal{B}^A_{\mathcal{L}^A(\pi),5}\ (1) &= \mathcal{B}^A_{3,5} \circ \mathcal{B}^A_{2,4} \circ \mathcal{B}^A_{0,3} \circ \mathcal{B}^A_{1,2} \circ \mathcal{B}^A_{0,1}\ (1) \\ &= \partial_2\partial_3\partial_4\ e_4^{(4)}\ \partial_2\partial_3\ e_3^{(3)}\ e_2^{(2)}\ \partial_1\ e_1^{(1)} = z_1^3 z_2 + z_1^3 z_3 + z_1^3 z_4 = X_\pi\ .\end{aligned}$$

In type C let $n = 2$ and $\pi = 3\bar{2}\bar{1}$. Then $\mathcal{L}^C(\pi) = \overline{310}$ and

$$\begin{aligned}\mathcal{B}_{\mathcal{L}^C(\pi),3}\ (1) &= \mathcal{B}^C_{3,3} \circ \mathcal{B}^C_{1,2} \circ \mathcal{B}^C_{0,1}\ (1) \\ &= \partial_0\partial_1\partial_2\ \mathcal{B}_{0,3}\ \partial_1\ \mathcal{B}_{0,2}\ \mathcal{B}_{0,1}\ (1) \\ &= \partial_0\partial_1\partial_2\ \mathcal{B}_{0,3}\ \partial_1\ (Q_{31} + z_1 Q_{21}) \\ &= \partial_0\partial_1\partial_2\ (\mathcal{B}_5 + (z_1+z_2)\mathcal{B}_4 + z_1 z_2 \mathcal{B}_3)\ Q_{21} \\ &= \partial_0\partial_1\partial_2\ (Q_{521} + (z_1+z_2)Q_{421} + z_1 z_2 Q_{321}) \\ &= \partial_0\partial_1\ (Q_{421} + z_1 Q_{321}) \\ &= \partial_0\ (Q_{321}) = \partial_0\ \mathcal{B}_3 \circ \mathcal{B}_2 \circ \mathcal{B}_1 \circ\ (1) \\ &= (\mathcal{B}_3 + z_1\mathcal{B}_2) \circ (\partial_0\ \mathcal{B}_2) \circ \mathcal{B}_1 \circ\ (1) \\ &= (\mathcal{B}_3 + z_1\mathcal{B}_2) \circ (\mathcal{B}_2 + z_1\mathcal{B}_1) \circ (\partial_0\ \mathcal{B}_1)(1) \\ &= (\mathcal{B}_3 + z_1\mathcal{B}_2) \circ (\mathcal{B}_2 + z_1\mathcal{B}_1)\ (1) \\ &= Q_{32} + z_1 Q_{31} + z_1^2 Q_{21} = X^C_\pi\ .\end{aligned}$$

In type D let $n = 2$ and $\pi = 321$. Then $\mathcal{L}^D(\pi) = \overline{\overline{2'10}}$ and

$$
\begin{aligned}
\bar{\mathcal{B}}_{\mathcal{L}^D(\pi),3}\ (1) &= \bar{\mathcal{B}}_{2',3} \circ \bar{\mathcal{B}}_{1,2} \circ \bar{\mathcal{B}}_{0,1}\ (1) \\
&= \partial_{1'}\partial_2\ \bar{\mathcal{B}}_{0,3}\ \partial_1\ \bar{\mathcal{B}}_{0,2}\ (1) \\
&= \partial_{1'}\partial_2\ \bar{\mathcal{B}}_{0,3}\ \partial_1\ (P_2 + z_1 P_1) \\
&= \partial_{1'}\partial_2\ (\bar{\mathcal{B}}_4 + (z_1 + z_2)\bar{\mathcal{B}}_3 + z_1 z_2 \bar{\mathcal{B}}_2)\ (P_1) \\
&= \partial_{1'}\partial_2\ (P_{41} + z_1 P_{31} + z_2 P_{31} + z_1 z_2 P_{21}) \\
&= \partial_{1'}\ (P_{31} + z_1 P_{21}) \\
&= \bar{\mathcal{B}}_3 \partial_{1'} \bar{\mathcal{B}}_1(1) + 2\bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) + (z_1 + z_2)\partial_{1'}\bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) + (z_1 + z_2)\bar{\mathcal{B}}_2(1) \\
&\quad + z_1 z_2 \bar{\mathcal{B}}_1(1) - \bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) - z_2\partial_{1'}\bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) \\
&= \bar{\mathcal{B}}_3(1) + \bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) + z_1\partial_{1'}\bar{\mathcal{B}}_2\bar{\mathcal{B}}_1(1) + (z_1 + z_2)\bar{\mathcal{B}}_2(1) + z_1 z_2 \bar{\mathcal{B}}_1(1) \\
&= P_3 + P_{21} + z_1\bar{\mathcal{B}}_2(1) + z_1(z_1 + z_2)\bar{\mathcal{B}}_1(1) + z_1^2 z_2 + (z_1 + z_2)P_2 + z_1 z_2 P_1 \\
&= P_3 + P_{21} + 2z_1 P_2 + z_1^2 P_1 + z_2 P_2 + 2z_1 z_2 P_1 + z_1^2 z_2 = X_\pi^D\ .
\end{aligned}
$$

5. Further properties of Schubert polynomials of type C and D

We discuss some basic properties of the terms occurring in Schubert polynomials of type A, C, and D. Recall from the introduction that the ring of integral cohomology $H^*(Fl_n, \mathbb{Z})$ of the flag manifolds $Fl_n = G/B$ is isomorphic to the ring $P_n = \mathbb{Q}[z_1, \ldots, z_n]$ factored by the ideal I_n of W-invariant polynomials without constant term. The $\mathbb{Z}$-modules of residues of the rings P_n/I_n have the following natural basis:

$$H_n \equiv H_n^A = \{x^l := z_1^{l_{n-1}} \ldots z_{n-1}^{l_1} \mid l \in \mathbb{L}_n\} \text{ with}$$

$$\mathbb{L}_n := \{l = l_{n-1} \ldots l_1 \mid 0 \le l_\nu \le \nu,\ \nu = 1, \ldots, n-1\ \}$$

in type A, i.e., the basis has cardinality $n! = |S_n|$;

$$H_n^C = \{Q_\mu x^l \mid \mu \in \mathcal{D},\ \mu \subset \delta_n,\ l \in \mathbb{L}_n\}$$

in type C, i.e., the basis has cardinality $n!\ 2^n = |\bar{S}_n|$; and

$$H_n^D = \{P_\mu x^l \mid \mu \in \mathcal{D},\ \mu \subset \delta_{n-1},\ l \in \mathbb{L}_n\}$$

in type C, i.e., the basis has cardinality $n!\ 2^{n-1} = |\bar{\bar{S}}_n|$. Moreover set

$$
\begin{aligned}
\widetilde{H}_n^C &= \{Q_\mu x^l \mid \mu \in \mathcal{D},\ \mu \subset \delta_n + \delta_{n-1},\ l \in \mathbb{L}_n,\ |\mu| + |l| \le n^2\ \}\ , \\
\widetilde{H}_n^D &= \{P_\mu x^l \mid \mu \in \mathcal{D},\ \mu \subset \delta_{n-1} + \delta_{n-1},\ l \in \mathbb{L}_n,\ |\mu| + |l| \le n(n-1)\ \}\ .
\end{aligned}
$$

It is well known that $\{X_\pi \mid \pi \in S_n\}$ is again a $\mathbb{Z}$-basis of H_n^A, since for every $l \in \mathbb{L}_n$ there exists a unique $\pi \in S_n$, such that x^l is the leading term (i.e., with coefficient 1) of X_π with respect to reverse lexicographic order. In addition every X_π is a linear combination with non-negative integer coefficients of elements of H_n^A.

In types C and D the situation is slightly more complicated: it is easy to see from Theorem 4.4 that for $\diamond = C, D$ and $\pi \in S_n^\diamond$ every $X_\pi^\diamond$ is a $\mathbb{Z}$-linear combination of elements of $\widetilde{H}_n^\diamond$, and — using in addition Lemma 5.1 below and the type A theory — that for every $Q_\mu x^l \in H_n^C$ $[P_\mu x^l \in H_n^D]$ there is a unique $\pi \in S_n^C$ $[\pi \in S_n^D]$, such that $Q_\mu x^l$ $[P_\mu x^l]$ is the leading term of X_π^C $[X_\pi^D]$ with respect to the term order defined in [BH, p.476]. It therefore follows that the set $\{X_\pi^\diamond \mid \pi \in S_n^\diamond\}$ is a $\mathbb{Z}$-basis of $H_n^\diamond\ (mod\ I_n)$ [BH, Thm.3C, Thm.4C]. The non-negativity of the coefficients occurring in the $X_\pi^\diamond$ is a deeper property, which is immediate from the (very involved) combinatorial constructions in [BH, Thm.3-4], but is not easily seen by our methods — to the contrary of the following final result:

Proposition 5.1. [BH, Thm.3B, Thm.4B] *Let $\mu = \mu_1 \dots \mu_l \in \mathcal{D}$, $\mu^- = \mu_2 \dots \mu_l$, $\bar{\pi}(\mu) = \bar{\mu_1}\ \dots\ \bar{\mu_l}\ 1 \dots \in \bar{S}_{\mu_1}$, and $\bar{\bar{\pi}}(\mu) = \bar{\nu_1}\ \dots\ \bar{\nu_l}\ 1 \dots \in \bar{\bar{S}}_{\mu_1}$, where $\nu_i = \mu_i + 1$, taking $\mu_l = 0$ if necessary to make the number of parts even, and where '$1 \dots$' in $\bar{\pi}(\mu)$ and $\bar{\bar{\pi}}(\mu)$ stands for the elements of the set $\{1, \dots, \mu_1\} \setminus \{\mu_1, \dots, \mu_l\}$ written in increasing order. Then*

$$X^C_{\bar{\pi}(\mu)} = Q_\mu \tag{5.1}$$

$$X^D_{\bar{\bar{\pi}}(\mu)} = P_\mu\ . \tag{5.2}$$

Proof. We proceed by induction over the length of μ, where the case $l(\mu) = 0$ is trivial. Using the recursive structure (Thm.4.4) and the stability (Prop.3.5) of Schubert polynomials, if $\mu_1 - \mu_2 > 1$, one computes

$$X^C_{\bar{\pi}(\mu)} = \partial_1 \dots \partial_{\mu_1 - 1} \mathcal{B}_{0,\mu_1}\ (Q_{\mu^-}) = \mathcal{B}_{\mu_1}\ (Q_{\mu^-}) = Q_\mu\ ,$$

and

$$X^D_{\bar{\bar{\pi}}(\mu)} = \partial_1 \dots \partial_{\nu_1 - 1} \bar{\mathcal{B}}_{0,\nu_1}\ (P_{\mu^-}) = \bar{\mathcal{B}}_{\nu_1 - 1}\ (P_{\mu^-}) = \bar{\mathcal{B}}_{\mu_1}\ (P_{\mu^-}) = P_\mu\ .$$

□

References

[B] A. Borel, Sur la cohomologie des espaces fibré principaux et des espaces homogènes des groupes Lie compacts, Ann. Math. **57** (1953), 115 - 207.

[B1] S. Billey, Transition equations for isotropic flag manifolds, Discrete Math., to appear.

[B2] S. Billey, Kostant polynomials and the cohomology ring for G/B, Duke J. Math., to appear.

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert Cells and cohomology of the spaces G/B, Russian Math. Surveys **28** (1973), 1-26.

[BH] S. Billey, M. Haiman, Schubert Polynomials for the Classical Groups, J. Amer. Math. Soc. **8** (1995), 443-482.

[EG] P. Edelman, C. Greene, Balanced Tableaux, Advances in Math. **63** (1987), 42 - 99.

[F] W. Fulton, *Young tableaux*, Cambridge University Press, Cambridge, 1997.

[FGP] S. Fomin, S. Gelfand, A. Postnikov, Quantum Schubert polynomials, J. Amer. Math. Soc. **10** (1997), 565 - 596. .

[FK] S. Fomin, A.N. Kirillov, Combinatorial B_n-analogues of Schubert polynomials, Trans. Amer. Math. Soc. **348** (1996), 3591-3620.

[HH] P.N. Hoffman, J.E. Humphreys, *Projective Representations of the Symmetric Groups*, Clarendon Press, Oxford, 1992.

[Hi] H. Hiller, *Geometry of Coxeter groups*, Pitman, Boston, 1982.

[Hu] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge, 1990.

[J] Naihuan Jing, Vertex Operators and Hall-Littlewood Symmetric Functions, Advances in Math. **87** (1991), 226 - 248.

[KKL] A. Kerber, A. Kohnert, A. Lascoux, SYMMETRICA, an object oriented computer-algebra system for the symmetric group, J. Symbolic Computation **14** (1992), 195 - 203.

[L] A. Lascoux, Polynômes de Schubert. Une approache historique, Discrete Math. **139** (1995), 303 - 317.

[LS] A. Lascoux, M.-P. Schützenberger, Décompositions dans l'algébre des différences divisées, Discrete Math. **99** (1992), 165 - 179.

[M1] I.G. Macdonald, Schubert polynomials, in: A.D. Kendwell (ed.), *Surveys in Combinatorics*, London Math. Soc. Lec. Notes Ser. **166**, 73 - 99, Cambridge University Press, Cambridge, 1991.

[M2] I.G. Macdonald, *Notes on Schubert polynomials*, Publications du L.A.C.I.M., vol. 6, Université du Québec, Montréal, 1991.

[M3] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press, Oxford 1979.

[Sa] B.E. Sagan, *The Symmetric Group*, Wadsworth & Brooks/Cole, Pacific Grove, California (1991).

[V] S. Veigneau, SP, a Package for Schubert Polynomials Realized with the Computer Algebra System MAPLE, J. Symbolic Computation **23** (1997), 413 - 426.

[W1] R. Winkel, Recursive and combinatorial properties of Schubert polynomials, Sem. Loth. Comb. **B38c** (1996), 29 pp.

[W2] R. Winkel, A combinatorial derivation of the Poincarè polynomials of the finite irreducible Coxeter groups, preprint (1997).

[W3] R. Winkel, On the expansion of Schur and Schubert polynomials into standard elementary monomials, Advances in Mathematics **136** (1998), 224 - 250.

[Z] A.V. Zelevinsky, *Representations of Finite Classical Groups*, Lecture Notes in Math. **869**, Springer, Berlin, 1981.

Institut für Reine und Angewandte Mathematik, RWTH Aachen, D-52056 Aachen, Germany, *papers/preprints*: http://www.iram.rwth-aachen.de/ ~winkel/

E-mail address: winkel@iram.rwth-aachen.de