# On algebraic and combinatorial properties of Schur and Schubert polynomials

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# Introduction

The content of the present work[1] is part of the fast growing field of *algebraic combinatorics*, which is the study of the interaction between algebraic structures — like polynomial rings, and groups and their representations — and combinatorial structures — like permutations, tableaux, and partially ordered sets —. *Schur polynomials* and their generalizations — like Jack and Macdonald and in particular *Schubert polynomials* — occupy an important place in the intersection of algebra and combinatorics, where all the above mentioned structures are intimately connected. Schur polynomials appear for example naturally in the representation theory of groups, in the enumeration of discrete structures under symmetries, and as generating functions (with non-negative integer coefficients) for irreducible characters and tableaux, whereas the generalizations of Schur polynomials have numerous applications in computer algebra, multivariate statistics, algebraic and differential geometry (intersection theory), numerics (multivariate Newton interpolation), and quantum physics.

The present work contains contributions to the structural theory of the above mentioned polynomials and as a result new algorithms, too. In particular we give

- recursive structures and algorithms for Schubert polynomials (valuable for both theory and computation);
- a description of the various polynomials in terms of 'Baxter operators', which yields new effective algorithms for their explicit computation;
- first results on the expansion of Schur and Schubert polynomials into 'standard elementary monomials' (important for mathematical physics);
- a very explicit description of the 'weak Bruhat order' for the finite irreducible Coxeter groups;
- formulas (in terms of Schubert polynomials) for the calculation of the number of reduced words of permutations;
- a new combinatorial bijection between 'standard Young tableaux' and 'reduced words' of 'Grassmannian permutations'.

Let us explain the above statements in more detail:

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The symmetric groups $S_n$ of permutations of the natural numbers $1, \ldots, n$ are a venerable subject of widespread use. Despite the fact that they are easily described, they are far from being completely understood, except for the smallest numbers $n$. Therefore they are a subject of continuing research. The same is true, though on a smaller scale, for *symmetric polynomials*, i.e. polynomials invariant under arbitrary permutations of their variables. The first appearance of symmetric polynomials was probably in the dependency of the coefficients of a polynomial from its roots: for

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} + \cdots + (-1)^n a_n = (x - x_1) \ldots (x - x_n) \ ,$$

the coefficients $a_1, \ldots, a_n$ are necessarily symmetric polynomials in the roots $x_1, \ldots, x_n$, namely the elementary symmetric polynomials. As discussed in more detail in Sections 1.1 and 1.2 the elementary symmetric and other types of symmetric polynomials (monomial, complete, power sum, and Schur symmetric polynomials) are indispensable companions of the development of many classical theories including the theory of determinants, of polynomial equations – especially Galois theory –, of groups and representations. Central to the theory of representations of the symmetric groups $S_n$ and the general linear groups $GL(n, \mathbb{C})$ of matrices over the complex numbers are the *Schur polynomials* $s_\lambda(x_1, \ldots, x_n)$, where the index $\lambda$ stands for a *partition* of a natural number $n \in \mathbb{N}$ into additive parts. (Usually one writes a partition $\lambda \equiv \lambda_1 \ldots \lambda_s$ as a non-increasing (finite) sequence of natural numbers: $\lambda_1 \geq \cdots \geq \lambda_s \geq 1$, and $\lambda \vdash N$ as a shorthand for: $\lambda$ is a partition with $|\lambda| := \lambda_1 + \cdots + \lambda_s = N$.)

We summarize some historical and mathematical facts about the representation theoretic origin of Schur polynomials, although we will not use it explicitly later on. Around 1900 Frobenius and Schur have developed the theory of linear representations of groups and have applied it to the symmetric groups and general linear groups: For the symmetric groups Frobenius observed that the partitions $\lambda \vdash n$ parameterize both the conjugacy classes of permutations in $S_n$ and the irreducible representations of $S_n$ (over $\mathbb{C}$); the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ — introduced much earlier by Jacobi — turned out to be the generating functions for the irreducible characters of $S_n$ upon expansion into power sum symmetric polynomials. For the general linear groups Schur investigated rational representations $D$ of $GL(n, \mathbb{C})$, i.e. the entries $a_{ij}$ of $A \in GL(n, \mathbb{C})$ are taken as variables and the entries of $D(A)$ are rational functions in the $a_{ij}$, and he showed first that the irreducible rational representation $D$ can be reduced to polynomial representations upon multiplication by a power of the determinant polynomial. In fact these polynomial entries are homogeneous of degree say $d$. Then the irreducible characters of the representation $D$ are the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ for some

$\lambda \vdash d$, more precisely

$$\text{trace } D(A) = s_\lambda(x_1, \ldots, x_n) \ ,$$

where the variables $x_1, \ldots, x_n$ stand for the $n$ eigenvalues of $A$. In particular the equivalence classes of these irreducible representations of $GL(n, \mathbb{C})$ are parameterized again by partitions $\lambda \vdash d$.

Right from the beginnings of representation theory around 1900 A. Young introduced combinatorial methods into the investigation of Schur polynomials, namely the tableau later named Young tableaux: let a partition $\lambda \vdash N$ be represented graphically by its *Ferrer diagram*, which is a left adjusted array of $N$ unit boxes in the plane with $\lambda_1$ boxes in the first row, $\lambda_2$ boxes in the second row below, etc. . Then the Young tableaux are certain fillings of the Ferrer diagram $\lambda$ with natural numbers, such that the monomials occurring in a Schur polynomial $s_\lambda$ can be characterized by a simple combinatorial rule involving the Young tableaux of shape $\lambda$ (cf. Section 1.2).

The importance of Schur polynomials is not at all exhausted by their appearance in representation theory: they are central in Redfield's theory of enumeration of discrete structures under symmetries (cf. [Kr]) and there are several one- and two-parameter families of symmetric polynomials, which contain the Schur polynomials as special cases. The basic reason for the ubiquity of symmetric polynomials is that situations abound, in which *indistinguishable* variables $x_1, \ldots, x_n$ are connected by a polynomial law – one example was that of $x_1, \ldots, x_n$ being the roots of a polynomial. A more recent example is that of $x_1, \ldots, x_n$ being stochastic variables, which has lead in multivariate statistics to the appearance of Jack symmetric polynomials, a family of polynomials depending on a real parameter $\alpha$, which specialize to Schur polynomials in case of $\alpha = 1$. Another example is the Calogero-Sutherland model describing the long-range interaction of $n$ indistinguishable quantum particles on a circle and – in this connection – of the Virasoro algebra of analytical vector fields on the circle. Therefore in quantum physics generalizations of Schur polynomials like the Jack polynomials or the Macdonald polynomials are much investigated, and it is virtually impossible to find a monthly volume of the electronic journal *quantum algebra* (`q-alg`), which does not contain a paper concerned with one of the generalizations of Schur polynomials — either the symmetric ones mentioned above or the non-symmetric Schubert polynomials discussed below.

In Chapter 1 and 2 of the present book the symmetric generalizations of Schur polynomials will be investigated in more detail: Chapter 1 contains a historical introduction to symmetric polynomials and describes the following generalizations of Schur polynomials: Q-Schur functions and Hall-Littlewood, Jack, and Macdonald polynomials, where emphasis is on their combinatorial definitions in terms of Young tableaux.

Endomorphism and derivations are certainly the most important equationally defined types of linear operators on an algebra $A$. There are only a few other equationally defined types of operators possible, one of which are the *Baxter operators*, which were introduced in the context of probability theory by G. Baxter [Ba] around 1960. Baxter operators have been shown by Rota to be closely connected to the theory of symmetric functions and may have further applications, for example, in the theory of hypergeometric functions and quantum groups (see Section 2.3). In Section 2.1 we will describe the Baxter operator approach of G. P. Thomas to Schur polynomials and in Section 2.2 our own results on how to represent Hall-Littlewood, Jack, and Macdonald polynomials in terms of Baxter operators. Baxter operators provide an effective method for the computation of these otherwise rather inaccessible families of polynomials: for a fixed partition $\lambda \vdash N$ the sequence of Schur polynomials $s_\lambda$ (and their generalizations) in an increasing number of variables can be computed easily by applying a sequence of $N$ Baxter operators to the sequence of variables $x = (x_1, x_2, x_3, \dots)$. Moreover Baxter operators provide a deeper understanding of the *combinatorics* of these symmetric polynomials — a subject, which demands further exploration. Therefore Section 2.4 contains a collection of open problems in this direction.

All remaining Chapters 3 to 9 of the present work are concerned with *Schubert polynomials*. These polynomials form a $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[x] = \mathbb{Z}[x_1, x_2, x_3, \dots]$ indexed not by partitions as in the symmetric case, but by (finite) permutations $\pi$. We briefly explain here, why $\mathbb{Z}$-bases of symmetric polynomials naturally have partitions as indices and the — in general non-symmetric — Schubert polynomials permutations $\pi$: a permutation $\pi \in S_n$ can be interpreted via its *Lehmer code* (cf. Section 3.2) — an important tool in the combinatorial theory of Schubert polynomials — as an arbitrary finite sequence $d = (d_1, \dots, d_n)$ of non-negative integers and vice versa. Clearly the set of all monomials $\{x^d = x_1^{d_1} \dots x_n^{d_n}\}$ is the natural basis for the free $\mathbb{Z}$-module $\mathbb{Z}[x_1, \dots, x_n]$, and equally clearly one gets a basis again, if one replaces every $x^d$ by a polynomial of the form $x^d+$ 'higher terms', where 'higher' means that for all additional terms $x^{d'}$ one has $d < d'$ in a suitable total order on all the $d$'s, e.g., in lexicographic order. Schubert polynomials are of this latter form ' $x^d+$ higher terms'. If one considers $\mathbb{Z}$-bases for the subring $\Lambda_n$ of symmetric polynomials in $\mathbb{Z}[x_1, \dots, x_n]$, then for every $d$ occurring as an exponent in a symmetric polynomial also every permutation of $d = (d_1, \dots, d_n)$ must occur. Hence one can choose as a representative the unique nowhere increasing sequence among all permutations of the $d_1, \dots, d_n$, which is a partition.

Schubert polynomials have their origin in the intersection theory of flags of subspaces of a vector space, and as such they are of interest to algebraic geometers and topologist as well as to differential geometers and physicists concerned with fiber bundles. The theory of Schubert polynomials blends in a beautiful way algebraic, combinatorial, and geometric ideas. To get a first impression how this might work, let us briefly consider the example of the Jordan normal form theorem for nilpotent matrices $M$ in

$$Nil(n, \mathbb{C}) := \{ \ n \times n \text{ matrices } M \text{ over } \mathbb{C} \text{ with } M^n = 0 \ \} \ .$$

In the first place there is an *algebraic* result saying that every $M \in Nil(n, \mathbb{C})$ is equivalent to a certain matrix of Jordan normal form; but in addition the theorem gives a classification of all $M \in Nil(n, \mathbb{C})$ by means of a *combinatorial* object, namely the partition $\lambda \vdash n$, which records the size of Jordan blocks; and finally, the partitioning of the set $Nil(n, \mathbb{C})$ into equivalence classes having the same Jordan normal form yields a *geometric* stratification of the $(n^2 - n)$-dimensional variety $Nil(n, \mathbb{C})$.

For the Schubert case let us begin with the projective complex $(n-1)$-dimensional space $\mathbb{CP}^{n-1}$, which can be identified with the space of all lines or 1-dimensional subspaces of a complex $n$-dimensional vector space $V$. A natural generalization of projective spaces are the *Grassmannian manifolds* $G(k, n) \equiv G_k(V)$ of all $k$-planes or $k$-dimensional subspaces of $V$, i.e., $\mathbb{CP}^{n-1} = G(1, n)$. (The manifold structure on $G_k(V)$ is induced from that of $Aut(V)$ resp. $GL(n, \mathbb{C})$ by taking the quotient with respect to the stabilizer of some fixed $k$-plane.) Grassmann manifolds appear not only in algebraic and geometric invariant theory (cf. [Stu, Chp.3]), but also as the target of the Gauss map in differential and algebraic geometry, which sends a $k$-dimensional tangent space to an element of a Grassmannian of $k$-planes. Still more generally one can consider the *flag manifolds* $F(V)$ of all flags $V_\bullet$ of a complex $n$-dimensional vector space $V$:

$$V_\bullet : \ \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V \ ,$$

where the $V_i$ are subspaces of $V$ of complex dimension $i$. (The manifold structure on $F(V)$ is induced from that of $Aut(V)$ resp. $GL(n, \mathbb{C})$ by taking the quotient with respect to the stabilizer of some fixed flag.) In generalization of the Gauss map they work as "classifying spaces" in the theory of fiber bundles over differentiable, analytic, or algebraic manifolds.

Projective, Grassmannian, and flag spaces all have a natural *cell decomposition.* To consider the simplest case of projective space $\mathbb{CP}^{n-1}$ first let us fix a flag $V_\bullet$ in $V = \mathbb{C}^n$

and define

$$\langle i \rangle := \{ l \in \mathbb{CP}^n \mid l \subset V_{i+1},\ l \not\subseteq V_i \} \ .$$

Then each cell $\langle i \rangle$ is homeomorphic to $\mathbb{C}^i$, for example:

$$\begin{aligned}\langle 1 \rangle = \{ l \in \mathbb{CP}^n \mid l \subset V_2,\ l \not\subseteq V_1 \} &= \{ l \subset V_2,\ l \neq V_1 \} \\ &\approx \mathbb{CP}^1 \setminus \{V_1\} \approx S^2 \setminus \{ \text{ one point } \} \approx \mathbb{C} \ .\end{aligned}$$

The more general decompositions of the Grassmann manifolds into *Schubert cells* (indexed by partitions) and of flag manifolds into their *Schubert cells* (indexed by permutations) as well as the historic origin of *Schubert calculus* as an intersection calculus of algebraic varieties is described in more detail in Sections 3.1-2.

The simple example of the cell $\langle 1 \rangle$ in $\mathbb{CP}^n$ above shows that a cell, namely $\langle 1 \rangle \approx \mathbb{C}$, may be cohomologically trivial, but that its closure, $\approx S^2$ in this case, is not. This is all the more true for the *Schubert varieties*, which are the topological closures of the Schubert cells. From intersection theory of algebraic topology it follows that the intersection of Schubert varieties can be computed by taking the cup product of their cohomology classes. Since the calculation with abstract cohomology classes is rather cumbersome even in the important special cases of Grassmannian and flag manifolds, it comes as a relief that there exist explicit polynomial representatives of the cohomology classes of 'Schubert varieties': the Schur polynomials in the Grassmann case and the Schubert polynomials in the flag case. The intersection of Schubert varieties can then be computed just by expanding the product of the associated Schur/Schubert polynomials into a linear combination of Schur/Schubert polynomials: the occurring Schur/Schubert polynomials correspond to the Schubert varieties occurring in the intersection and their (necessarily non-negative) coefficients are the respective multiplicities. The complicated geometric features of Schubert varieties then translate into algebraic and combinatorial properties of Schur resp. Schubert polynomials.

Since all Grassmann case–Schubert varieties are embeddable as flag case–Schubert varieties, one concludes that Schur polynomials are special Schubert polynomials. Indeed, exactly the Schubert polynomials associated to *Grassmannian permutations* are Schur polynomials, where a Grassmannian permutation is defined as follows: let $\pi(\lambda) \in S_n$ $(n := s + \lambda_1)$ be the permutation associated to a partition $\lambda$ of length $s$ defined by[2]

$$\pi(\lambda) := \lambda_s + 1 \quad \lambda_{s-1} + 2 \quad \dots \quad \lambda_1 + s \quad 1\ 2\ 3\ \dots \quad ,$$

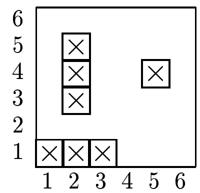
where 1 2 3 ... means that all letters $1, \dots, n$ not occurring on the first $s$ places are appended in increasing order. One can obtain all Grassmannian permutations in this

[2]A permutation $\pi \in S_n$ is given here as a 'list' $\pi \equiv \pi(1) \dots \pi(n)$.

way, if one admits $\lambda_s \geq 0$ instead of $\lambda_s \geq 1$. Equivalently, a Grassmannian permutation is a permutation $\pi$ with a unique descent, i.e., $\pi(i) > \pi(i+1)$ for exactly one $i \in \{1, \ldots, n-1\}$.

The important point now is that Schubert polynomials can be computed in many different ways, most basically by the application of a $\pi$-dependent sequence of divided difference operators to a certain "top-polynomial". This has been discovered in 1982 by A. Lascoux and M.-P. Schützenberger. Therefore one can forget about the rather complicated geometric and cohomological background of Schubert polynomials and develop a purely algebraic and combinatorial theory, which facilitates not only the intersection calculus of Schubert varieties, but even the classical calculations for the representation theory, invariant theory, and combinatorics of the symmetric groups: for example, the multiplication of Schur polynomials according to the Littlewood-Richardson rule, which involves manipulations of Young tableaux can be substituted by simpler computations with Schubert polynomials and manipulations of permutations. This is the basic idea behind the computer algebra package SYMMETRICA [KKL]. In addition Schubert polynomials have an application in numerics: they generalize Newton interpolation to several variables [LS6].

Chapter 4 ([W2]) gives an account of the basic Lascoux-Schützenberger theory of Schubert polynomials under the perspective of two new recursive methods for their calculation (Prop.4.3.3). One of these methods is used to give new relatively simple proofs of the basic properties of Schubert polynomials. Since full proofs of all results are given, this chapter can be read as an introduction to the Lascoux-Schützenberger theory complementary to [KKL], which has no proofs, and [M1, M2], which contains additional material on orthogonality and 'double Schubert polynomials'. Special attention is placed on calculations with Lehmer codes and reduced sequences, whereas in [M2] 'vexillary permutations' are investigated in greater depth. In addition a fast and simple method for the recursive calculation of Schubert polynomials is presented (Section 4.6), which avoids completely divided differences (and thereby the computation of intermediary terms, which eventually cancel). At the end of Section 4 we describe a combinatorial rule for the generation of Schubert polynomials previously conjectured by Kohnert in his thesis [Ko]. This rule uses the — also otherwise important — *diagram* $D(\pi)$ of a permutation $\pi \in S_n$, which is a certain subset of an $n \times n$-array of unit squares or boxes in the plane, e.g., for $\pi = 263154 \in S_6$ one has $D(\pi) =$



(cf. Section 3.2). Kohnert's conjecture says that all monomials occurring in a Schubert polynomial $X_\pi$ can be found by looking at the set of box diagrams derivable from $D(\pi)$ according to a simple combinatorial rule. The main tools for the proof of Kohnert's conjecture in [W3] are the recursive structure of Schubert polynomials from Section 4.3 and a in addition a partial order on the set of box diagrams. (As a byproduct one obtains (combinatorial) proofs of two other rules for the generation of Schubert polynomials (based on box diagrams), namely, the more complicated rule of Bergeron, and the rule of Magyar, which is in fact a simplification of Bergeron's rule.) Because of its elegance, Kohnert's rule has attracted a great deal of attention, but our proof (cf. [W3]) is the first and only one so far.

In Chapter 5 (parts of [W6, W7]) the Baxter operator approach to graded Schur functions of G.P. Thomas (Section 2.1) is generalized to the Schubert case and leads there to the introduction of (graded) Schubert functions and skew Schubert polynomials and functions.

We should mention here that the symmetric group $S_n$ of permutations can be viewed as special case of a *Coxeter groups* defined by generators and relations. Namely, the generators are the elementary transpositions $\sigma_\nu = (\nu, \nu+1)$, and the relations are:

$$\begin{aligned} &\text{(i) } \sigma_\nu^2 = \mathbf{1}, \\ &\text{(ii) } \sigma_\nu\sigma_{\nu'} = \sigma_{\nu'}\sigma_\nu \ , \quad \text{if } |\nu - \nu'| \geq 2, \text{ and} \\ &\text{(iii) } \sigma_\nu\sigma_{\nu+1}\sigma_\nu = \sigma_{\nu+1}\sigma_\nu\sigma_{\nu+1} \ . \end{aligned}$$

Every permutation $\pi \in S_n$ can be written as a product of minimal length $l = l(\pi)$ in the generators: $\pi = \sigma_{a_1} \ldots \sigma_{a_l}$. In this case $\sigma_{a_1} \ldots \sigma_{a_l}$ is called a *reduced word* and $a \equiv a_1 \ldots a_l$ a *reduced sequence* for $\pi$.

Since the computation of the formulas for graded Schubert functions in Chapter 5 relies fundamentally on the knowledge of the set of reduced words for a permutation, it seems natural that in turn the number of reduced words of a permutation can be determined with the help of Schubert functions: we describe algebraic formulas and a combinatorial procedure, which allow the *effective* determination of the number of reduced words for an arbitrary permutation in terms of Schubert polynomials. Comparison of the Baxter operator formula for Schur functions and its analog for Schubert functions shows that the two sets of 'standard' Young tableaux of shape $\lambda$ and of reduced words for the Grassmannian permutation $\pi(\lambda)$ must have equal cardinality. In Section 5.7 an especially simple combinatorial bijection between these sets is described.

As discussed in more detail in Section 3.3 below there are not only Schubert polynomials associated to the symmetric groups $S_n$ — more exactly called 'Schubert polynomials of type A' — but also Schubert polynomials of types B, C, and D, which are associated to other Weyl groups. They were first described by Billey and Haiman [BH] with the help of a rather cumbersome combinatorial method. In Chapter 7 ([W10]) we give a unified algebraic treatment of Schubert polynomials of types A — D in the style of the Lascoux-Schützenberger theory. We use the operator formulas for Q-Schur and P-Schur functions of Section 1.3 to give for Schubert polynomials of type B, C, D: (1) simple and natural forms of the 'top polynomials', (2) formulas for the easy computation with all divided differences, (3) recursive structures (in extension of the recursive structure for type A of Chapter 4), and (4) simple derivations of some basic properties.

Inspired by the close relationship between the recursive structures of Schubert polynomials of types A (Section 4.3), and B, C, D (Section 6.4) on one hand and codes for (signed) permutations and 'standard reduced words' on the other hand we study in Chapter 7 ([W5]) the combinatorics of Coxeter groups in greater depth. (For an account of the relations between classical matrix groups, Weyl and Coxeter groups, and groups of (signed) permutations see Section 3.3 .) More specifically we explore the consequences of a simple partitioning property of the weak Bruhat order into order-isomorphic parts, where 'weak Bruhat order' is a partial order on Coxeter groups naturally associated to reduced words. This leads to the following results: First, the Poincaré polynomials of the finite irreducible Coxeter groups and the Poincaré series of the affine Coxeter groups on three generators are derived by an elementary combinatorial method avoiding the use of Lie theory and invariant theory. And second, non-recursive methods for the computation of 'standard reduced words' for (signed) permutations are described.

In theoretical physics one studies string theory as a unified model for all elementary particles and their interactions. The recent tremendous efforts to understand the mathematics of string theory and mirror symmetry have lead to the basic concepts of Gromov-Witten invariants and of *quantum cohomology*. The latter is a deformation of ordinary cohomology, where a "common point" in the intersection of varieties is replaced by an "equivalence class of rational curves", with which the varieties intersect. Here the "curves" are models of a "string" $\mathbb{CP}^1$ and in the limit, where the strings degenerate to points, one recovers ordinary cohomology. One of the rare non-trivial cases, in which quantum cohomology has been computed and studied explicitly, is that of the Schubert varieties for flag manifolds.

Fomin, Gelfand, and Postnikov [FGP] have found recently (based on [GK]) that quantum Schubert polynomials, which are multi-parameter deformations of (ordinary) Schubert polynomials, can be computed through a simple quantization procedure from (ordinary) Schubert polynomials in case the latter are given as a linear combination of 'standard elementary monomials' (SEM ). Therefore the "understanding" of quantum Schubert polynomials is reduced to the "understanding" of the SEM expansion of (ordinary) Schubert polynomials. Chapter 8 ([W9]) contains first results in this direction: in the special case of Schur polynomials a rather complete understanding has been reached through a determinantal formula and a combinatorial rule; in the general case there are so far only partial results available, but several surprising conjectures substantiated by computer calculations are likely to stimulate further research.

There are currently many different (and of course strongly interconnected) approaches for the understanding and computation of Schubert polynomial $X_\pi$ ($\pi \in S_n$) available:

1. The original definition 3.(3.4) of Lascoux and Schützenberger using divided differences, which permit also the recursive methods of Section 4.3.
2. The recursive generation of the $X_\pi$ without divided differences based on Monk's rule and the use of the Bruhat order on permutations. There are two variants of this method: (1) the 'transition formula' variant of Lascoux and Schützenberger departing from the identity permutation [LS2] (see also [B1]), and (2) the 'ascent-descent' variant departing from one of the top-permutations $\omega_n$ (see Section 4.6).
3. The "generating function" or "generating product" approach using the nil-Coxeter relations [FS] (see Remark 7.2.9).
4. The formula of Billey, Jockusch, and Stanley using the set of reduced words for $\pi$ [BJS] (see Section 5.2).
5. The approach via sums of 'mixed shift and multiplication operators' applied to the sequence of variables $(x_1, x_2, x_3, \dots)$ (see Section 5.3).
6. The approach via 'balanced labeled tableaux' [FGRS].
7. The approach via 'configurations of labeled pseudo-lines' [FK2] or 'rc-graphs' [BB].
8. The approach via 'flagged Schur modules' associated to a diagram [KP].
9. The approach via 'Kostant polynomials' [B2].

10. And finally the combinatorial generation via sets of ‘box diagrams’ discussed in Section 4.7.

Whereas the whole theory of (Schur and) Schubert polynomials is developed on the basis of orthogonality by Macdonald [M1-6] (see also Sections 1.4 and 8.3), the emphasis in our presentation is on recursiveness and associated partial orders. We therefore think that the present work makes a worthwhile complementary reading to Macdonald’s re-known presentations of the subject. Of course, it is well possible to develop everything on the basis of pseudo-lines ([FS,FK1-4]) or Kostant polynomials [B2] and we urge the reader to consult at least these references to get a more complete picture. Anyway, the theory of Schubert polynomials has turned out to be a most vivid and multi-facetted subject with many connections — just like the more traditional Schur polynomials — and more approaches are likely to come!

Chapters 5 to 8 can be read independently from each other, but some reference is made to elementary properties of Schubert polynomials from Chapters 3 and 4 as well as to some definitions of symmetric polynomials and functions from Chapter 1. Chapter 2 relies on Chapter 1. Items are numbered ‘locally’ without the number of the chapter, if referred to in the same chapter — like Proposition 4.3, formula (1.6), Section 2 —, but ‘globally’ with the number of the chapter, if referred to in another chapter — like Proposition 4.4.3, formula (1.1.6), Section 3.2 —.

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