FROM QUANTUM COHOMOLOGY TO ALGEBRAIC COMBINATORICS — THE EXAMPLE OF FLAG MANIFOLDS

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Abstract. The computation and understanding of quantum cohomology is a very hard problem in mathematical physics (string theory). We review in non-technical terms how in case of the flag manifolds this problem turns out to be at its core a non-trivial problem in algebraic combinatorics.

The basic quest of physics in the 20th century is that for a unified theory of the physical world. During the 80's *string theory* emerged as the most promissing candidate of such a "Theory of Everything". In string theory the classical point particle is replaced by a closed string (a copy of the "circle" S^1). This has at least two advantages: 1. the singularities (infinities of potentials) of the classical theory are avoided, and 2. strings may have vibrational eigenstates (just as mechanical strings) thereby representing different kinds of particles. In 1988 Witten [Wt1] has established the notion of the *σ-model* as a fundamental ingredient of a string theory and subsequently *quantum cohomology* as the core of the technical machinerie of σ-models [Wt2]. To give a first idea what quantum cohomology is about consider the cup product of ordinary cohomology classes: its structure constants represent the intersection properties of submanifolds or subvarieties of a given manifold or variety, where intersection means *intersection in points.* Now with the deformation of classical point particles to strings the new quantum version of intersection has the meaning of *intersection with a common rational curve* (without changing the manifold or varitety itself and without requiering that the subobjects meet at all in a common point). The main difficulty with this new and "axiomatically" introduced quantum cohomology (cf. [KM]) was to show the *associativity* of the quantum cup product. Thus it came as a major breakthrough, when in 1995 Ruan and Tian first established this associativity and therefore the existence of quantum cohomology for a reasonably broad class of manifolds (semi-positive symplectic manifolds and Ricci-flat Kähler = Calabi-Yau manifolds [RT]).

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With the rigorous foundations of quantum cohomology being laid the next problem of course was the actual computation of the structure constants (= Gromov Witten invariants) of the quantum cohomology ring. In fact this had been attempted before (assuming existence) and as a result it had turned out that the seemingly inaccessable computation of the quantum cohomology ring for one manifold could be replaced by the much easier *counting of equivalence classes of embedded rational curves* on a completely different but physically equivalent manifold, the *mirror manifold.* This gave rise to the now highly developed topic of *mirror symmetry.* (For uninitiated mathematicians the introductionary accounts given in [V] and [Mo1, Mo2] are usefull; see also the electronic journals *q-alg* and *alg-geom* for recent developments).

In the present paper we follow a different line of reasoning: in case of the flag manifolds it is possible to identify as the "really hard part" of the computation and understanding of Gromov Witten invariants certain non-trivial problems of algebraic combinatorics .

For any fixed natural number n let G/B denote the *flag manifold*, where G is the group $GL_n(\mathbb{C})$ of invertible $n \times n$-matrices over the complex numbers and B is the subgroup of upper triangular matrices. The topology of the quotient is induced from the natural topology of G. The name flag manifold can be justified as follows: take any element of G with column vectors $f^1, \dots, f^n$; then the sequence of subspaces of $\mathbb{C}^n$

$$\{0\} \subset \langle f^1 \rangle \subset \langle f^1, f^2 \rangle \subset \cdots \subset \langle f^1, \dots, f^n \rangle = \mathbb{C}^n$$

is the associated flag. Since the same flag can be represented by many different ordered bases of $\mathbb{C}^n$ resp. elements of G, we factorise by B, which contains all representatives of the flag generated by the canonical basis vectors of $\mathbb{C}^n$ resp. the unit matrix of G. A classical result of Borel [Bo] says that the cohomology of the flag manifolds over the integers is isomorphic to a factor ring of polynomials:

$$H^*(Fl_n, \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n]/(e_1^{(n)}, \dots, e_n^{(n)}) \ , \tag{1}$$

where

$$e_i^{(n)} \equiv e_i(x_1, \dots, x_n) := \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \dots \ x_{j_i} \tag{2}$$

is the *elementary symmetric polynomial* of degree i in the variables $x_1, \dots, x_n$. By a slight variation of Gauss elimination one sees that every $g \in G$ can be decomposed as $g = b\hat{\pi}b'$, where b and b' are upper triangular and $\hat{\pi}$ is a uniquely determined permutation matrix corresponding to an element π of the symmetric group S_n. This induces a decomposition of G/B into submanifolds $(B\hat{\pi}B)/B$, where the decomposition bears the name of Bruhat (due to his generalization of this decomposition to the finite irreducible Coxeter groups) and where the submanifolds are known as Schubert varieties (due to their appearence in Schubert's calculus of enumerative geometry). Now the Bruhat decomposition induces under the Borel isomorphism a distinguished basis for the cohomology: namely the basis of *Schubert polynomials* $X_\pi \in \mathbb{Z}[x_1, \dots, x_n]/(e_1^{(n)}, \dots, e_n^{(n)})$ indexed by permutations $\pi \in S_n$, which represent the cohomology classes of the Schubert varieties. Since the Schubert polynomials are invariant under the natural embedding of the symmetric groups

$$S_n \hookrightarrow S_{n+1},\ \pi \mapsto \pi(1) \dots \pi(n)\ (n+1)\ ,$$

they can be identified with polynomials in $\mathbb{Z}[x_1, x_2, \dots]$. To give a quick idea about how Schubert polynomials look like and how they can be computed — there are now at least a dozen different approaches! — consider first the (right) weak Bruhat order on S_n: π covers π' in this order iff $\pi(k) > \pi(k+1)$ and $\pi' = \pi\sigma_k$, where σ_k is the elementary transposition of the numbers $\pi(k)$ and $\pi(k+1)$. Therefore the top permutation in this order is $n\ (n-1)\ \dots 2\ 1$ and the bottom permutation is $1 \dots n = id_n$. For the top permutation one "chooses" as the corresponding Schubert polynomial $x_1^{n-1} x_2^{n-2} \dots x_n^0$ (this lucky choice is due to Lascoux and Schützenberger (1982), who subsequently developed the basics of the beautifull algebraic-combinatorial theory of Schubert polynomials) and for the bottom element one has $X_{id} = 1$ as a consequence of the following prescription: if π covers π' in weak Bruhat order with $\pi' = \pi\sigma_k$, then $X_{\pi'} = \partial_k X_\pi$, where ∂_k is the divided difference operator acting on the variables x_k and x_{k+1} (the introduction of divided differences is due simultaneously to Bernstein, Gelfand, and Gelfand (1973) and Demazure (1973-74)). For more thorough information about Schubert polynomials the reader is invited to consult [L,M1,M2,W1,W3] and the references therein.

Let H_n denote the free $\mathbb{Z}$-module underlying $\mathbb{Z}[x_1, \dots, x_n]/(e_1^{(n)}, \dots, e_n^{(n)})$; so far we know that the set $\{X_\pi \mid \pi \in S_n\}$ of Schubert polynomials is a $\mathbb{Z}$-basis of H_n. A different more familiar basis is given by the monomials

$$\{x^k := x_1^{k_{n-1}} \dots x_{n-1}^{k_1} \mid k \in K_n\}\ ,$$

where

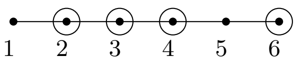
(3) $$K_n := \{k = k_1 \dots k_{n-1} \mid 0 \le k_\nu \le \nu;\ \nu = 1, \dots, n-1\}\ .$$

(Note that $|K_n| = n!$.) A third $\mathbb{Z}$-basis, already described by Lascoux and Schützenberger in 1989 [LS], but not much used since then, has turned out to be of utmost importance for the quantum cohomology of flag manifolds: the basis of *standard elementary polynomials* (SEM)

(4) $$\{e_k := e_{k_{n-1}}^{(1)} \dots e_{k_1}^{(n-1)} \mid k \in K_n\}\ .$$

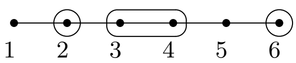
(Note that a SEM is non-symmetric in general.) Of course it is possible to express any of the three bases in terms of the others; for example, as indicated above the expansion of Schubert polynomials into ordinary monomials can be achieved by application of divided differences to the top Schubert polynomial; also the expansion of Schubert polynomials into SEM can be done by application of divided differences to the top polynomial $x_1^{n-1}x_2^{n-2}\dots x_{n-1} = e_{1\ 2\dots(n-1)}$.

To see what happens under quantization with all the items introduced so far we recall next the natural combinatorial interpretation of the elementary symmetric polynomials. The monomials occuring in $e_i^{(n)}$ correspond to all possible coverings of the n points of $[1,n] \subset \mathbb{Z}$ by i disjoint monomers; as an example let $n = 6$ and $i = 4$, then



gives $x_2x_3x_4x_6$.

If the monomers $\{k\}$, which correspond to variables x_k, are supplemented by *dimers* $\{k, k+1\}$, which correspond to variables q_k of weight 2, then the set of all possible coverings of $[1,n] \subset \mathbb{Z}$ by i disjoint monomers and dimers gives all the monomials of the *quantized elementary symmetric polynomials* $\widetilde{e}_i^{(n)} = \widetilde{e}_i(x_1,\dots,x_n,q_1,\dots,q_{n-1})$. For example:



gives $q_3x_2x_6$.

Clearly, $e_i^{(n)} = \widetilde{e}_i(x_1,\dots,x_n,0,\dots,0)$, which explains why $q_1,\dots,q_{n-1}$ are called deformation parameters. In 1995 Givental and Kim [GK] published the following quantized form of Borel's result: the intgral quantum cohomology ring of the flag manifolds is isomorphic to a factor ring of polynomials with deformation parameters $q_1,\dots,q_{n-1}$:

$$QH^*(Fl_n,\mathbb{Z}) \cong \mathbb{Z}[x_1,\dots,x_n,q_1,\dots,q_{n-1}]/(\widetilde{e}_1^{(n)},\dots,\widetilde{e}_n^{(n)}) \ . \tag{5}$$

On the basis of [GK] Fomin, Gelfand, and Postnikov found a fantastically simple characterization of quantum Schubert polynomials $\widetilde{X}_\pi$, whose structure coefficients for multiplication are the Gromov-Witten invariants of the flag manifolds:

Theorem 1. [FGP] *Assume that the* SEM *expansion of a Schubert polynomial*

$$X_\pi = \sum_{k\in K_n} \alpha_k(\pi)\ e_k \quad \text{with } \alpha_k(\pi) \in \mathbb{Z} \tag{6}$$

is known. Then the quantum Schubert polynomial $\widetilde{X}_\pi$ can be computed by simply quantizing the elementary symmetric polynomials occuring on the r.h.s. of (6)*, i.e., replacing every $e_i^{(n)}$ by $\widetilde{e}_i^{(n)}$.*

This result brings up immeadiately the following two problems:

Problem 1. *Is there in general a* quantization adapted expansion *of ordinary cohomology classes, which "trivializes" quantization ?*

And, since the "blind" computation of SEM expansions of Schubert polynomials by application of divided differences (see above) yields almost no understanding of the outcome:

Problem 2. *Investigate and understand the properties of the quantization adapted expansion of cohomology classes.*

A positive answer to the first problem could lead to an extraordinarily important and interesting interaction between (algebraic) combinatorics and theoretical physics. — Subsequently we restrict to the second problem in case of the flag manifolds.

For algebraic geometers the importance of Schubert polynomials is due to their role in intersection theory, but algebraic combinatorialists are attracted by a different fact: *Schur polynomials are special Schubert polynomials.* (This is a natural consequence of the relations between Schubert varieties, flag manifolds, and Grassmannian manifolds, and has been proved first by Lascoux and Schützenberger.) More precisely:

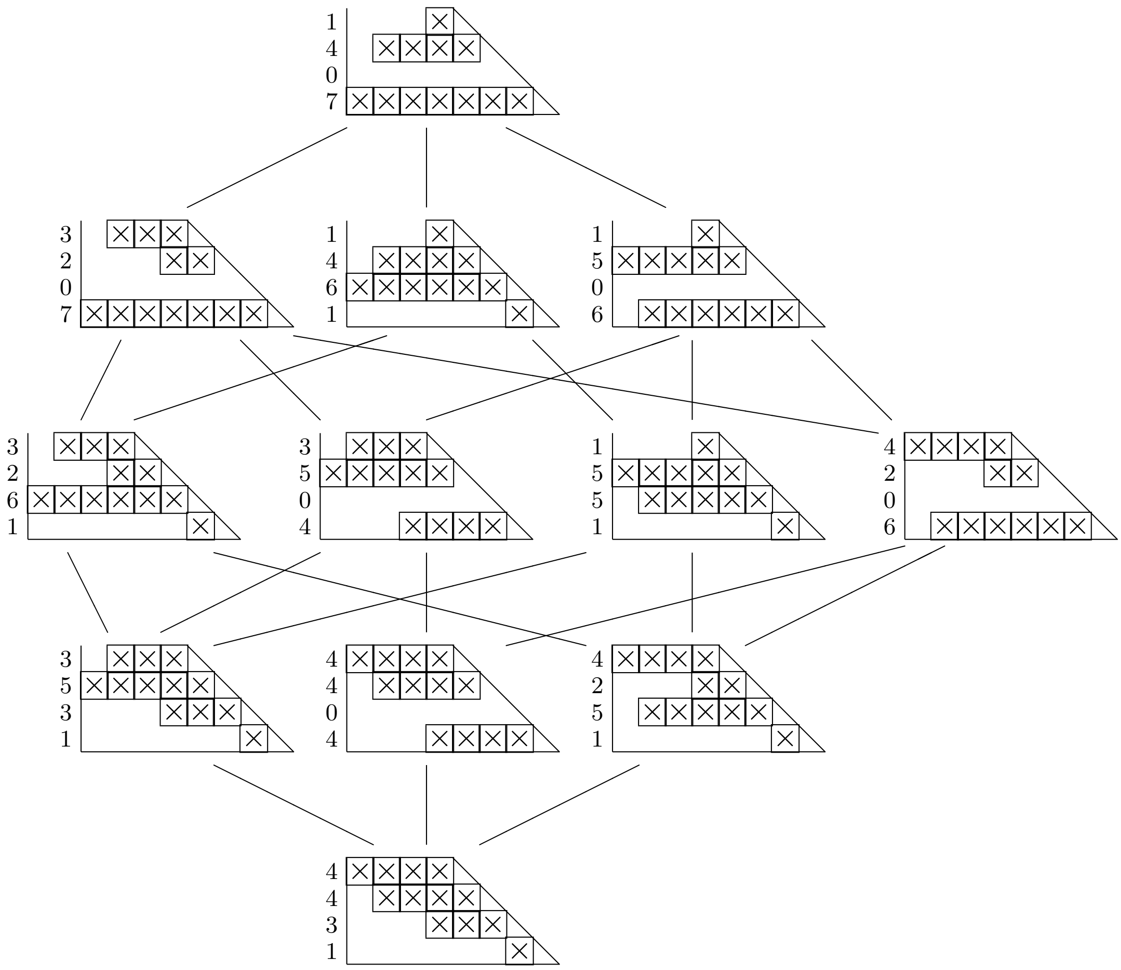
> X_π is a symmetric polynomial $\Longleftrightarrow$
> π is a Grassmanian permutation, i.e., a permutation with a unique descent ($\pi(m) > \pi(m+1)$) at some place $m \leq n$ $\Longleftrightarrow$
> X_π equals $s_\lambda^{(m)} := s_\lambda(x_1, \ldots, x_m)$, the Schur polynomial in $x_1, \ldots, x_m$ associated to the partition $\lambda = \lambda_1 \ldots \lambda_t$ ($\lambda_1 \geq \cdots \geq \lambda_t > 0$) (cf. [M3,Sa]).

In the special case of Schur polynomials Problem 2 is almost completely solvable: directly form the classical Jacobi-Trudi formula for $s_{\lambda'}^{(m)}$, where λ' denotes the conjugate partition of λ, it follows that [W2, Theorem 2.1]

$$s_{\lambda'}^{(m)}(x) = \det\left(e_{\lambda_i - i + j}^{(m+j-1)}\right)_{1 \leq i,j \leq t}, \tag{7}$$

and from this explicit determinatal formula one can deduce a simple combinatorial rule, which we explain with the help of an example:

Let λ =4 4 3 1. Then we can derive the following ranked poset $\mathbb{D}(\lambda)$ of *staircase box diagrams* (SBD):



Begin with the staircase shape of "positions" $\{(i,j) \mid 1 \leq i+j \leq n\} \subset \mathbb{N} \times \mathbb{N}$ — to save space we have depicted above only the lower parts of the staircase shapes —. Take as bottom element for $\mathbb{D}(\lambda)$ filling of the the staircase shape in a rightadjusted fashion by 1 boxes in row 1, 3 boxes in row 2, 4 boxes in row 3, and 4 boxes in row 4. The other SBD occuring in $\mathbb{D}(\lambda)$ are reached by shifting downwards boxes from one row to a lower row whithout changing their column position, where only the SBD 's with gaplessly filled rightadjusted rows are *admissable*. An admissable SBD D *covers* another admissable SBD D' in $\mathbb{D}(\lambda)$ iff in addition all the traversed positions in one row are either empty or completely filled with boxes.

Having completed the construction of $\mathbb{D}(\lambda)$ we associate a term $\Phi(D) := (-1)^{rk(D)} e_D$ to every $D \in \mathbb{D}(\lambda)$, where $rk(D)$ is the rank of D and e_D is a SEM , whose factors

$e_i^{(n)}$ are determined by the number of boxes contained in the different rows of D (the staircase shape corresponds to the defining inequalities of K_n). For example, the three SBD of rank 1 in $\mathbb{D}(\lambda)$ contribute the terms $-e_{0003531}$, $-e_{0004404}$, and $-e_{0004251}$. Then we have:

Theorem 2. [W2, Theorem 2.7]

$$s_{\lambda'}^{(m)}(x) = \sum_{D\in\mathbb{D}(\lambda)} \Phi(D) \ . \tag{8}$$

The ranked posets $\mathbb{D}(\lambda)$ have many interesting properties: in fact they are rank-symmetric lattices — probably even self-dual (= anti-isomorphic to themselfs) —, which "interpolate" between two extrems: the full weak Bruhat order on the symmetric groups and their maximal embedded Boolean sublattices [W2, Proposition 2.13, Corollary 2.11]. But more seems to be true: a similar type of poset $\mathbb{D}(\pi)$ does possibly exist for every π [W2, Conjecture 5.2]. And in addition these posets seem to provide an "alternating approximation" of the Schubert polynomials X_π: let

$$\Sigma_{\leq\nu}(\pi) := (-1)^\nu \left[\sum(\text{ terms associated to items of rank} \leq \nu \text{ in } \mathbb{D}(\pi)) - X_\pi\right] \ ;$$

then $\Sigma_{\leq\nu}(\pi)$ has non-negative integer coefficients for all ν and π [W2, Conjecture 5.4].

Finally, we mention one more of the numerous interesting properties, which a closer study of the SEM expansions of Schubert polynomials might reveal: for the expansion (6) set

$$\xi_\pi(i) := \sum_{k\in K_n} \alpha_k \ k_i \ , \quad \zeta_\pi := \sum_{i=1}^{n-1} \xi_\pi(i) \ , \quad \overline{\zeta}_\pi := \sum_{i=1}^{n-1} |\xi_\pi(i)| \ ,$$

and for all $n \in \mathbb{N}$:

$$C_n := \sharp\{\pi \in S_n \mid \zeta_\pi = 0\} \ , \quad \overline{C}_n := \sharp\{\pi \in S_n \mid \overline{\zeta}_\pi = 0\} \ .$$

Since clearly $[\ \overline{\zeta}_\pi = 0 \Longrightarrow \zeta_\pi = 0\ ]$, $\overline{\zeta}_{id} = 0$, and $\zeta_{\omega_n} > 0$, it follows

$$1 \leq \overline{C}_n \leq C_n < n! \ .$$

But results on the Grassmanian case and extensive computer calculations suggest that [W2, Conjecture 5.9]

$$\lim_{n\longrightarrow\infty} C_n/n! = \lim_{n\longrightarrow\infty} \overline{C}_n/n! = 1 \ , \tag{9}$$

which we suspect to have a physical interpretation, too.

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