A TRANSFER PRINCIPLE IN THE REAL PLANE FROM NON-SINGULAR ALGEBRAIC CURVES TO POLYNOMIAL VECTOR FIELDS

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Abstract. For every non-singular algebraic curve C of degree m in the real plane a polynomial vector field of degree $2m-1$ is constructed, which has exactly the ovals of C as attracting limit cycles. Therefore, every progress on the algebraic part of Hilbert's 16th problem automatically yields progress on its dynamical part.

Hilbert's 16th problem [H2] is concerned with (1) the topology of real plane projective curves having the maximal number of ovals ("algebraic part") and (2) the topology of real planar polynomial vector fields, in particular it asks for the maximal number and position of limit cycles possible for fixed degree ("dynamical part"). The very simple *transfer principle* introduced in the present paper associates to every non-singular plane algebraic curve C of degree m a planar polynomial vector field of degree $2m-1$ having exactly the ovals of C as limit cycles, where all of them are either attracting or repelling. If therefore a certain topological configuration of ovals is known to occur for a real plane algebraic curve of degree m, then it is automatically known to occur as topological configuration of limit cycles for all plane polynomial vector fields of degree at least $2m-1$. Whereas the transfer principle introduced here does not yield new insights on the possible *number* of limit cycles, it thereby dramatically increases our knowledge on their possible *mutual positions*. Before we state, prove, and discuss further our main result we will review concisely the current knowledge on both parts of Hilbert's 16th problem:

For a polynomial $f \in \mathbb{R}[x, y]$ of degree m the set of zeros

$$V(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$$

is the *affine curve* $C = V(f)$ associated to f; the *projective cone* for f is given by the set of zeros of $z^m f(x/z, y/z)$ in $\mathbb{R}^3$; and the *projective completion* $\bar{C}$ for f results from rewriting the projective cone in homogeneous cordinates $(x_0 : x_1 : x_2)$. Geometrically the projective completion $\bar{C}$ of C in the projective plane $\mathbb{R}P^2$ can be viewed as resulting from the identification in $\mathbb{R}^2$ of the pair of points "at infinity" in polar directions. It is much more economical to study projective completions than affine curves: since

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antiquity it is known that the three non-degenerate types of affine curves of degree 2 are represented by one cone, and already Newton found that the 99 types of affine curves of degree 3 are represented by only 5 cones. It is also well known that the topological model for $\mathbb{R}P^2$ is the sphere with the cross cap and that the singularity-free components of $\bar{C}$ are of two topologically different types: *two-sided ovals*, which have an 'inside' homeomorphic to a disk and an 'outside' homeomorphic to a Möbius strip, and *one-sided components* behaving like the projective line. For a non-singular curve, i.e., the gradient of f does not vanish on C, the mutual positions of ovals and the (possible) existence of a one-sided component togther constitutes the *(real) scheme* of the curve — two one-sided components necessarily intersect in a singular point.

In 1876 Harnack [H] proved that the maximal number of components, which a real plane projective curve of degree m may have, is

$$h(m) := \frac{(m-1)(m-2)}{2} + 1$$

and he constructed examples of curves attaining this maximum for every m. Later I. Petrovsky [P] coined the name "M-curves" for curves having the maximal number of components. It is not hard to see that M-curves are non-singular and that for even degree they have $h(m)$ ovals and for odd degree $h(m) - 1$ ovals and one one-sided component. For degree $m = 6$ Harnack described one possible real scheme of the 11 ovals of an M-curve. Hilbert [H1] found another scheme in degree 6, which — together with the geometric appeal of these problems — prompted him to pose the question about the possible real schemes of M-curves as a part of his 16th problem. Since M-curves are non-singular and non-singular curves have only ovals and possibly one one-sided component, it is natural to extend the question about the possible real schemes to all non-singular curves. When Hilbert posed his problems in 1900 the classification of real schemes for non-singular curves was known for all $m \leq 5$. Gudkov [GU] solved the case $m = 6$ with a complete list of the 56 possible real schemes. He also showed that a third scheme for M-curves in degree 6 occurs, a scheme which Hilbert had conjectured to be impossible. With the help of new gluing techniques Viro [V1] succeded to give a complete list of the 121 real schemes in degree $m = 7$.

For all degrees $m \geq 8$ no such complete lists are known, but many schemes are known to occur: the methods of Harnack, Hilbert, Gudkov, Viro, and others (cf. [V2,IV]) allow their construction. The problem of completing a known list of real schemes for some degre m enforces the invention of evermore powerfull construction techniques. For example, the methods of Harnack and Hilbert allowed to find all 56 schemes of degree 6, and Viro had to add "only" 16 schemes to the list already confirmed by the methods of Harnack, Hilbert, and Gudkov in degree 7. For higher degrees there is an increasing number of schemes not covered by the known construction techniques, and therefore the algebraic part of Hilbert's 16th problem is far from being solved.

But things are much worse for the the dynamical part — the older surveys [CT,Y1] are still worth reading —. Let

$$\begin{aligned} \dot{x} =& P(x,y) \\ \dot{y} =& Q(x,y) \end{aligned} \tag{1}$$

be a polynomial vector field or system of ordinary differential equations given by polynomials $P, Q \in \mathbb{R}[x,y]$ of maximal degree m. Then Hilbert's question is twofold: first to find the maximal possible number $H(m) \in \mathbb{N} \cup \{\infty\}$ of limit cycles (= isolated periodic orbits) for all systems of degree m and second to describe all possible topological configurations of limit cycles. Certainly, the analogy between ovals of real plane algebraic curves and limit cycles of planar polynomial vector fields explains, why Hilbert has put them side by side in his 16th problem, but he had more in mind: both questions seemed to be accessable to a "method of continous change of coefficients"[1]. In fact, most of the results about real schemes are obtained by perturbation (dissipation) of singular curves [H,H1,GU,V2] and many results on the topology of planar polynomial vector fields by bifurcation methods, mainly perturbations of Hamiltonian systems and Hopf bifurcations (cf. [Ll,Y2,Z] and several articles in [Sch]). The disadvantage of bifurcation methods is, of course, that they are essentially local methods ("local" in the usual analytical sense and "local" in the parameter space of polynomial systems), whereas the occurence and location of limit cycles is a global phenomenon. Therefore, even a complete topological classification of quadratic planar systems is unknown: a few topological configurations of limit cycles have been obtained and for $H(2)$ the quite unsatisfying estimate $4 \leq H(2) \leq \infty$ is established. For degree $m = 3$ there are only scattered results and for higher degrees next to nothing is known.— A major progress several years ago was the proof of the finiteness of the number of limit cyles for every fixed planar polynomial vector field — which does not exclude the possibility $H(m) = \infty$. This has been achieved independently by Ecalle and Il'yashenko with the help of summation methods and "transseries" (cf. [E,I2,I3] and their papers in [Sch]).

For plane algebraic curves methods of algebraic geometry provide estimates for the possible real schemes [W,V2] and with the appearance of *gluing* [V1] and *patching* [IV] the first global methods of construction have surfaced. For planar polynomial vector fields genuinly global methods are still missing, so that the *transfer principle* between polynomial curves and systems described by the Theorem below seems to be the first contribution in this direction: a simple explicit formula shows how to associate to every real scheme or curve C of degree m a polynomial vector field of degree $2m-1$ having

[1] "... Methode der kontinuierlichen Koeffizientenänderung ..." [H2]

exactly the ovals of C as attracting limit cycles. This construction is a specialization of system (2) below:

Proposition. *Let $C = V(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ for some polynomial $f \in \mathbb{R}[x,y]$ (or C^∞-function on $\mathbb{R}^2$) be non-singular, i.e., $\nabla f(x,y) \neq 0$ for all $(x,y) \in V(f)$, then C is invariant under the flow of the planar polynomial vector field*

$$\begin{aligned} \dot{x} &= -\, p f_y + q_1 f \\ \dot{y} &= \quad p f_x + q_2 f \ , \end{aligned} \tag{2}$$

where $p, q_1, q_2 \in \mathbb{R}[x,y]$ are arbitrary polynomials (or C^∞-functions on $\mathbb{R}^2$) and f_x and f_y are the partial derivatives of f with respect to x and y, respectively.

Proof. The gradient field ∇f of f is perpendicular to the level sets of f. Therefore the Hamiltonian system

$$\begin{aligned} \dot{x} &= -\, f_y \\ \dot{y} &= \quad f_x \end{aligned} \tag{3}$$

associated to f leaves invariant all level sets. Clearly, the factor p in (2) affects only the length or orientation of vectors in (3) — it may produce additional fixed points —, whereas the summands $q_i f$ change (3) only on the complement of C. This shows the assertion. □

Despite the simplicity of its proof the following *transfer principle* does not seem to have been observed before:

Theorem. *Let $C = V(f)$ be a non-singular plane curve for some polynomial $f \in \mathbb{R}[x,y]$ [C^∞-function on $\mathbb{R}^2$]. Then the planar polynomial [C^∞-] vector field*

$$\begin{aligned} \dot{x} &= -\, f_y - f\ f_x \\ \dot{y} &= \quad f_x - f\ f_y \end{aligned} \tag{4}$$

has the components of C as attractors and in particular the ovals of C as attracting limit cycles. There are no other limit cycles.

Proof. Since (4) is a special case of (2) with $p = 1$, the components of C are invariant under the flow and fixpoint free. For small α the matrix

$$R_\alpha = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$$

is a good approximation of the rotation matrix

$$D_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \ ,$$

and for large modulus of α the effect of R_α is close to a rotation by $\pm\pi/2$ combined with expansion. Therefore, the application of R_f to the Hamiltonian vector field (3) — which yields (4) — leaves the flow on C invariant. Since (3) is a rotation of the gradient field ∇f in counterclockwise direction, f has positive [negative] values in the right [left] hand neigborhood of the flow on C. Therefore the application of R_f to (3) rotates all vectors on the positive right [negative left] hand side of C in positive [negative] direction, such that C is an attracting limit set for (4). Moreover, "far away" from C the vector field (4) resembles the gradient flow, which prevents the occurence of other limit cycles besides the ovals of C. □

Proof. (alternative) We investigate $\frac{d}{dt}f(x(t),y(t)) = f_x\dot{x} + f_y\dot{y}$, which is the scalar product of ∇f with the vector field (4). Clearly,

$$\dot{f} = -f(f_x^2 + f_y^2) \ ,$$

whence the vector field has an obtuse [acute] angle with ∇f on the left [right] hand side of C, where $f > 0$ [$f < 0$]. This shows that C is an attracting limit set. Let now Γ be a periodic solution of (4) different from every oval of C. Then by the uniqueness of solutions Γ does not have a common point with C and either $f > 0$ or $f < 0$ on all of Γ. Since the orbit Γ is fixpoint free, the above expression for $\dot{f}$ implies that either $\dot{f} < 0$ or $\dot{f} > 0$ on all of Γ, too. But any orbit having always an obtuse or acute angle with the gradient field ∇f cannot be closed, whence there is no periodic orbit of (4) different from the ovals of C. □

Remark 1. The proofs of the Theorem use ideas from [D] (Duff's rotated vector fields) and [Ly] (Lyapunov's stability theory).

Remark 2. Another spezialisation of (2) containing (4) as a special case has been considered in [GMR] (see also [Vb]): for $f, q \in C^\infty(\mathbb{R}^2)$, non-singular $C = V(f)$, and $q(x,y) \neq 0$ for all $(x,y) \in C$ the smooth vector field

$$\begin{aligned} \dot{x} &= -\ f_y + q\ f\ f_x \\ \dot{y} &= \quad f_x + q\ f\ f_y \ , \end{aligned} \tag{5}$$

is shown to have an attracting [repelling] oval, if $q \leq 0$ [$q \geq 0$] in a full neighborhood of the oval. (In fact, this is easily seen with the above methods upon observing that (5) is the Hamiltonian vector field (3) with $\frac{1}{2}q\nabla f^2$ added.) However, (a) this result has not been related to algebraic curves in [GMR,Vb], (b) the "obvious" choices $q = \pm 1$ have not been considered, and (c) the proofs are much more complicated.

Remark 3. Consider system (5) for a non-singular polynomial $f \in \mathbb{R}[x,y]$ of degree m. By the Theorem and Remark 2 it has exactly the ovals of $C = V(f)$ as limit cycles, where all of them are attracting [repelling] upon setting $q = -1$ [$q = 1$]. This choice of q results in the degree $2m-1$ of the vector field (5). Instead of having for all limit cycles the same type of stabilitity it is, of course, possible to have some limit cycles attracting and others repelling by appropriately choosing q — but only at the expense of a higher degree of the vector field. We have not considered here the questions arising from these different choices of q, because our primary aim is to provide examples of the possible topological configurations of limit cycles (in accordance with Hilbert's original questions).

The obvious consequence $H(2m-1) \geq h(m)$ of the Theorem is not very exciting — except for the ease with which we have derived it —, because of the following stronger estimate

$$H(m) \geq (m+2)(m-1)/2 \tag{6}$$

of Il'yashenko [I1]. The proof of (6) given in [I1] is non-constructive and proceeds by identifying and investigating the subspace of Hamiltonian perturbations in the space of all degree m polynomial perturbations of the Hamiltonian system (3). Inequality (6) suggests that for given f or C of degree m there might exist polynomial vector fields of degree less than $2m-1$ (perhaps even of degree m), which are topologically equivalent to the system (4) associated to f. But the "rigidity" of the algebraic variety $C = V(f)$ suggests that

Conjecture. *The degree $2m-1$ of the Theorem is sharp in the following sense: for given C of degree $m \geq 4$ there is in general no polynomial vector field of degree less than $2m-1$ leaving invariant C and having exactly the ovals of C as limit cycles.*

As mentioned already in the first paragraph of the present paper the real strength of the Theorem is to provide for all degrees m many concrete non-trivial examples how the limit cycles of a planar polynomial vector field (1) may be positioned in the plane. In fact, the number of possible arrangements of limit cycles grows very fast with growing degree, as indicated by the following table (extracted from [V2] — general estimates are not yet available):

m	number of real schemes
1	1
2	2
3	2
4	6
5	7
6	56
7	121
8	$\gg 686$

Here the number 686 for degree $m = 8$ is a much too small lower bound for the actual number of real schemes: it only gives the number of schemes with 19 — 22 ovals, which can be generated by dissipation of the union of 4 ellipses having a second order tangency at two points [V2].

Observe that in general the non-singular polynomial f representing a certain real scheme is irreducible: assume that $f = ab$ for some polynomials $a, b \in \mathbb{R}[x, y]$. Then the degrees $m_a, m_b \geq 1$ of a and b, respectively, sum up to the degree m of f. Since f is non-singular it follows that (see e.g. [CLS])

$$C = V(f) = V(ab) = V(a) \cup V(b) \ , \quad V(a) \cap V(b) = \emptyset \ .$$

But since $h(m_a)+h(m_b)$ is much smaller than $h(m)$ for large m, all real schemes having a number of components greater than

$$\max_{2 \leq k \leq m-2} h(m-k) + h(k) = h(m-2) + h(2) = h(m-2) + 1$$

are necessarily associated to irreducible polynomials f. In addition it is unclear, whether every scheme with less components can be generated as the union of two lower degree schemes.

For any given $f \in \mathbb{R}[x, y]$ the topological type of its associated affine curve or its associated real scheme can be determined by the computer algebraic method of *cylindrical algebraic decomposition*, which runs in polynomial time [C,ACM1,ACM2,AB].

Under very special circumstances — such as high symmetries — system (4) may have "lines" of fixed points instead of having just a finite number of them. This happens exactly when the greatest common divisor of f_x and f_y is a non-constant polynomial. The greatest common divisor (GCD) of two multivariate polynomials can be computed e.g. with the help of Gröbner bases [CLS, Chp.4 §3]. One reason for the rare occurence of "lines" of fixed points is that they are easily destroyed under perturbations, which retain the topological configurations of attractors:

Example. Let $f_\varepsilon(x, y) = ((x-\varepsilon)^2 + y^2 - 1)\ (x^2 + y^2 - 9)$. Then $C_0 = V(f_0)$ consist of to circles with radius 1 and 3 centered at the origin. As expected from the circular symmetry one has $\mathrm{GCD}(f_{0,x}, f_{0,y}) = (x^2 + y^2 - 5)$ and for the corresponding system (4) a circle with radius $\sqrt{5}$ of repelling fixed points occurs in the annular region enclosed by the attracting circles of C_0. Under slight perturbations ($\varepsilon \neq 0$) this circle of fixed points decays into a repelling node and a saddle, which are connected by two heteroclinic orbits running around the inner circle in the two possible directions.

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