A COMPLETE TOPOLOGICAL CLASSIFICATION OF PLANE POLYNOMIAL VECTOR FIELDS DERIVED FROM NON-DEGENERATE COMPLEX POLYNOMIALS

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Abstract. The present paper contains a study of the dynamics of planar polynomial vector fields that result from non-degenerate complex polynomials of arbitrary degree by realification.

The possible topological types of these vector fields are described by a new kind of combinatorial diagrams. Since all possible diagrams can be shown to be realised by a vector field of the given class, a complete topological classification of these vector fields is obtained.

Moreover, the topological type of a given vector field can be determined from the zeros of an explicitely calculable transcendental function or, alternatively, by a method that uses a new family of global bifurcations.

1. Introduction

In this paper we study and classify the dynamics of planar polynomial vector fields of arbitrary degree m

$$\dot{x_j} = p_j(x) \ , \quad p_j \in \mathbb{R}[x,y], \quad m := \max \ \deg \ p_j \quad (j = 1, 2) \ , \tag{1.1}$$

that are derived from the complex equation

$$\dot{z} = Q(z) \ , \qquad Q \in \mathbb{C}[z], \quad m := \deg Q \quad (m \in \mathbb{N}) \tag{1.2}$$

through the usual identification of $z = x + iy = \Re(z) + i\Im(z) \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ (of course $\Re$ and $\Im$ denote the real and imaginary part, respectively):

$$\begin{aligned} \dot{x} &= \Re(Q)(x,y) \\ \dot{y} &= \Im(Q)(x,y) \end{aligned} \qquad \Re(Q), \Im(Q) \in \mathbb{R}[x,y] \ . \tag{1.3}$$

To keeps matters simple we restrict to non-degenerate complex polynomials without multiple roots, but the approach to be described in the following paragraphs should work for the degenerate case and probably for many more systems (1.1), too.

First of all, let $f(t) = \sum_{n=0}^{\infty} a_n t^n$ be a real analytic function defined on an maximal interval $I \subset \mathbb{R}$ containing the origin. We recall first that *the global behaviour of f is determined by the infinite tail $(a_K, a_{K+1}, \dots)$ of coefficients* ($K \in \mathbb{N}$ arbirarily large): this is seen in undergraduate calculus when the increase of the degree of the the Taylor polynomials of, e.g., $\cos(t)$ adds more and more minima and maxima in ever greater

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distance from the origin. On a more sophisticated level one sees this from classical summation techniques such as the Mittag-Leffler-method

$$\sum_{n=0}^{\infty} \frac{a_n}{\Gamma(1+\delta n)} t^n \ .$$

In the limit $\delta \longrightarrow 0$ when the higher coefficients gain more and more their full weight the function f converges inside its Mittag-Leffler star, i.e., in every point of its domain of definition that is "visible" from the origin [H, B], and in particular on the interval I. To the contrary, the local behaviour of f around $t = 0$ is influenced mostly by the lowest terms, which is the basis of approximation.

Assume now that (a_n) has a neatly describable asymptotic behaviour, say, for $n \longrightarrow \infty$ it is approximated well by a “simple” sequence $(\bar{a}_n)$. Then instead of trying to continue analytically the original function f we can apply the well known

Power Series Transformation. (PST) *Assume that the power series* $\sum_{n=0}^{\infty} a_n(t-t_0)^n$ *converges inside a circle of radius* r *and* t_1 *is any point with* $|t_1 - t_0| < r$. *Then for all* t *with* $|t - t_1| < r - |t_1 - t_0|$ *one has*

$$\sum_{n=0}^{\infty} a_n(t-t_0)^n = \sum_{n=0}^{\infty} b_n(t-t_1)^n \quad \textit{with} \;\; b_k := \sum_{n=k}^{\infty} \binom{n}{k} a_n (t_1 - t_0)^{n-k} \ . \tag{1.4}$$

on the asymptotical sequence $(\bar{a}_n)$. More precisely: We can continue f analytically by *matching the asymptotics of the coefficients of* f *for different base points in accordance with* PST. Since the expression for b_k in (1.4) works for arbitrarily great k, it is not necessary that we know the sequence (a_n) exactly, but the simpler asymptotic approximation $(\bar{a}_n)$ can do the job as well. Even though we use an approximation of the asymptotics the orbits resulting from matching points by PST are exact!

Of course, a sufficient degree of simplicity of the sequence $(\bar{a}_n)$ is necessary in order to be able to calculate the b_k's from (1.4). It will turn out that for the systems (1.2-3) the asymptotics of coefficients of the solution series is extremely simple (Section 2) and can be given explicitely for any initial value x_0 in phase space. More precisely, the asymptotics for the the given system and all initial values can be characterised by certain parameters comprising a (two-dimensional) pseudo phase space $\mathcal{P}$. Assume further, that we have a *transfer function* $R : \mathbb{R}^2 \longrightarrow \mathcal{P}$, which associates to every point x_0 of phase space a point of $R(x_0) \in \mathcal{P}$, which captures the asymptotics of solution coefficients at x_0. The points of $\mathcal{P}$ will now be matched to form *pseudo orbits* by virtue of the *global algebraic law* derived from formula (1.4) of PST. Clearly, the global behaviour of the pseudo orbit based at $R(x_0)$ gives us an information about the global behaviour of the orbit based at x_0: if, for example, a pseudo orbits is periodic, then the associated orbit must be periodic. But the reverse is not true, since in general the inverse R^{-1} of the transfer function R is multi-valued and can only be understood as a “lift back” to phase space of the information gained in pseudo phase space.

This leads to a very important conclusion: if $R^{-1} \circ R$ is *not* the identity on the phase space, *its natural interpretation is that of a lifting map to a covering space of the phase space. Moreover, the lifting of orbits, i.e., the reverse images of pseudo orbits, behave* *on the the covering space like usual orbits of a gradient field. Therefore the original vector field and its flow is the projection of a gradient vector field and its flow on the covering space.* We therefore call this covering space the *pseudo potential* of the vector field and show that every polynomial vector field (1.2-3) has such a pseudo potential. Pseudo phase space is in this case a complex domain, the transfer function R is explicitly calculable – but as a transcendental multivalued function not easily interpreted – ; the matching law in pseudo phase space is of utmost simplicity, and so we can derive a lot of new results:

(1) Local and global questions can be treated on the same footing: Every system (1.2) is determined completely by the roots r_j of Q, which correspond to the fixed points. We compute a certain *characteristic number* $\chi_j \in \mathbb{C}$ for every r_j, which characterises the type of the fixed point r_j, and at the same time contains the part of the information, which r_j contributes to the global dynamics of the system (Section 2). This is reasonable, since polynomials are very rigid objects, determined by a finite number of parameters, e.g., their roots.

    It is exactly this algebraic rigidity of polynomial systems, which made everyone in the past feel confident that a nice description of the global dynamics of general systems (1.1) should be possible — and at the same time feel irritated about the seeming impossibility to make explicit the wealth of information encoded in such a simple manner.

(2) The transfer function reveals by a very simple argument that limit cycles do not exist (Section 3). Armed with this knowledge and some additional facts about fixed points and the boundaries of basins of attraction we derive a complete description of the possible topological types. This description is given in terms of certain planar diagrams or, equivalently, partitions of sets of signed integers and is therefore of *combinatorial* character (Section 4).

(3) In Section 5 we show that indeed every possible combinatorial type found in Section 4 is realizable by a system (1.2-3), whence we arrive at a complete topological classification of all these vector fields. This is done by realizing the recursive generation of combinatorial types as a sequence of perturbations of vector fields.

(4) Of course the topological classification does not say how to determine the topological type of any given system. This can be done by a careful interpretation of the transfer function R for the system. From R one derives an (explicitly calculable) associated transcendental function, whose set of zeroes determines the topological type uniquely by separating all *sectors* of attraction and repellation and the occurring centre sectors (Section 5).

(5) Alternatively it is possible to construct a given system by a sequence of controlled global bifurcations starting from a system of well known topological type (Section 5).

Of course many questions for equations (1.2) remain open. Fourteen of them are singled out as problems. For general polynomial vector fields (1.1) the applicability of the new concepts remains to be evaluated.

Since for equations (1.2) almost all of our results seems to be new, the bibliography is very short. In fact, the historical line of research on these differential equations had a focus completely different from ours, namely to identify those differential equations, which have well-behaved, e.g., meromorphic, solutions. This lead to the study of Riccati equations, and later to Painleve equations and to Nevanlinna theory.

An interesting feature of the new approach to algebraic dynamics is its very classical spirit. Except for some ideas from combinatorics not much 20th century mathematics is needed to understand the paper. On the other hand it would be misleading to neglect the decisive role of computer experiments and verifications using highly developed mathematical software (MAPLE in particular), which helped a lot in providing enough "experience" to push research in the right direction.

2. Basic results

First of all, one observes that it is enough to investigate the complex equation (1.2), because it contains all information about the real case (1.3). (This offers not only a more thorough understanding through the use of complex variables, but also reduces dimensions from two to one.) More precisely: if

$z(t;z_0)$ denotes the solution of (1.2) for initial value $z_0 \in \mathbb{C}$ and
$x(t;x_0)$ and $y(t;y_0)$ the solutions of (1.3) for $(x_0, y_0) \in \mathbb{R}^2$

then $x(t;x_0) = \Re(z(t;z_0))$ and $y(t;y_0) = \Im(z(t;z_0))$, if $(x_0, y_0) = (\Re(z_0), \Im(z_0))$. In particular one gets the coefficients x_n and y_n of the series expansions of $x(t;x_0)$ and $y(t;y_0)$ as real and imaginary part, respectively, of the coefficients z_n of the series expansion of $z(t;z_0)$.

Since $Q \in \mathbb{C}[z]$ is determined mainly by the vector (or set) of roots $r = (r_1, \ldots, r_m)$, we sometimes indicate this dependency directly in our notation:

$$Q(z;r) = c\prod_{j=1}^{m}(z - r_j) \quad \text{with } c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}. \tag{2.1}$$

The next lemma collects some useful facts about the symmetries of the vector field Q with respect to elementary operations on the roots.

Lemma 2.1. *For every $Q(z;r) \in \mathbb{C}[z]$ one has:*
(a) $Q(z; r + (c, \ldots, c)) = Q(z - c; r)$ for all $c \in \mathbb{C}$;
(b) $Q(z; \rho r) = \rho^m Q(\rho^{-1} z; r)$ for all $\rho \in (0, \infty) \subset \mathbb{R}$;
(c) $Q(z; \gamma r) = \gamma^m Q(\gamma^{-1} z; r)$ for all $\gamma = e^{i\varphi} \in S^1$.

Hence the translation of the the defining roots of the vector field $Q(z;r)$ gives a translated (congruent) vector field; a similarity transform of the roots a similar vector field; a rotation of the roots a simultaneous rotation of all single vectors. Particular

cases of (c) are the following:
(d) if γ is a multiple of a $(m-1)^{th}$-root of unity ($k \in \mathbb{Z}$)

$$\zeta_{m-1}^k = \exp(\frac{2k\pi i}{m-1}), \qquad \zeta_{m-1} = \zeta_{m-1}^1 \ ,$$

i.e., if $\gamma = \zeta_{m-1}^k$, then

$$\gamma^m = \gamma \quad \textit{and} \quad Q(z; \gamma r) = \gamma Q(\gamma^{-1} z; r) \ .$$

This means: $Q(z;\gamma r)$ is congruent to $Q(z;r)$.
(e) if $m \geq 3$ and $\gamma = \zeta_{2(m-1)}^{2k+1}$, then — for $m=2$ nothing happens — :

$$\gamma^m = -\gamma \quad \textit{and} \quad Q(z; \gamma r) = -\gamma Q(\gamma^{-1} z; r) \ .$$

This means: $Q(z;\gamma r)$ is congruent to $Q(z;r)$ with orientation reversed.
(f) If $\gamma = \zeta_{4(m-1)}^{2k+1}$, then:

$$\gamma^m = -\gamma \ , Q(z; \gamma r) = \pm i Q(\gamma^{-1} z; r) \ ,$$

where the plus occurs, if $2k+1 \equiv 1(\mod 4)$, and the minus occurs, if $2k+1 \equiv 3(\mod 4)$. This means: $Q(z;\gamma r)$ is congruent to the dual *vector field $Q^*(z;r)$ of Q, where all orbits cross the orbits of $Q(z;r)$ in perpendicular direction.*
(g) If $c \in \mathbb{R}$ in formula (2.1), then the conjugation of the roots r_j results in the reflection of the vector field with respect to the real axis. In particular, if Q has only real coefficients, the system is self-conjugate.

Proof. (a), (b), and (c) are proved by straightforward calculations; $(c) \Longrightarrow (d), (e), (f)$. (g) results from $Q(\overline{z}; \overline{r}) = \overline{Q(z;r)}$. □

From Lemma 2.1 (b,c) we conclude that appropriate scaling of the form $z \longmapsto \lambda z$ with $\lambda \in \mathbb{C}^*$ transforms any vector field (1.2) to an affinely equivalent vector field $Q(z)$ in ${}_m\mathbb{C}[z]$, where ${}_m\mathbb{C}[z]$ is the set of *monic* (= the coefficient of the leading term is 1) polynomials in $\mathbb{C}[z]$. Therefore we will restrict our investigation to the case

$$\text{(2.2)} \qquad \dot{z} = Q(z) \ , \quad Q \in {}_m\mathbb{C}[z] \quad \text{or} \quad Q(z;r) = \prod_{j=1}^{m} (z - r_j) \quad (m \in \mathbb{N}) \ .$$

From Lemma 2.1 (e) it follows also that for every vector field $Q(z) \in_m \mathbb{C}[z]$ of degree ≥ 3 there is a vector field congruent to $-Q(z)$ in ${}_m\mathbb{C}[z]$.

Subsequently we will assume in addition that $Q \in {}_m\mathbb{C}[z]$ is *non-degenerate*, which means that the roots $r_1, \dots, r_m$ of Q are all *different*, because the case of multiple roots can be treated by *confluence* of roots on the basis of a properly understood non-degenerate case. More precisely, by confluence we mean the following: let r be a root of multiplicity $k \geq 2$ of Q, i.e., Q contains a factor $(z-r)^k$. Replace now the k-fold root r by k different simple roots, which are close to r, e.g., replace $(z-r)^k$ by a product

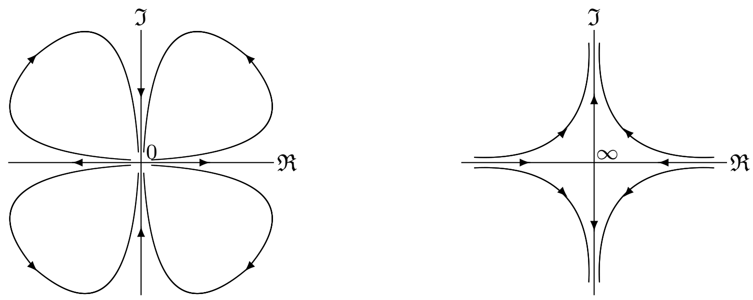
$$\prod_{j=1}^{k} (z - (r + \varepsilon e^{i\varphi_j})) \quad \text{with } -\pi \leq \varphi_1, \dots, \varphi_k < \pi \text{ different, } 1 \gg \varepsilon > 0.$$

Then the phase portrait of the perturbed vector field is "close" to the original unperturbed vector field for small ε and the two can not be distinguished for $\varepsilon \longrightarrow 0$.

We consider next the behaviour of the vector field (2.2) for "large" $z \gg \max\{|r_j|\}$. As for every polynomial vector field this behaviour is determined by the homogeneous part of highest degree m:

$$\dot{z} = z^m \ . \tag{2.3}$$

It follows from Lemma 2.1 (d) that (2.3) is invariant under a rotation of the complex plane by any ζ_{m-1}^k and from Lemma 2.1 (e) that (2.3) is changes only the orientation under a rotation by any ζ_{m-1}^{2k+1}. These facts in addition to homogeneity imply that all lines in the directions ζ_{m-1}^{2k+1} are invariant under the flow and in between there are $2(m-1)$ homoclinic sectors with respect to 0 and $2(m-1)$ hyperbolic sectors with respect to ∞. Moreover, the orientation alternates from sector to sector beginning with the positive real axes as an outgoing orbit of 0. As an example we depict the flow of (2.3) for $m = 3$ below (left figure at 0, right figure at ∞):



Let r be a root of multiplicity $k \geq 2$ of Q, then in a small neighbourhood of r the phase portrait of Q looks like the one for $\dot{z} = z^k$ at 0, because Q is at r approximately of the form $c(z-r)^k$ with $c \in \mathbb{C}^*$ given by the the evaluation of $Q(z)/(z-r)^k$ at r. This brings up the question what is known about the fixed points r_j of the non-degenerate vector fields (2.2) in general.

Lemma 2.2. *The fixed points r_j of non-degenerate vector field* (2.2) *are either nodes, foci, or centres. Saddle points are absent. The linearised vector field at some r_j has always full S^1-rotational symmetry and is given by*

$$Q'(r_j) = \prod_{\substack{\nu=1 \\ \nu \neq j}}^{m} (r_j - r_\nu) := \chi_j \ . \tag{2.4}$$

We call χ_j the characteristic number *of r_j.*

Proof. Formula (2.4) follows from the product rule and the other assertions from the Cauchy-Riemann equations $\Re(Q)_x = \Im(Q)_y$ and $\Re(Q)_y = -\Im(Q)_x$: the determinant of the linearisation is positive, whence one has either two equal real non-zero eigenvalues or a pair of conjugate complex ones. □

More explicitly: r_j is a sink, if $\Re(\chi_j) < 0$, and a source, if $\Re(\chi_j) > 0$; r_j is a node, if $\Im(\chi_j) = 0$, and a focus, if $\Re(\chi_j) \neq 0$ and $\Im(\chi_j) \neq 0$; it is a weak focus or a centre — we will see later that it is actually always a centre — , if $\Re(\chi_j) = 0$; and, finally, in case of a focus the orbits spiral clockwise when approaching r_j, if $\Im(\chi_j) < 0$, and counter-clockwise, if $\Im(\chi_j) > 0$.

So far everything in this section has been a straightforward application of well known elementary methods. Lucky circumstances allow now the derivation of an exact solution for (2.2) in implicit form. Let w be the complex extension for the real variable t.

Theorem 2.3. *The exact solution $z = z(w; z_0)$ for the initial value problem* (2.2) *with $z(0) = z_0 \in \mathbb{C}$ is given in implicit form by*

$$w - \widetilde{C} = \sum_{j=1}^{m} c_j \log(z - r_j) \ , \tag{2.5}$$

$$\text{with } c_j = \chi_j^{-1}, \quad \widetilde{C} = -\sum_{j=1}^{m} c_j \log(z_0 - r_j), \text{ and} \tag{2.6}$$

$$\log(z) = \ln|z| + i \arg(z) \ . \tag{2.7}$$

Here log *is the full logarithm with values on the whole Riemannian surface with infinitely many sheets,* ln *is the logarithm for positive real numbers, and* arg *is the the full argument function on all of $\mathbb{R}$. Arg and Log will denote the restriction of* arg *and* log, *respectively, to their principal branches, where $-\pi \leq \varphi < \pi$. A useful short form of* (2.5-7) *is*

$$Ce^w = \prod_{j=1}^{m} (z - r_j)^{c_j} =: R(z) \ , \qquad \text{with } C = R(z_0) \ . \tag{2.8}$$

(The exact meaning of $R(z)$ is of course $R(z) = \exp(\sum_{j=1}^{m} c_j \log(z_0 - r_j))$.)

Proof. For z_0 not a fixed point let $w = z^{[-1]}(\widetilde{w})$ be the local inverse of $z(w) = z(w; z_0)$. Then $\dot{z}(w) = Q(z(w))$ yields $z'(z^{[-1]}(\widetilde{w})) = Q(\widetilde{w})$ and by the theorem about the derivative of the inverse function one has:

$$(z^{[-1]})'(\widetilde{w}) = \frac{1}{z'(z^{[-1]}(\widetilde{w}))} = \frac{1}{Q(\widetilde{w})} = \sum_{j=1}^{m} \frac{c_j}{\widetilde{w} - r_j} \tag{2.9}$$

with c_j as defined above. But this gives by indefinite integration

$$w = z^{[-1]}(\widetilde{w}) = \int \frac{d\widetilde{w}}{Q(\widetilde{w})} + \widetilde{C} = \sum_{j=1}^{m} c_j \log(\widetilde{w} - r_j) + \widetilde{C} \ .$$

As a check we set

$$F(w, z) = \sum_{j=1}^{m} c_j \log(z - r_j) - w + \widetilde{C} \tag{2.10}$$

and compute the derivative of the implicitly defined function $z(w)$ as

$$z' = -\frac{F_w}{F_z} = \frac{1}{\sum_{j=1}^{m} \frac{c_j}{z-r_j}} = (\frac{1}{Q(z)})^{-1} = Q(z) \ .$$

$\square$

Remark 2.4. For $m = 1$ the initial value problem $\dot{z} = z - r$, $z(0) = z_0 \in \mathbb{C}$ one has the solution $z(t; z_0) = (z_0 - r)e^t + r$ in accordance with Theorem 2.3.

Following our program of "pseudo orbits in pseudo phase space" we are interested in the asymptotics $n \longrightarrow \infty$ of the coefficients z_n of the solution

$$z(w; z_0) = \sum_{n \geq 0} z_n w^n \tag{2.11}$$

and therefore in the structure of singularities of $z(w; z_0)$.

Lemma 2.5. *For any given initial value problem* (2.2) *with* $z(0) = z_0 \in \mathbb{C} \setminus \{r_1, \ldots, r_m\}$ *the only singularities of the solution* $z(w; z_0)$ *are infinities.*

Proof. Fix some $z_0 \in \mathbb{C} \setminus \{r_1, \ldots, r_m\}$. As seen already in the proof of Theorem 2.3 one has $F_z(w, z) = 1/Q(z)$, which implies by the implicit function theorem the existence of a unique local analytic function $z(w; z_0)$ for all $(w, z) \in \mathbb{C}^2$, where $1/Q(z) \neq 0$. But this means — since Q is a polynomial — that singularities occur only for $z \longrightarrow \infty$. $\square$

Lemma 2.6. *Let* $c_1, \ldots, c_m$ *be the numbers computed from any* m*-tuple* $r = (r_1, \ldots, r_m)$ *of* m *different complex numbers according to* (2.6) *and* (2.4). *Then*

$$\sum_{j=1}^{m} c_j = 0 \ . \tag{2.12}$$

Proof. Let $V(r)$ denote the Vandermonde determinant with rows $(r_j^{m-1}, \ldots, r_j, 1)$ for $j = 1, \ldots, m$ and similarly $\widetilde{V}(r)$ the determinant with rows $(r_j^{m-2}, \ldots, r_j, 1, 1)$. With the help of

$$V(r) = \prod_{1 \leq l < h \leq m} (r_l - r_h) \neq 0$$

and the expansion of $\widetilde{V}(r)$ with respect to the last column

$$0 = \widetilde{V}(r) = \sum_{j=1}^{m} (-1)^{m+j} V(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m)$$

one than computes

$$V(r) \sum_{j=1}^{m} c_j = \sum_{j=1}^{m} \frac{\prod_{1 \leq l < h \leq m} (r_l - r_h)}{(-1)^{j-1} \prod_{j < \nu \leq m} (r_j - r_\nu) \prod_{1 \leq \nu < j} (r_\nu - r_j)} =$$

$$\sum_{j=1}^{m} (-1)^{j-1} \prod_{\substack{1 \leq l < h \leq m \\ l,h \neq j}} (r_l - r_h) = \sum_{j=1}^{m} (-1)^{j-1} V(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m) \ ,$$

which implies (2.12). $\square$

Proof. (Second proof)[1] From (2.9) one concludes

$$1 = \sum_{j=1}^{m} c_j \frac{Q(w)}{w - r_j} = (\sum_{j=1}^{m} c_j) w^{m-1} + \text{ terms of lower order in } w.$$

Comparing coefficients gives (2.12). □

Theorem 2.7. *Given any initial value problem* (2.2) *with* $z(0) = z_0 \in \mathbb{C} \setminus \{r_1, \dots, r_m\}$*. Then the asymptotics of the coefficients of the solution* $z(w; z_0)$ *in* (2.11) *is determined by the solutions* w_0 *of*

$$e^{-w} = R(z_0) \ , \tag{2.13}$$

which are closest to the origin. In general w_0 *is unique, but two complex conjugate solutions* w_0 *and* $\overline{w_0}$ *are possible, too.*

Proof. Since the only singularities of $z(w; z_0)$ are infinities by Lemma 2.5, we calculate

$$\lim_{n \longrightarrow \infty} R(z) = \lim_{n \longrightarrow \infty} \prod_{j=1}^{m} (z - r_j)^{c_j} = \lim_{n \longrightarrow \infty} \prod_{j=1}^{m} z^{c_j} = \lim_{n \longrightarrow \infty} z^{\sum_{j=1}^{m} c_j} \stackrel{(2.12)}{=} 1 \ .$$

Together with (2.8) this implies that the asymptotics of the coefficients is determined by the solutions w_0 of $e^w R(z_0) = 1$ which are closest to the origin. The second assertion is obvious from the period $2\pi i$ of the exponential function. □

Remark 2.8. Note that through the elimination of $R(z)$ from (2.8) we get rid of the difficulties implied by this implicit form. At the same time we dismiss with the exact solution: the obtained asymptotics is different from the true asymptotics of the z_n in (2.11); but we retain everything necessary for our "semi-exact" approach. In fact, it is still possible to determine the orbits of (2.2) exactly, but in general with a different parametrisation.

Definition 2.9. *In the setting of Theorem* 2.7 *the set* $C = C_Q$ *of all* z_0*, such that two complex conjugate numbers* w_0 *and* $\overline{w_0}$ *exist as solutions of* (2.13) *closest to the origin, is called the* cut set *of* Q*.*

Theorem 2.7 says that the asymptotics we are looking for are determined by a single complex number w_0 for every given Q and z_0, hence the pseudo phase space is a subset of the complex plane. We will show below in this section that the pseudo orbits are straight lines parallel to the real axis and that the transfer functions are given by R from (2.8), respectively, the right hand side of (2.5).

Lemma 2.10. *Given a function* f*, which is analytic at the origin and has the expansion* $\sum_{n=0}^{\infty} a_n t^n$ *there. Assume that* $a_n = \eta^{n+1}$ *for some* $\eta \in \mathbb{C}$ *and let* $I \subset \mathbb{R}$ *be the maximal interval, on which this expansion of* f *can be continued analytically. Let* $\sum_{n=0}^{\infty} a_n(\tau)(t - \tau)^n$ *be the expansion of* f *at any point* $\tau \in I$ *(*$a_n = a_n(0)$*). Then*

$$a_n(\tau) = \eta(\tau)^{n+1} \quad \text{with } \eta(\tau) = (\frac{1}{\eta} - \tau)^{-1} \ . \tag{2.14}$$

[1]provided by Wolf Jung

In addition one has:

$$(2.15)\quad \textit{if}\quad \eta = \frac{1}{u+iv},\ \eta(\tau) = \frac{1}{u(\tau)+iv(\tau)} \quad \textit{then}\quad u(\tau) = u-\tau,\ v(\tau) = v = \text{const.}$$

Proof. With formula (1.4) of the power series transformation theorem one calculates

$$a_k(\tau) = \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k}\tau^n = \sum_{n=0}^{\infty} \frac{(n+k)\cdots(n+1)}{n!}\eta^{n+k+1}\tau^n$$
$$= \eta^{k+1}\sum_{n=0}^{\infty} \frac{(n+k)\cdots(n+1)}{n!}(\eta\tau)^n = (\frac{\eta}{1-\eta\tau})^{k+1}\ ,$$

where the last equation is valid for $|\eta\tau| < 1$ (binomial series). This gives (2.14) with the the restriction $|\eta\tau| < 1$ on τ. But since one easily verifies that $\eta(\tau)$ is a one-parameter group with respect to τ, formula (2.14) is valid for all $\tau \in I$. (2.15) follows immediately from (2.14). □

Problem 1. *Search for and investigate systematically other families of power series, where the building laws for the coefficients a_n lead to equally closed form continuation formulas. (Promising are other hypergeometric functions.) Investigate applications in differential equations, functional equations, etc., where analytic continuation is an issue.*

Theorem 2.11. *For any vector $r = (r_1, \ldots, r_m)$ of distinct roots let $c_j = a_j + ib_j$ ($a_j = \Re(c_j)$, $b_j = \Im(c_j)$, $j = 1, \ldots, m$) be the inverses of the characteristic numbers* [cf. (2.4),(2.6)] *and $Q(z;r)$ the associated non-degenerate polynomial vector field* [cf. (2.1),(2.2)]. *Let $\mathcal{O}(z_0) \subset \mathbb{C}$ denote the orbit of $Q(z;r)$ with initial value $z_0 \in \mathbb{C}$. Then for any $v_0 \in \mathbb{R}$ and fixed r the set of solutions $z_0 \in \mathbb{C}$ of the equation*

$$(2.16)\qquad v_0 = V(z_0;r) := -\sum_{j=1}^{m} [\ b_j \ln|z_0 - r_j| + a_j \arg(z_0 - r_j)\ ]$$

is either empty or a finite collection of orbits $\mathcal{O}(z_0)$ of Q; we denote the closure (in Euclidean topology) of such a set by $\mathcal{C} = \mathcal{C}(r, v_0)$ and call it the (plane) algebro-differential curve *for r and v_0.*

Similarly, for any $u_0 \in \mathbb{R}$ and fixed r the set of solutions $z_0 \in \mathbb{C}$ of the equation

$$(2.17)\qquad u_0 = U(z_0;r) := -\sum_{j=1}^{m} [\ a_j \ln|z_0 - r_j| - b_j \arg(z_0 - r_j)\ ]$$

is either empty or a finite collection of orbits $\mathcal{O}^*(z_0)$ of the dual vector field Q^* of Q [cf. Lemma 2.1 (f)]; we denote the closure of such a set by $\mathcal{C}^* = \mathcal{C}^*(r, v_0)$ and call it the dual curve *for r and v_0.*

Proof. Let us contemplate again the exact solution (2.8) of Theorem 2.3. In Lemma 2.5 we have seen that all singularities of the solution $z(w;z_0)$ are infinities and that they lay periodically on a line, which is parallel to the imaginary axis. But much more can be said: since the periodicity includes all values and not just ∞ on the the left hand side of (2.5), all singularities are of the same type; and since w and z_0 are on one side of (2.5) and z on the other, it is possible to write

$$z(w;z_0) = f \circ g(w;z_0) \ . \tag{2.18}$$

Here $g(w;z_0)$ has as singularities simple poles exactly in those places, where $z(w;z_0)$ has its infinities and the "enveloping" function f does not depend on z_0 and specifies the true type of infinities of $z(w;z_0)$. If now $g(w;z_0')$ originates from $g(w;z_0)$ through analytic continuation along the real axes, then also $z(w;z_0')$ from $z(w;z_0)$. It is therefore possible to restrict the investigation of $z(w;z_0)$ to the simpler $g(w;z_0)$.

Assume that some meromorphic function h has the expansion $\sum_{n=0}^{\infty} a_n w^n$ around the origin; assume further that $w_1,\ldots,w_s$ are the singularities of h, which are closest to the origin — all have the same modulus —, and that they are simple poles. Then

$$\sum_{j=1}^{s} \frac{1}{w_j - w} = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{s} \frac{1}{w_j^{n+1}} \right) w^n \tag{2.19}$$

is a good approximation of h around $w_1,\ldots,w_s$ and

$$a_n \sim \sum_{j=1}^{s} \frac{1}{w_j^{n+1}} \quad \text{for } n \longrightarrow \infty. \tag{2.20}$$

From Theorem 2.7 we know that in case of $g(w;z_0)$ generically $s=1$, where the position of the pole is denoted by w_0, and possibly $s=2$ with a conjugate pair w_0 and $\overline{w_0}$. Let first $s=1$ and $w_0 = u_0 + iv_0$ [$u_0 = \mathfrak{R}(w_0)$, $v_0 = \mathfrak{I}(w_0)$]; then v_0 is constant by Lemma 2.9 on the orbits $\mathcal{O}(z_0)$ of $Q(z;r)$. Using (2.13) of Theorem 2.7 and the definition of $R(z_0)$ from Theorem 2.3 one calculates

$$\begin{aligned} w_0 = -\sum_{j=1}^{m} c_j \log(z - r_j) &= -\sum_{j=1}^{m} (a_j + ib_j)(\ln(z - r_j) + i \arg(z - r_j)) \\ &= U(z_0;r) + iV(z_0;r) \ . \end{aligned}$$

This proves (2.16). Observing that v_0 is nowhere constant on an open region of the complex plane assures that $V(z_0;r)$ is a discrete set; if in addition one takes into consideration that $Q(z;r)$ behaves like the homogeneous vector field (2.3) outside a bounded region of the complex plane, finiteness follows. Finally, the orthogonality of the level lines of the real and imaginary part of a (locally) conformal function proves the assertion about (2.17) and $U(z_0;r)$.

What happens in case of $s=2$? An easy calculation with the polar form of w_0 shows

$$a_n \sim \frac{1}{w_0^{n+1}} + \frac{1}{\overline{w_0}^{n+1}} = \frac{2}{|w_0|^{n+1}} \mathfrak{R}(i^{n+1}) \quad \text{for } n \longrightarrow \infty. \tag{2.21}$$

Taking real parts in Lemma 2.9 leads to the conclusion that in case of $s=2$ we have automatically $v_0 = 0$. The associated solutions z_0 of (2.6) for $v_0 = 0$ give the curve $\mathcal{C}(r, v_0)$, hence (in case of $v_0 = 0$) it is not necessary to distinguish between $s=1$ and $s=2$. □

3. Consequences

In this section we investigate more closely the possible types of fixed points (3.1-3.2) and orbits (3.3-3.8), the problems due to the occurrence of the full arg-function in Theorem 2.11 (Problems 2-5, 3.9-3.16), and the systems associated to a given m-tuple of characteristic numbers (Problems 6-7, 3.17-3.19).

First of all one observes that the facts formulated in Lemma 2.2 and its succeeding paragraph can be recasted in terms of the $c_j = a_j + ib_j$. For this it is only necessary to take into account

$$a_j = \frac{\Re(\chi_j)}{|\chi_j|^2} \quad \text{and} \quad b_j = -\frac{\Im(\chi_j)}{|\chi_j|^2} \quad \text{for } j = 1, \dots, m. \tag{3.1}$$

This implies: r_j is a sink, if $a_j < 0$, and a source, if $a_j > 0$; r_j is a node, if $b_j = 0$, and a focus, if $a_j \neq 0$ and $b_j \neq 0$; it is a weak focus or a centre, if $a_j = 0$; and, finally, in case of a focus the orbits spiral clockwise when approaching r_j, if $b_j > 0$, and counter-clockwise, if $b_j < 0$.

Theorem 3.1. *Every non-degenerate vector field $Q(z; r)$ has as fixed points only nodes, foci, and centres.*

Proof. In view of Lemma 2.2 it remaines to be shown that no weak foci occur, i.e., every fixed point, whose linearisation has a pair of conjugate imaginary eigenvalues, is a centre. Thus let $a_\nu = 0$ and $b_\nu \neq 0$ for some root r_ν. Since $V(z; r)$ is constant on any orbit $\mathcal{O}(z_0)$ by Theorem 2.11 and for all z in a small neighbourhood U of r_ν the sum

$$\sum_{\substack{j=1 \\ j\neq\nu}}^{m} [\ b_j \ln|z_0 - r_j| + a_j \arg(z_0 - r_j)\ ]$$

is nearly constant, we conclude that the summand $b_j \ln|z_0 - r_j|$ for ν in $V(z; r)$ is bounded. Hence the orbit $\mathcal{O}(z_0)$ with $z_0 \in U$ can not approach (forwards or backwards in time) r_ν, but lies in an annulus around r_ν, which becomes arbitrarily narrow for $z_0 \longrightarrow r_\nu$. (Alternatively, use the next Theorem to avoid considering annuli.) □

Remark 3.2. It is instructive to reconsider the forgoing the arguments in case of a node r_ν, i.e., $b_\nu = 0$ and $a_\nu \neq 0$. Again we conclude from (2.16) that $a_\nu \arg(z_0 - r_\nu)$ is bounded to a narrow sector with apex r_ν for any orbit $\mathcal{O}(z_0)$, if z_0 is in a small neighbourhood U of r_ν, and that $\mathcal{O}(z_0)$ approaches r_ν under an angle, which finally becomes constant.

In addition we examine in both cases the orthogonal curves, too; here one observes that $t = -u$ can take arbitrarily large values in the positive or negative direction. First, if r_ν is a centre, then the term $b_\nu \arg(z_0 - r_\nu)$ must become unbounded for $z_0 \in U$, which is achieved through a permanent cycling of the orbit $\mathcal{O}(z_0)$: recall that arg is the full argument function. Second, if r_ν is a node, then the term $a_\nu \ln|z_0 - r_\nu|$ must become unbounded for $z_0 \in U$, which is achieved through $z_0 \longrightarrow r_\nu$.

Finally, if r_ν is a focus, then for $z_0 \in U$ the orbits $\mathcal{O}(z_0)$ and the dual orbits $\mathcal{O}^*(z_0)$ both spiral in opposite directions and the degree of spiralling versus the degree of "approaching" as given by the ratio of b_ν and a_ν is inverse proportional for orbits and dual orbits.

An important consequence of Theorem 3.1 and Remark 3.2 above is that our formulas (2.16) and (2.17) reproduce faithfully the usual local dynamics at the fix points, but they contain in addition global information, whose full extend will become clearer as we progress. In fact, our method allows to *treat local and global dynamics in one unified setting.*

Theorem 3.3. *Let $Q(z;r)$ be a non-degenerate vector field* (2.2) *and $\mathcal{O}(z_0)$ any non-periodic orbit. Then the α- and ω-limits $\alpha(z_0)$ and $\omega(z_0)$ of $\mathcal{O}(z_0)$ are contained in*

$$\overline{Fix}(Q) = Fix(Q) \cup \{\infty\} = \{r_1, \ldots, r_m\} \cup \{\infty\} \; . \tag{3.2}$$

In particular, Q has no limit cycles.

Proof. Assume that some orbit $\mathcal{O}(z_0)$ accumulates on a cycle $\mathcal{P}$. By Theorem 2.11 there is a real number v_0 such that $\mathcal{O}(z_0) \subset \mathcal{C}(r, v_0)$. Since $V(z;r)$ is clearly locally analytic and non-constant for every $z \in \mathcal{O}(z_0)$, we derive a contradiction upon taking some Poincarè section P of $\mathcal{P}$: the intersection of $\mathcal{O}(z_0)$ with P gives a sequence of points z converging to $P \cap \mathcal{P}$, where $V(z;r)$ is constant equal to v_0. This completes the proof of the assertion. $\square$

Corollary 3.4. *There are only three types of orbits: fixed points, periodic orbits around centres, and orbits connecting points in $\overline{Fix}(Q)$.*

Orbits, which have ∞ as α- or ω-limit deserve special attention:

Definition 3.5. *Let $Q(z;r)$ be a non-degenerate vector field. Then the set of stable and unstable manifolds of ∞ will be called the* separator set $\mathcal{S} = \mathcal{S}_Q$ *of Q. Set $\overline{\mathcal{S}}_Q := \mathcal{S}_Q \cup Fix(Q)$. (By Theorem* (3.3) *$\overline{\mathcal{S}}_Q$ is the closure of $\mathcal{S}_Q$.) The orbits in $S \in \mathcal{S}_Q$, which connect ∞ and some point in $\overline{Fix}(Q)$, will be called* separators*; if ∞ is connected with ∞ again, the separator will be called* pure. *The* valence $\nu(z_0)$ *of a fixed point z_0 is the number of separators having z_0 as α- or ω-limit: $\nu(z_0) \in \mathbb{N}$, if z_0 is a source or sink, and $\nu(z_0) = 0$, if z_0 is a centre.*

The connected components of the complement of $\mathcal{S}_Q$ will be called the sectors *of Q. The union of all sectors, which have a fixed sink z_0 on their boundaries, together with the incoming separators of z_0 is the* basin of attraction *of z_0. A sector, which contains a centre, is a* centre sector*. (Centre sectors are bounded by pure separators.)*

The separator set and separators of the dual vector field Q^* will be called dual separator set $\mathcal{S}_Q^*$ *and* dual separators S^*, *respectively.*

Corollary 3.6. *The topological type of a vector field* (2.2) *is completely determined by the closure of its separator set $\overline{\mathcal{S}}_Q$.*

Since the asymptotic directions of the separators of a vector field (2.2) are determined by the roots of unity $\zeta^k_{2(m-1)}$ $(k = 0, \ldots, 2m-3)$ it is possible to approximate them numerically with orbits $\mathcal{O}(z_0)$, where $z_0 = re^{i\varphi}$, $\varphi = \zeta^k_{2(m-1)}$, and $r \gg \max(|r_j|)$. But this numerical approximation can be backed up by exact information about appropriate initial values z_0 — not necessarily with "large" r — or it can be replaced completely by a contour-plot of a certain transcendental function associated to Q:

Theorem 3.7. *Let $Q(z;r)$ be a non-degenerate vector field* (2.2). *Then the separator set $\mathcal{S}_Q$ can be computed from the set of zeros z_0 of $V(z_0;r)$ by taking only those lines, which are connected to ∞. The cut set C_Q of Q is given by*

$$C_Q = \{z_0 \in \mathbb{C} \mid V(z_0;r) = 0\} \setminus \mathcal{S}_Q \ . \tag{3.3}$$

Similarly, the dual separator set $\mathcal{S}_Q^*$ can be computed as the set of zeros of $U(z_0;r)$.

Proof. Let S be a separator and z_0 a point on it, which approaches ∞. From Theorem 2.7 and its proof follows that $\lim_{z_0 \longrightarrow \infty} R(z_0) = 1$ and $e^{-w_0} = 1$, where w_0 depends on z_0. But Theorem 2.11 says that $v_0 = \mathfrak{I}(w_0)$ is constant on orbits, whence $v_0 = 0$ on S and (2.16) implies the first assertion. Note, that according to the proof of Theorem 2.11 exactly the bounded orbits contained in $\{z_0 \mid V(z_0;r) = 0\}$ constitute the cut set C_Q. The proof of the dual case is analogous. $\square$

At this point we pause the development of the theory and study some examples.

Example 3.8. The easiest non-linear non-degenerate vector fields $Q(z;r)$ are quadratic. By suitable scaling and translation (cf. Lemma 2.1) every such system can be brought to the form

$$\dot{z} = z^2 - \gamma^2 = (z-\gamma)(z+\gamma) \quad \text{with } \gamma = e^{i\varphi} \in S^1 \quad (0 \le \varphi < \pi) \ , \tag{3.4}$$

i.e., $r = (\gamma, -\gamma)$. The two "extreme" cases $\varphi = 0$ and $\varphi = \pi/2$ ($\varphi = \pi$ is the same as $\varphi = 0$) are easily visualised: in case of $\varphi = 0$ one gets the "dipole"

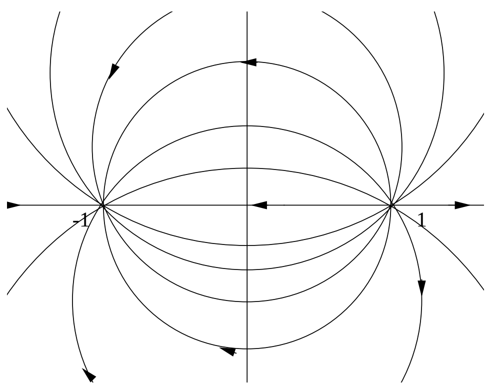


Fig. 1: $\dot{z} = z^2 - 1$

and in case of $\varphi = \pi/2$ one gets the "dual dipole"

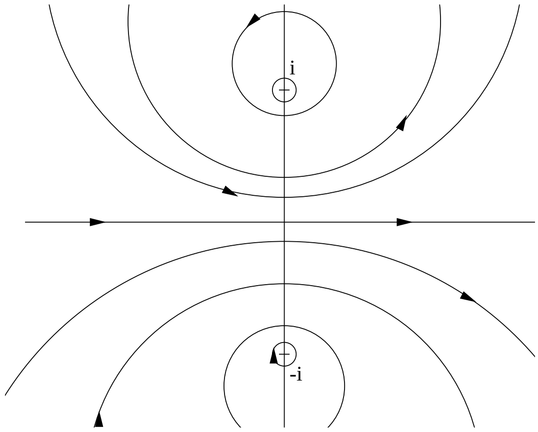


Fig. 2: $\dot{z} = z^2 + 1$

(In fact by Lemma 2.1 (f) the phase portraits of $\dot{z} = z^2 - 1$ and $\dot{z} = z^2 + 1$ are dual to each other modulo a rotational factor $\pi/2$.) How do the other flows look like? The phase portrait for $\varphi = 0$ is depicted in Fig. 1. If one increases φ slightly, the phase portrait is first very similar to the case $\varphi = 0$ except for a slight spiralling of orbits near the fixed points. Increasing φ further results in a higher degree of spiralling (Fig. 3) until at $\varphi = \pi/2$ the various turns of the spirals "close" to form periodic orbits (Fig. 2). A further increase of φ reverses this process until at $\varphi = \pi$ the flow is again the same as for $\varphi = 0$.

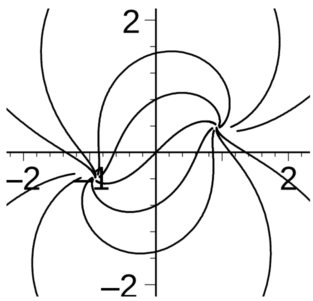


Fig. 3: $\dot{z} = z^2 - (e^{i\pi/8})^2$

What does our theory say about the quadratic family (3.4)? For $r_1 = \gamma = -r_2$ one computes according to formulas (2.4), (2.6), and (2.12)

$$c_1 = \frac{1}{r_1 - r_2} = \frac{1}{2\gamma} = \frac{1}{2} e^{-i\varphi} = -c_2 \ .$$

Setting $\gamma = \alpha + i\beta$, i.e., $\alpha = \cos(\varphi)$ and $\beta = \sin(\varphi)$, this gives with $c_1 = a_1 + ib_1 = -a_2 - ib_2 = -c_2$

$$a_1 = \frac{\alpha}{2} = -a_2 \ , \quad b_1 = -\frac{\beta}{2} = -b_2 \ .$$

Note that by the first paragraph of this section we have two nodes in case of $\varphi = 0$, two centres in case of $\varphi = \pi/2$, and foci otherwise; in addition for $\varphi \neq \pi/2$ the fixed point in the right half plane is a source and in the left half plane a sink. Formulas (2.16) and

(2.17) yield

$$V(z_0;(\gamma,-\gamma)) = -\frac{\beta}{2}\ln\left|\frac{z_0+\gamma}{z_0-\gamma}\right| + \frac{\alpha}{2}\arg\left(\frac{z_0+\gamma}{z_0-\gamma}\right)$$
$$U(z_0;(\gamma,-\gamma)) = \frac{\alpha}{2}\ln\left|\frac{z_0+\gamma}{z_0-\gamma}\right| + \frac{\beta}{2}\arg\left(\frac{z_0+\gamma}{z_0-\gamma}\right)$$

and with $z_0 = x_0 + iy_0$ one computes

$$F(z_0;\gamma) := \frac{z_0+\gamma}{z_0-\gamma} = \frac{x_0^2+y_0^2-\alpha^2-\beta^2+2i(\beta x_0-\alpha y_0)}{(x_0-\alpha)^2+(y_0-\beta)^2} \ .$$

Let us investigate the case $\varphi = 0$, i.e., $\gamma = \alpha = 1$ and $\beta = 0$, in greater detail: since $V(z_0;(1,-1))$ is constant on orbits (Theorem 2.11) one computes

$$V(z_0;(1,-1)) = \text{const.} \Longleftrightarrow \arg(\frac{z_0+\gamma}{z_0-\gamma}) = \arg(F(z_0;1)) = \text{const.}$$
$$\Longleftrightarrow \frac{\Re(F(z_0;1))}{\Im(F(z_0;1))} = \frac{x_0^2+y_0^2-1}{2y_0} = \text{const.} =: h \ .$$

But the last equation is equivalent to

$$x_0^2 + (y_0-h)^2 = h^2+1 \ ,$$

which shows that all orbits are circular arcs with centre $(0,h)$ and radius h^2+1 between the points $(1,0)$ and $(-1,0)$ in $\mathbb{R}^2$. If the constant $h = 0$, then the whole real axis is obtained with cut set C_Q the straight line segment between $(1,0)$ and $(-1,0)$ and separator set $\mathcal{S}_Q$ the rest. Note that the algebraic-differential curves for $\dot{z} = z^2-1$ are all algebraic, namely the real axes and circles.

Similarly, since $U(z_0;(1,-1))$ is constant on dual orbits one computes

$$U(z_0;(1,-1)) = \text{const.} \Longleftrightarrow \ln|F(z_0;1)| = \text{const.} \Longleftrightarrow |F(z_0;1)|^2 = \text{const.}$$
$$\Longleftrightarrow \frac{(x_0+1)^2+y_0^2}{(x_0-1)^2+y_0^2} = \text{const.} =: h' \ .$$

But from

$$(1-h')\left[y_0^2 + \left(x_0 - \frac{h'+1}{h'-1}\right)^2 - \frac{4h'}{(h'-1)^2}\right] = (x_0+1)^2+y_0^2-h'((x_0-1)^2+y_0^2)$$

one concludes that except for $h' = 1$ all dual orbits are circles with centre $(0, \frac{h'+1}{h'-1})$ and radius $\frac{2\sqrt{|h'|}}{|h'-1|}$, which enclose either $(1,0)$ or $(-1,0)$. In the limit $h' \longrightarrow 1$ one gets the imaginary axes, which is the dual separator set $\mathcal{S}_Q^*$ for $\dot{z} = z^2-1$. Note that $\mathcal{S}_Q^*$ is *not* the set of zeros of $U(z_0;(1,-1))$. In fact, the real axes is the set of zeros of $V(z_0;(i,-i))$, in which case the cut set is empty. The treatment of the case $\varphi = \pi/2$ is in general completely analogous modulo a rotation by $\pi/2$ around the origin: the dual curves are now the primary orbits and vice versa.

For the general case $0 \le \varphi < \pi$ we first investigate the equation

$$-\frac{\beta}{2}\ln|Z| + \frac{\alpha}{2}\arg(Z) = \text{const.}$$

The curves of solutions Z are rays emanating from the origin, if $\alpha = 1$ and $\beta = 0$, and circles with the origin as centre, if $\alpha = 0$ and $\beta = 1$; they are spirals emanating from the origin, which have wide turns, if $1 > \alpha \gg \beta > 0$, and narrow turns, if $0 < \alpha \ll \beta < 1$. Considering this curves on the Riemannian sphere $\overline{\mathbb{C}}$ and observing that the factor $1/2$ makes the arg-term π-periodic gives a more complete and symmetric picture: the former rays are in fact great circles through the "south pol" 0 and the "north pol" ∞, the former circles remain circles, and the former spirals become two interwoven spirals, which make wide or narrow turns around ∞ of the same kind — but opposite direction — as around the origin.

Having settled the form of the solution curves in different cases it remains to "lift" them to true orbits. For this we have only to consider their images under the inverse mapping of $F(z_0;\gamma)$:

$$F^{[-1]}(Z;\gamma) = (-\gamma)\frac{Z+1}{-Z+1} \ .$$

Clearly $F^{[-1]}$ maps 0 and ∞ to γ and $-\gamma$, respectively. Since $F^{[-1]}$ is a Möbius transform, it preserves circles on $\overline{\mathbb{C}}$ and hence spirals and their winding behaviour. The result is exactly the phase portrait as discussed on a pictorial basis above. Note that the great circle for const. $= 0$ in case of $\alpha = 1$ and $\beta = 0$ splits into a separator part connecting 1 and -1 via ∞ and a cut set part connecting 1 and -1 via 0; moreover the two associated spirals for const. $= 0$ in case of $0 < \alpha, \beta < 1$ split into spiralling separators connecting γ and $-\gamma$ with ∞ and spiralling "cut set"-orbit connecting γ and $-\gamma$ via 0; since in the case of $\alpha = 0$ and $\beta = 1$ the circles do not touch 0 and ∞ the "equator" of $\overline{\mathbb{C}}$ is mapped to the real axes by $F^{[-1]}$ and the cut set is empty. $\square$

A drawback of formula (2.16) in case of $v_0 = 0$ (or any other v_0) is the use of the full arg-function. In accordance with the notation of Theorem 2.3 let

$$\arg(z) = \mathrm{Arg}(z) + 2\pi k \quad \text{with } k \in \mathbb{Z}, \tag{3.5}$$

where Arg is the principal branch of the arg-function. Let $k = (k_1, \ldots, k_m)$ be a vector of integers and set

$$V(z_0; r; k) := -\sum_{j=1}^{m} [\ b_j \ln|z_0 - r_j| + a_j(\mathrm{Arg}(z_0 - r_j) + 2\pi k_j)\ ] \ . \tag{3.6}$$

Since for example MAPLE accepts as input only expressions like $V(z_0; r; k)$, but not the full $V(z_0; r)$, the following problem is natural:

Problem 2. *Write software, which handles the full* arg*-function, e.g., traces the contours of a function involving* arg.

If all a_j in (2.16) vanish, i.e., if all fixed points are centres, then the k-problem disappears: (2.16) reduces to

$$e^{-v_0} = \prod_{j=1}^{m} |z_0 - r_j|^{b_j} \ , \tag{3.7}$$

and if all numbers b_j are rational, this equation can be brought to a purely polynomial form in z_0. Hence in this case our differential-algebraic curves are in fact algebraic curves.

Problem 3. *Investigate the differential-algebraic curves for fixed degree m or fixed Q. What are the possible topological types? Compare with the topological types of usual plane algebraic curves. Find conditions on r for the occurrence of at least one, finitely many, or infinitely many algebraic curves among the differential-algebraic curves of $Q(z;r)$. (Clearly, a necessary condition for a differential-algebraic curve to be algebraic is that no focus is contained.)*

We collect some further observations, definitions, examples and problems concerning the numbers k_j, respectively, the m-tuple k:

Observation 3.9. If z_0 is a solution of (2.16) for some v_0 and k, then there exist a u_0 such that also (2.17) has the solution z_0 for the same k and vice versa: in fact, both equations are just real and imaginary part of one complex equation.

Observation 3.10. Theorem 2.7 says that for any given z_0 not in the cut set the number $w_0 = u_0 + iv_0$ is uniquely determined by the minimal modulus requirement. But it may well be possible to choose the vector k in different ways, without changing the (minimum) modulus of w_0. Clearly, this can happen only on certain 1- or 0-dimensional sets.

Note that for two commensurable numbers a_j and $a_{j'}$, i.e., if their quotient is rational, there are infinitely many choices of k_j and $k_{j'}$, which all give the same sum $a_j k_j + a_{j'} k_{j'}$. It is therefore necessary to choose a unique representative $(k_j, k_{j'})$ for such a family, for example, the pair with smallest sum of absolute values.

Definition 3.11. *Let $Q(z;r)$ be a non-degenerate vector field* (2.2)*. Then a set of points z_0, where the integers k_j are uniquely determined by the minimal modulus requirement on w_0 and the requirement to have smallest possible sum of absolute values, is called a k-*region*. Sets of points inside a k-region or between two k-regions are called k-*boundaries *and points where two or more k-regions meet are called k-points.*

Observation 3.12. To determine the k-regions and k-points for a given $Q(z;r)$ it is only necessary to find the k-boundaries. Two situations for a k-boundary are conceivable: either it is a boundary between different k-regions or it lays inside a k-region.

Observation 3.13. If all fixed points of a non-degenerate vector field (2.2) are centres, the k-problem disappears; hence assume that r_1 is a source or sink. Since $\sum_{j=1}^{m} a_j = 0$ by (2.12), it is possible then to assume that k_1 vanishes. On the other hand, if $a_{j_1}, \ldots, a_{j_s} = a_\nu$, it is possible to use one term $a_\nu 2\pi k_\nu$ instead of several different terms with integers $k_{j_1}, \ldots, k_{j_s}$. Whether the first or second choice is more convenient depends on the concrete situation.

Example 3.14. Consider again the quadratic family (3.4). With the same notations as in Example 3.8 one computes

$$4(V^2+U^2) = (\alpha^2+\beta^2)(\ln^2\left|\frac{z_0+\gamma}{z_0-\gamma}\right| + \arg^2\left(\frac{z_0+\gamma}{z_0-\gamma}\right)) .$$

This expression is minimal for any given $z_0 \in \mathbb{C}\setminus Fix(Q)$, if $|\mathrm{Arg}(\dots)+2k\pi|$ is minimal. Since $|\mathrm{Arg}(\dots)| \le \pi$, the appropriate choice of k is usually $k=0$, but in case of $|\mathrm{Arg}(\dots)| = \pi$ it is $k=\pm1$. Therefore the k-boundary of a system from the quadratic family (3.4) contains all z_0, where

$$\mathrm{Arg}(z_0+\gamma) - \mathrm{Arg}(z_0-\gamma) = \pm\pi .$$

But this is the case exactly on the straight line segment joining γ and $-\gamma$. Note that this set coincides with the cut set, if $\gamma=1$, and is different otherwise; in particular it exists also for $\gamma = i$, where the cut set is empty. This is in accordance with the fact that the orbits revolve infinitely often around the fixed points, if $\gamma \neq 1$, thereby increasing or decreasing infinitely the angle $\arg(z_0 \pm \gamma)$.

Example 3.15. Let $r=(0,-i,i)$. Then $c_1=a_1=1$ and $c_2=c_3=a_2=-1/2$, i.e., 0 is a source and both i and $-i$ are sinks. $U(z_0;r)$ contains only ln-terms and

$$V(z_0;r;k) = -\mathrm{Arg}(z_0) + 2\pi k_1 + \frac{1}{2}\mathrm{Arg}(z_0^2+1) + \pi k_2 .$$

We investigate V on the axes: for $s\in\mathbb{R}$ let first $s\le 0$; then $V(s;r;(0,0))=0$. Second, if $s<0$, then $V(s;r;k) = \pm\pi + 2\pi k_1 + \pi k_2$, and the choices $k=(0,\mp1)$ make V vanishing. Third, if $s>1$, then $V(is;r;(0,0)) = \mp\pi/2 + \mathrm{Arg}(1-s^2)/2 = \mp\pi/2 \pm \pi/2 = 0$. And fourth, if $0<s<1$, then $V(is;r;k) = \mp\pi/2 + 2\pi k_1 + \pi k_2$, there is no choice for k such that V vanishes.

All in all one has the real axis and the imaginary axis except for the line segment joining i and $-i$ as separator set; and the negative real axes as (part of (?)) the k-boundary.

Example 3.16. Here we present some more complex examples in connection with Problem 2 (the tracing of separators, when the full arg is involved). In the quest for the separator set it is — according to Observation 3.13 — always a good choice to begin with $k=(0,\dots,0)$. But as the previous example has shown already, not all points can be found as solutions of $V(z_0;r;(0,\dots,0))$.

If for example $r=(2i,0,1+i,1-i)$, then $2i$ is a source, 0 and $1+i$ are centres, and $1-i$ a sink. Let the "upper half" of the pure separator bounding the centre sector of 0 be all points on it with imaginary part greater -1. This is the complete branch tending to infinity in the negative real direction. It can not be found with the choice $k=(0,0,0,0)$, but with $k=(0,0,0,-1)$.

Of course, spiralling of orbits around r_j leads to a new k_j with every new turn, but it is possible to have on a single orbit different several different values of k without spiralling, too. If for example $r=(1-i,0,-1+2i,1+2i,2-2i)$, then $1-i$ is a source, 0 a centre, and all other fixed points sinks. Again the centre sector of 0 is bounded by one pure separator, but this time with 3 "k-sections": $k=(0,0,0,0,0)$,

$k = (0,0,0,-1,0)$, and $k = (0,0,-1,-1,0)$; the first for points below the line $-2i$, the second between the lines $-2i$ and $-i$, and third above $-i$. Obviously, every time the separator crosses a ray emanating from a fixed point in the negative real direction, k changes, but it is not so easy to come by with general rule, which predicts the correct k values for all systems (2.2) and orbits.

Problem 4. *Investigate the location of k-boundaries for general vector fields $Q(z;r)$. In particular, find a rule, which predicts the correct k for all "sections" on an orbit.*

Problem 5. *Is there a simple local procedure to determine the valence of a fixed point? (The problem here is that a fixed point may be a k-point.) Are there k-points where more than three k-regions meet?*

Finally in this section we discuss a problem concerning the characteristic numbers χ_j defined in (2.6). Namely, what can be said about vector fields $Q(z;r)$ or simply vectors $r = (r_1, \ldots, r_m)$, which yield the same m-tuple $\chi = (\chi_1, \ldots, \chi_m)$ of characteristic numbers?

Observation 3.17. By the translation property of Lemma 2.1 (a) one can assume $r_1 = 0$ and, correspondingly, χ_1 as completely determined by formula (2.12). Moreover, for our non-degenerate vector field $Q(z;r)$ all χ_j are non-zero. It is therefore enough to consider for given $\chi_2, \ldots, \chi_m \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ the solutions $r_2, \ldots, r_m \in \mathbb{C}^*$ of the system

$$P_j(r_2, \ldots, r_m) := r_j \prod_{\substack{\nu=2 \\ \nu \neq j}}^{m} (r_j - r_\nu) = \chi_j \qquad (j = 2, \ldots, m) \ . \tag{3.8}$$

Observation 3.18. By the rotation property of Lemma 2.1 (d) the number of solutions of (3.8) — if there are any — is divisible by $(m-1)$. Clearly, for $m = 2$ one always has the solution $r_2 = \chi_2$, and for $m = 3$ one computes that

$$r_2^2 = \frac{\chi_2^2}{\chi_2 + \chi_3} \ , \quad r_3 = -\frac{\chi_3 r_2}{\chi_2} \quad \text{ if } \chi_2 + \chi_3 \neq 0 \ ,$$

whereas $\chi_2 + \chi_3 = 0$ implies $r_2 = r_3$. Whence, for $m = 3$ there are exactly 2 solutions, which are related by a rotation ζ_2, or none.

Example 3.19. For $m = 4$ and $\chi = (2+2i, i, 2-2i, -2i)$ there are two solutions $r = (0, 1, 2, 1+i)$ and $(0, \frac{W}{50}(7+i), \frac{W}{25}(1+3i), W)$ with $W = (650 + 450i)^{1/3}$ of (3.8), which are not related by a rotation ζ_3 or ζ_3^2.

Problem 6. *Is* (3.8) *generically solvable, i.e., except for χ laying on an affine variety in $\mathbb{C}[\chi]$? If yes, characterise this variety. If* (3.8) *is solvable, is the number of solutions finite? If yes, what is the exact number?*

The preceding problem is of purely algebraic nature, whereas the following involves dynamics, too.

Problem 7. *Let $S(\chi)$ be the set of solutions of* (3.8) *for given χ and $m \geq 4$. Are all vector fields $Q(z;r)$ defined by the vectors $r = (0, r_2, \ldots, r_m)$ for $(r_2, \ldots, r_m) \in S(\chi)$ topologically (or even affinely) equivalent?*

4. Combinatorics

The combinatorial type of a non-degenerate vector field $Q(z;r)$ should be a faithful combinatorial representation of its topological type, which itself is completely determined by the relative location of separators and fixed points in the complex plane (cf. Corollary 3.6). The first and most natural such combinatorial representation is given by the plane *diagram* $D(Q)$ of Q. If Q has degree m, then $D(Q)$ is a planar graph with some additional structure:

(1) A unit circle S^1, on which the $2(m-1)$ roots of unity $\zeta^k_{2(m-1)}$ are marked by points. These are the *outer vertices*, which represent the $2(m-1)$ asymptotic directions of separators. The outer vertices are connected by *outer edges* forming the circle.

(2) Inside the circle there are m points, the *inner vertices*, which represent the m fixed points of Q and which carry signs {+,0,-} according to signs of the a_j, i.e., they represent the sources, centres, and sinks present. For simplicity we will speak of the inner vertices of positive, zero, or negative sign as sources, centres, or sinks, respectively. Instead of labelling the inner vertices with signs we will use the following convention: centres are depicted as small circles, the other inner vertices as dots; the distinction between sources and sinks will be a consequence of the next item.

(3) The outer vertices carry $+$ and $-$ signs in an alternating fashion, where the point $\zeta^0_{2(m-1)} = 1$ is negative. Every inner point, which is a source or sink, is connected by at least one *inner edge* to an outer vertex, which is always of opposite sign. Any outer vertex not connected to an inner vertex is connected to exactly one other outer vertex of opposite sign; these edges are called *pure.* The inner vertices of sign 0 are isolated, i.e., not incident with any edge.

(4) The connected components inside the unit circle of the planar graph characterised by 1.-3. above are called *cells.* They represent the sectors of Q. Note that the drawing of the diagram can be and will be assumed to have no crossings of edges. The boundary of a cell has two parts: the *outer boundary* consisting of outer vertices and edges and the *inner boundary* consisting of inner vertices and edges.

(5) Every centre vertex is contained in a cell with one or two pure edges as inner boundary. If there is one pure edge, which necessarily connects two adjacent outer vertices, the cell is called a *drop*; if there are two pure edges, the cell is called a *belt.* The union of all sectors containing a given source or sink in their closure does not contain any other source or sink on its inner boundary.

(6) A source or sink of valence 1 together with its single incident inner edge is called a *thorn*. We speak of a positive [negative] thorn, if the involved inner vertex has positive [negative] sign. A cell with one pure edge, which is not a drop, contains exactly two thorns of opposite sign. A cell with non-pure inner boundary contains exactly one thorn.

Conditions 5. and 6. on a diagram are combinatorial translations of Definition 4.1 and Lemma 4.2 below.

Definition 4.1. *A fixed point of valence 1 together with the separator leading to it is called a* thorn*; a thorn is called positive [negative], if the involved fixed point is a source [sink]. A centre sector is called a* drop *or a* belt*, if it is bounded by one or two pure separators, respectively.*

The names occurring in Definition 4.1 are chosen to reflect the geometry of the respective objects on the Riemannian sphere.

Lemma 4.2. *Every sector of a given non-degenerate vector field is either a centre sector or it contains exactly one or two thorns. More precisely: a sector, which is bounded by one pure separator, but is not a drop, contains exactly two thorns of opposite sign; a sector, which is bounded by non-pure separators, contains exactly one thorn. Alternatively: the boundary of a sector contains either no fixed point (in case of a centre sector) or exactly one source and one sink.*

Proof. From the results of Section 3 it is clear that for the boundary of a non-centre sector there are only two possibilities: either it consists of a pure separator and at least two thorns, or it consists of two separators meeting in a common fixed point and at least one additional thorn. In either case we label the thorns and their associated fixed points consecutively (with regard to asymptotic direction of their separators) by $T_1, \ldots, T_s$ and $F_1, \ldots, F_s$, respectively. Then the sign of thorns alternates and we may assume T_1 to be positive.

In case of a pure separator there are orbits leaving the source F_1, which move closely along the pure separator and end up in some sink F_ν, whence ν is even. If $\nu \neq 2$, then there exist orbits arriving a F_2, which come from some $F_{\nu'}$, where $1 \leq \nu' < \nu$. Since the family of all orbits from F_1 to F_ν fills an open connected region in the sector and the family of all orbits from $F_{\nu'}$ to F_2, too, there must be an additional boundary line between these two regions. However this boundary line must be a separator, because it can not be bounded; contradiction. If $\nu = 2$ but $s > 2$, a similar argument applies to the orbits from F_1 to F_2 and F_3 to F_ν ($\nu > 3$).

The case of two separators meeting in a common fixed point F is similar: then F_1 and F_s are both sources, F is a sink, and for $s \neq 1$ the two regions filled by the orbits from F_1 to F and the orbits from F_s to F would be divided by at least one additional separator.

The last (alternative) assertion follows by the same arguments as above. □

We give an example of a vector field Q and its associated diagram $D(Q)$:

Example 4.3. For $r = (1-i, 0, -1+2i, 1+2i, 2-2i)$ the fixed point $1-i$ is a source, 0 is a centre, and the other three are sinks of $\dot{z} = Q(z; r)$. It is easy to derive the diagram $D(Q)$ from the following picture of the vector field Q (— for better visibility all vectors have the same length —):

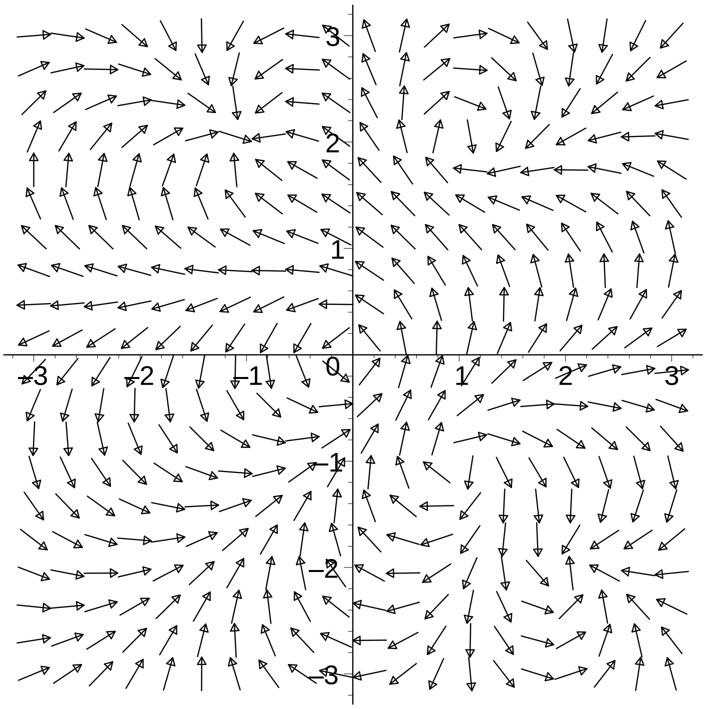
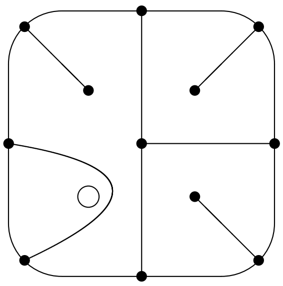


Fig. 4: $\quad \dot{z} = Q(z; r)$ with $\mathcal{S}_Q$.

Since the centre sector obviously occupies the upper half of the third quadrant, the single source $1-i$ must be connected to the remaining outer vertices of negative sign. But this determines already completely the diagram[2] $D(Q)$:



In general, it can of course be difficult to determine by mere inspection of the vector field the associated diagram without computing explicitly the separator set.

Note, that for the derivation of $D(Q)$ from a given Q one needs only items 1.-4. of the characterisation of diagrams. But the goal is of course to characterise completely in combinatorial terms all *realizable* diagrams; by this we mean all diagrams D, which are the diagrams $D(Q)$ of some Q. We have seen above that items 5. and 6. are necessary

[2]For typographical reasons we have to use rounded squares instead of circles.

conditions for realisability, and we will see in Section 5 below that they are in fact sufficient. But before this is done, it is good to explore the combinatorics of diagrams in greater detail.

The following definition is natural from Lemma 2.1 (d), (e), and (g):

Definition 4.4. *Two diagrams are* equivalent, *if they result from each other by a graph isomorphism, which rotates the outer vertices by ζ_{m-1}^k. They are* strongly equivalent, *if even rotations $\zeta_{2(m-1)}^k$ are allowed. The equivalence classes of diagrams are called* combinatorial types, *the strong equivalence classes* un-oriented combinatorial types. *The* conjugate diagram $\overline{D}$ *results from a given D by a graph isomorphism, which reflects the outer vertices along the real axis.*

Recall that two vector fields Q and Q' are *topologically equivalent*, if there exists a homeomorphism of the plane, which maps orbits to orbits while preserving their orientation. Any quantity associated to a vector field, which remains constant on topological equivalence classes, is called a *topological invariant.* The degree m of a non-degenerate vector field Q or its number of fixed points is the most obvious topological invariant. The next definition gives some more refined invariants:

Definition 4.5. *Let $Q(z; r)$ be a non-degenerate vector field. Then the* signatur *of Q is $\sigma = \sigma(Q) = (\sigma_+, \sigma_0, \sigma_-)$, where σ_+, σ_0, and σ_- are the numbers of sources, centres, and sinks, respectively. The* refined signatur *of Q splits the number σ_0 into a pair $\sigma_0 = (\sigma_{0+}, \sigma_{0-})$, where*

$$\sigma_{0\pm} := \sharp\{j \mid a_j = 0 \ , \ b_j \gtrless 0\} \ .$$

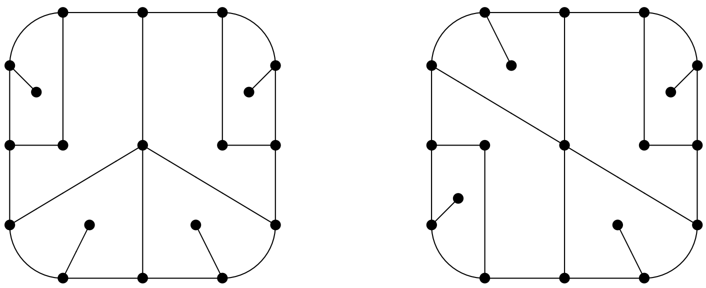
The complete signatur *of Q splits in addition the numbers σ_+ and σ_- into partitions λ^+ and λ^- of σ_+ and σ_-, respectively, where the non-increasingly ordered parts of λ^+ and λ^- are the valences of the occurring sources and sinks, respectively. The above definitions apply analogously to combinatorial types.*

Corollary 4.6. *Signature, refined signatur, and complete signatur are topological invariants of a non-degenerate vector field Q.*

Proof. For the signatur this is obvious and for refined and complete signaturs this follows from the fact (Corollary 3.6) that the topological type of Q is completely determined by the configuration of separators and fixed points, namely, its combinatorial type: observe that the sign of b_j for nodes and foci is not a topological invariant, but that the rotational direction of periodic orbits near a centre induces this direction to all periodic orbits in the same sector and therefore to the pure separator(s) of the boundary. □

Whereas the signatur and the refined signatur are calculated easily, the determination of the partitions λ^+ and λ^- underpins the significance of Problem 5. Vector fields of different topological type may well have the same signatur (ordinary, refined, or complete): for the ordinary and refined signatur it is an easy matter to find examples in the tables below; for the complete signatur an example of smallest possible degree is described in

Example 4.7. For $m = 7$ there are two non-equivalent combinatorial types having the same complete signatur $((2, 2, 1, 1), (0, 0), (4, 1, 1))$. Note that the two diagrams coincide in the right half, whereas they are mirror images of each other in the left half.

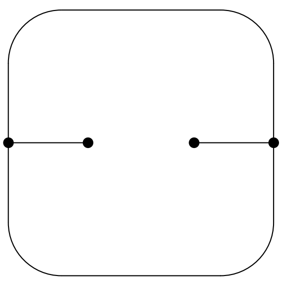


A step towards the solution of Problem 7, which is of independent interest, is:

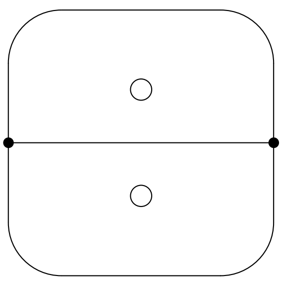
Problem 8. *Characterise the stratification of the space of m-tuples of characteristic numbers and of roots, which is induced by the different kinds of signaturs.*

The tables below contain diagrams, which represent all combinatorial types occurring for degree $2 \leq m \leq 5$. A mark (▼) at a diagram indicates that there is an additional conjugate diagram, which is not depicted. Below each diagram the signatur σ is given and the roots r of a vector field $Q(z; r)$, whose topological type is represented by the diagram. The roots are ordered such that the σ_+ sources come first, then the σ_0 centres follow, and finally the σ_- sinks conclude.

Table 4.8. Diagrams representing all combinatorial (and topological) types of vector fields of degree 2:

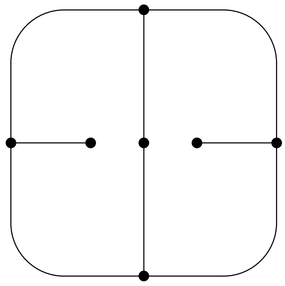


$\sigma = (1, 0, 1)$
$r = (1, -1)$

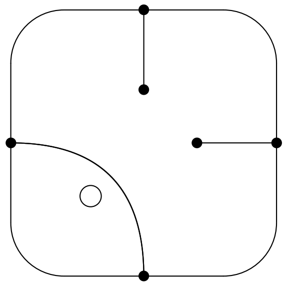


$\sigma = (0, 2, 0)$
$r = (i, -i)$

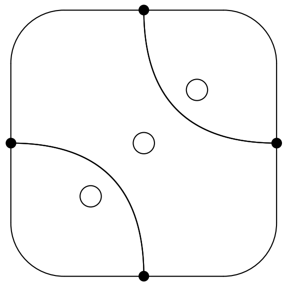
Table 4.9. Diagrams representing all combinatorial (and topological) types of vector fields of degree 3:



$\sigma = (2, 0, 1)$
$r = (1, -1, 0)$

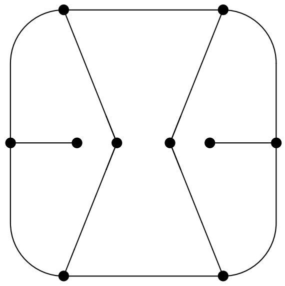


$\sigma = (1, 1, 1)$
$r = (1, 0, i)$

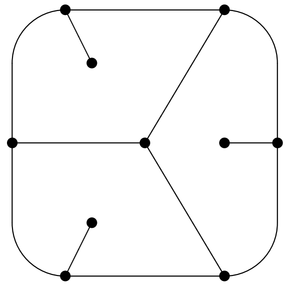


$\sigma = (0, 3, 0)$
$r = (-1 - i, 0, 1 + i)$

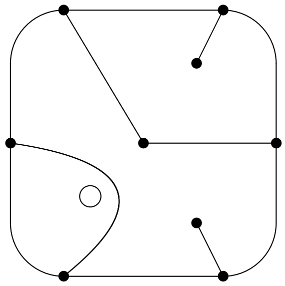
Table 4.10. Diagrams representing all combinatorial (and topological) types of vector fields of degree 4:



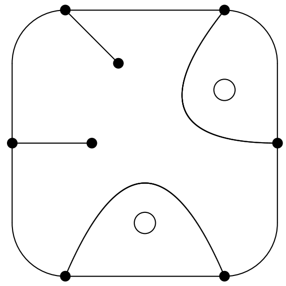
$\sigma = (2, 0, 2)$
$r = (-1, 1, 0, 2)$



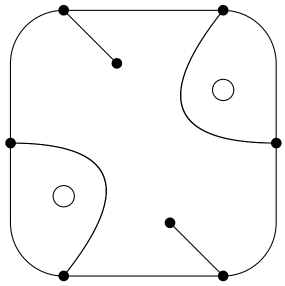
$\sigma = (3, 0, 1)$
$r = (1, i, -i, 0)$



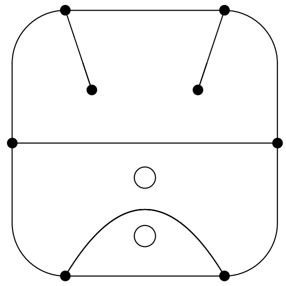
$\sigma = (1, 1, 2)$
$r = (0, -1 - i, -i, 2 + 2i)$



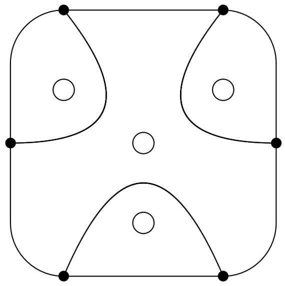
$\sigma = (1, 2, 1)$
$r = (0, -i, 2 + i, -1)$



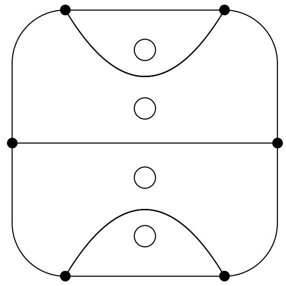
$\sigma = (1, 2, 1)$
$r = (2i, 0, 1 + i, 1 - i)$



$\sigma = (1, 2, 1)$
$r = (-1 + i, 0, -i, 1 + i)$

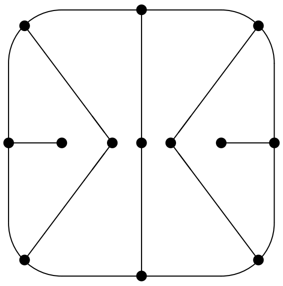


$\sigma = (0, 4, 0)$
$r = (-2i, 1, 2 + 2i, -2 + 2i)$

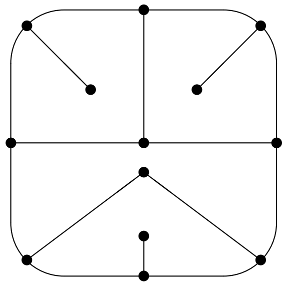


$\sigma = (0, 4, 0)$
$r = (-i, 0, i, 2i)$

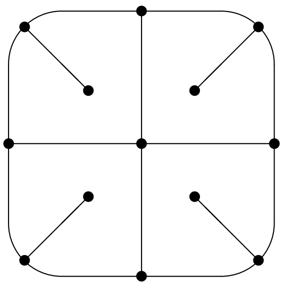
Table 4.11. Diagrams representing all combinatorial (and topological) types of vector fields of degree 5:



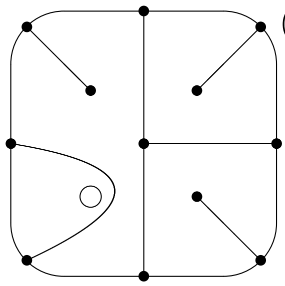
$\sigma = (3, 0, 2)$
$r = (-2, 0, 2, -1, 1)$



$\sigma = (3, 0, 2)$
$r = (-2i, 0, -i, 1 + i, -1 + i)$

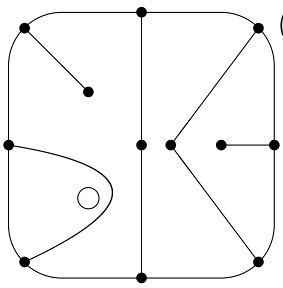


$\sigma = (1, 0, 4)$
$r = (0, 1, -1, -i, i)$



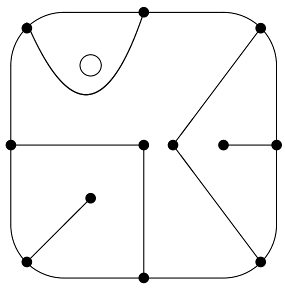
(▼)

$\sigma = (1, 1, 3)$
$r = (1 - i, 0, -1 + 2i, 1 + 2i, 2 - 2i)$

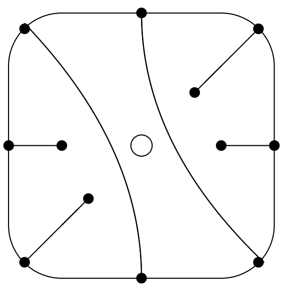


(▼)

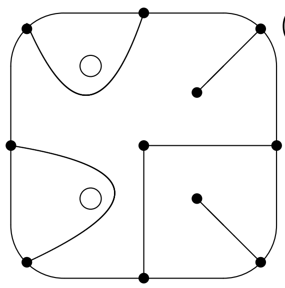
$\sigma = (2, 1, 2)$
$r = (1 - i, 2, 0, i, 2 + 2i)$



$\sigma = (2, 1, 2)$
$r = (2 - i, -1 - i, 0, 1, -1 - 3i)$



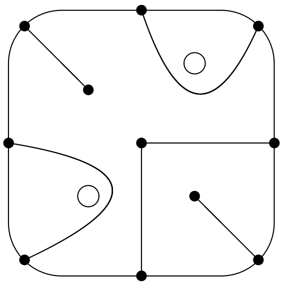
$\sigma = (2, 1, 2)$
$r = (-1, 1, 0, 1 + i, -1 - i)$



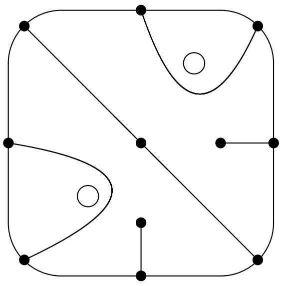
(▼)

$\sigma = (1, 2, 2)$
$r = (1 - i, 0, 1 + 3i, 2, \frac{3}{2} + 3i)$

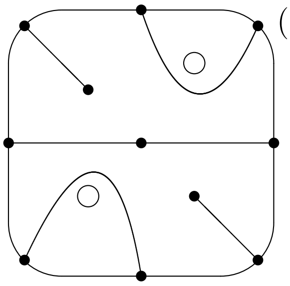
Diagrams of degree 5 (continuation):



$\sigma = (2,2,1)$
$r = (-1, 0, 1, -1+i, 2-2i)$

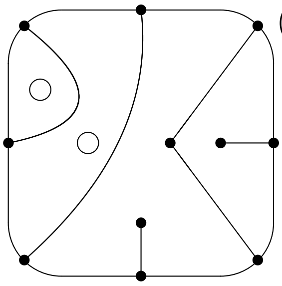


$\sigma = (2,2,1)$
$r = (6, -3i, 0, 3+3i, -1+i)$



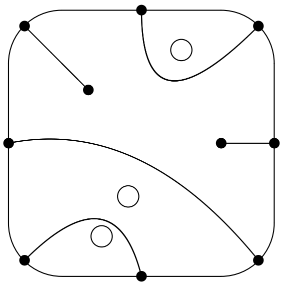
(▼)

$\sigma = (1,2,2)$
$r = (i, 0, 1+3i, 1, \frac{1}{2}(-1+3i))$

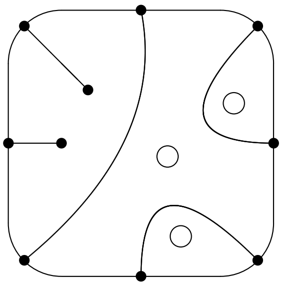


(▼)

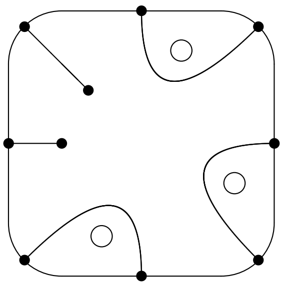
$\sigma = (2,2,1)$
$r = (1, 0, -1-2i, -1+2i, i)$



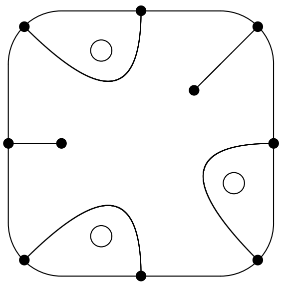
$\sigma = (1,3,1)$, $r =$
$(\frac{1}{25}(14-2i), 0, -1-i, -1+2i, \frac{1}{5}(-13+26i))$



$\sigma = (1,3,1)$
$r = (-7, 0, -1-i, -1-2i, \frac{1}{2}(-9+3i))$

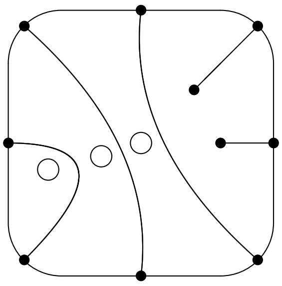


$\sigma = (1,3,1)$, $r =$
$(\frac{1}{25}(-2+14i), 0, 1+i, 1+2i, \frac{1}{35}(27+54i))$

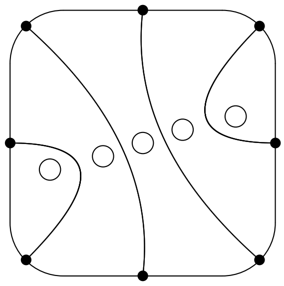


$\sigma = (1,3,1)$
$r = (-2, 0, 1, -1-2i, 2+i)$

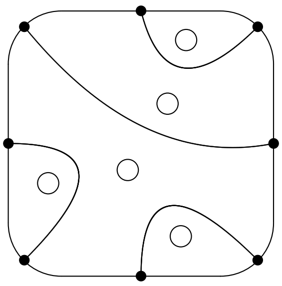
Diagrams of degree 5 (continuation):



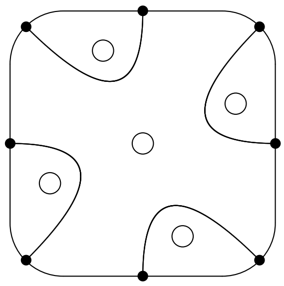
$\sigma = (1,3,1)$
$r = ?$ (see Remark 5.2)



$\sigma = (0,5,0)$
$r = (-2q, -q, 0, q, 2q),\ q = e^{i\pi/8}$



$\sigma = (0,5,0)$
$r \approx (0, 0.091 + i, -0.0455 + 0.5i,$
$0.69826i, -1.223)$



$\sigma = (0,5,0)$
$r = (0, 1 - (1+\sqrt{2})i, (1-\sqrt{2}) - i,$
$-(1+\sqrt{2}) - i, -(1+\sqrt{2}) - (3+2\sqrt{2})i))$

Problem 9. *Let $\tau(m,k)$ be the number of un-oriented combinatorial types of degree m with $\sigma_0 = k$. Investigate the properties of this triangle of numbers, in particular, whether there is a (nice) recursion or generating function. Are the numbers $\tau(m,k)$ essentially ($\tau(m,m-1) = 0$ excluded) unimodal for fixed m? What can be said about the sequence 2, 3, 9, 23, ... of cumulative numbers $\overline{\tau}(m) = \sum_{k=0}^{m} \tau(m,k)$? Are there connections with the Catalan and Motzkin numbers [Be]? What can be said about the same questions for (oriented) combinatorial types and diagrams?*

A first small step in solving Problem 9 is the following

Theorem 4.12. *$\tau(m,0) = \tau(m,m)$ in the notation of Problem 9. In fact, there is an explicit bijection called* combinatorial duality *between the un-oriented combinatorial types (or diagrams) of degree m with $\sigma_0 = 0$ or $\sigma_0 = m$, which extends to a duality between vector fields of degree m with zero or m centres, where in case of zero centres all fixed points are nodes.*

Proof. Combinatorial duality for the diagrams in question is similar to the usual duality for planar graphs: First let D be any diagram of degree m with $\sigma_0 = m$; replace every centre by a dot (representing a node) and draw within any centre sector (non-crossing) lines from the dot to all connected components of the outer boundary. Then the resulting figure is planar and has trivially the same number of inner vertices; but is has the same number of outer vertices, too, because from item 6. of the definition of a diagram or Lemma 4.2 the connected components of the outer boundary of a centre sector are simply outer edges. Thus we arrive at a diagram D^* of degree m with $\sigma_0 = 0$. Reversely, the original D can be restored from D^*: replace the dots for the inner vertices by the small circles for centres, every outer vertex by an outer edge and vice versa, and finally "blow up" every inner edge to an open neighbourhood containing exactly one inner vertex together with all incident edges and the new bounding outer edge. Thus the closed sets between the open neighbourhoods can be replaced by pure edges, and the construction of the desired bijection between diagrams is finished.

Since the construction does not depend on the special diagram representing an unoriented combinatorial type, the equality $\tau(m, 0) = \tau(m, m)$ follows.

If one considers vector fields Q of degree m with centres only, then the above description of combinatorial duality translates to a description using dual orbits of the perpendicular vector field: centres are replaced by nodes and every "lobe" of a centre sector between pure separators contains exactly one separator connected to the new node. The resulting vector fields Q^* of degree m has therefore only nodes as fixed points and the original Q can of course be recovered as $(Q^*)^*$. □

Problem 10. *Find an effective algorithm, which enumerates (recursively) the unoriented combinatorial types.*

For the construction of the desired algorithm it is useful to introduce a more symbolic description of diagrams. Let a diagram D of degree m with signatur information $(\lambda^+, \sigma_0, \lambda^-)$ be given and label the outer vertices of D consecutively in counter clockwise direction by $1, \overline{1}, 2, \overline{2}, \ldots, \overline{2(m-1)}$ such that 1 has label 1. Then the inner edges induce a *outer partition* $\pi = \pi(D) = (\pi_+, \pi_0, \pi_-)$ of this set of labels with the following properties:

(1) Let $L_m := \{1, \overline{1}, 2, \ldots, \overline{2(m-1)}\}$ be the set of labels. Then the set of all elements of π_+, π_0, and π_- forms a set partition of L_m into non-empty disjoint blocks.
(2) The blocks in π_+ are the unions of all outer vertices, which are adjacent to the same source. Thus only labels without bars occur and the vector of occurring block sizes can be ordered to form λ^+. Similarly, the blocks in π_- are formed from all outer vertices adjacent to the same sink; thus only labels with bars occur and the vector of occurring block sizes can be ordered to form λ^-. Finally, π_0 contains σ_0 sets, which consist of a two numbers each — one with and one without bar —.

This already completely defines how to associate a outer partition to a diagram, and we omit the translation of the further conditions and results about diagrams stated above into the language outer partitions. (Especially there is a cyclically invariant "non-crossing" condition, which reflects the planarity of the diagram.)

Example 4.13. The two diagrams of Example 4.7 have the outer partitions $\pi = (12|34|5|6, \emptyset, \overline{2456}|\overline{1}|\overline{3})$ and $\pi = (12|45|3|6, \emptyset, \overline{2356}|\overline{1}|\overline{4})$, respectively, where we have used the usual (self explanatory) notation for set partitions. The diagram of Example 4.3 has the outer partition $\pi = (124, 3\overline{3}, \overline{1}|\overline{2}|\overline{4})$.

Theorem 4.14. *The outer partition $\pi = (\pi_+, \pi_0, \pi_-)$ of a diagram is completely determined by either π_+ and π_0 or π_- and π_0. If in particular no centres are present, then $\pi = (\pi_+, \emptyset, \pi_-)$ is determined by π_+ or π_- alone.*

Proof. For a diagram of degree m consider first the case of no centres, i.e., $\sigma_0 = 0$ and $1 \leq \sigma_+ = s \leq m-1$. Assume that π_+ is given; then also λ^+ and the positive part D^+ of a diagram containing all sources and the edges to the outer vertices with labels $1, \dots, m-1$ is determined. To determine the number of connected components of D^+ inside the circle one observes that beginning with the empty interior every addition of a source with valence ν adds $\nu - 1$ new components. Hence D^+ contains

$$1 + (\lambda_1^+ - 1) + \cdots + (\lambda_1^+ - s) = (\lambda_1^+ + \cdots + \lambda_s^+) + 1 - s = (m-1) + 1 - s = \sigma_-$$

connected components. Since D is planar, every such component has to contain exactly one sink, which is adjacent to some of the $m-1$ outer vertices $\overline{1}, \dots, \overline{m-1}$. Thus the blocks of π_- are completely determined. (Of course, the argument is the same, when one starts with π_-.)

For the general case, assume that π_+ and $\pi_0 \neq \emptyset$ are given. In the diagram $D^{+,0}$ determined so far contract all pure edges and suitable pairs of adjacent outer edges to a single outer edge, such that one or several new smaller diagrams result, which retain only the sources and the outer vertices connected to them. Since these smaller diagrams do not contain centres and since they are determined by suitable pieces of π_+, the assertion of the theorem follows from the case of no centres already proven. ☐

Remark 4.15. Knowledge of π_+ and π_- does not determine π_0: if there are only centres, then $\pi_+, \pi_- = \emptyset$, but for $m \geq 4$ there are more than one combinatorial types of degree m with $\sigma_0 = m$ (cf. Tables 4.8-11).

A useful alternative to diagrams and outer partitions, which does not completely characterise the topological type, but is combinatorially more tractable, is the *adjacency graph* G of a diagram D or vector field Q, constructed as follows:

1. The vertices of G represent the inner vertices of D or fixed points of G. Thus it is convenient to speak of sources , centres, and sinks again.
2. A source and a sink are adjacent in G, if they are contained in the border of a single cell in D or if they are the α- and ω-limits of an orbit of Q.
3. Two vertices of G at least one of which is a centre are adjacent, if their respective cells or sectors contain in their boundaries a common pure edge or pure separator.
4. There are no other adjacencies.

Remark 4.16. (a) If D or Q has no centres, then G is bipartite.

(b) The adjacency of point 3. is well defined, because the common part of the boundary between two neighbourly centre sectors or a centre sector and a non-centre sector is pure. In fact, it follows from condition 6. on the possible type of cells of a diagram or from Lemma 4.2 for sectors of a vector field, that if a centre is adjacent to a non-centre in G, then there is in fact a triangle in G consisting of a source, a centre, and a sink.

(c) The combinatorial or topological type is not uniquely determined by the adjacency graph: if there is only one source an one sink present together with at least two drops, then the two drops can be separated by the two thorns or not; in either case the adjacency graph is the same.

(d) Problems 9 and 10 apply accordingly to isomorphism types of adjacency graphs.

5. Topology

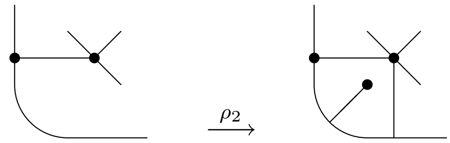
The description of a diagram in Section 4 characterises first the possible adjacencies between vertices in the planar graph (1.-3.) and then the possible kinds of cells (4.-6.). Lemma 4.2 shows that in fact all possible topological types of vector fields Q can be represented by diagrams. In this section we will prove that the reverse is true, too:

Theorem 5.1. *Let D be any diagram formed in accordance to the rules* 1. - 6. *of Section* 4. *Then there exists a non-degenerate vector field $Q(z;r)$ of degree $m \geq 2$ such that $D = D(Q)$, i.e., D represents the topological type of Q. In other words: the topological types of non-degenerate vector field $Q(z;r)$ are completely classified by the possible combinatorial types. (Since it is a simple matter to check whether a given planar graph is in fact a correct diagram, the problem of topological classification of non-degenerate vector fields $Q(z;r)$ is completely solved.)*

Proof. By definition any diagram contains at least one drop or one thorn (see also Lemma 4.2): if no drop is present, then any occurring centre sector is a belt, and the belts divides the diagram into sub diagrams with no centre sectors; but if a diagram has no centre sector and every inner vertex has valence greater 1, then one gets a contradiction:

$$2(m-1) = \sum_{j=1}^{m} \nu(j) \geq \sum_{j=1}^{m} 2 = 2m \ .$$

Therefore every diagram of degree $m+1$ can be obtained by one of the following three *insertion operations* from a diagram of lower degree. The first operation ρ_1 is the insertion of a drop; the second operation ρ_2 is the insertion of a thorn, where a suitable inner vertex increases its valence by one to match the second of the two generated outer vertices:



The third operation ρ_3 inserts two thorns of opposite sign, such that a new cell with a new pure boundary is generated. Note that ρ_1 and ρ_2 increase the degree of the digram by one, whereas ρ_3 increases it by two.

It is now possible to reproduce the recursive combinatorial insertion operations ρ_1, ρ_2, ρ_3 by topological insertion operations $\rho_1', \rho_2', \rho_3'$ in such a way that the newly generated topological type is represented by the newly generated combinatorial type. This is done by suitable *perturbations*. Assume that for some vector field Q a big enough disk $\Delta(Q)$ contains the fixed points $Fix(Q)$ and the essential dynamical features of Q, i.e., outside of $\Delta(Q)$ the vector field Q behaves like the homogeneous field (2.3) with an arbitrary degree of precision. Then the insertion ρ_1' of a drop, or a thorn ρ_2', or two thorns ρ_3' corresponding to ρ_1, ρ_2, ρ_3, respectively, does affect the interior of $\Delta(Q)$ up to an arbitrarily small correction. In fact, if Q has no centres, then Q is structurally stable inside $\Delta(Q)$, and if Q has σ_0 centres, then there are σ_0 complex parameters available to restore the σ_0 centres after the perturbations have taken place.

Having argued that the topological type of the original system can be retained sufficiently well inside a large enough disk $\Delta(Q)$, it remains to be shown how the perturbative insertion operations $\rho_1', \rho_2', \rho_3'$ can be realized. Note first of all that orientation is not an issue, because it can be adjusted through a rotation (Lemma 2.1 (e)). Subsequently, we use primes to denote the newly generated vector field of degree $m+1$, its fixed points, etc.

For ρ_2' let r_ν be the fixed point of Q, which increases its valence by one through the insertion of a thorn. This means that one of the separators leading to r_ν has to be "split" into two separators, thereby including a new sector with the new thorn. A suitable rotation $\zeta_{2(m-1)}^k$ brings this separator in a position, where its asymptotic direction is $\pm\pi$. Insertion of a negative real fixed point $r'_{m+1} \ll 0$ now has the following effect: one new separator S' connects r'_{m+1} to ∞ in the direction $\pm\pi$ and the old separator splits into the two having asymptotic directions $\zeta_{2m}^{m\pm1}$; inside of $\Delta(Q)$ this splitting and other changes are merely visible, but outside of $\Delta(Q)$ the separators are bend into the new asymptotic directions. To back up these assertions one computes

$$\chi'_{m+1} \approx \prod_{j=1}^{m} (r'_{m+1} - r_j) \approx (r'_{m+1})^m \ ,$$

which means that r'_{m+1} is repelling for even m and attracting for odd m, just as necessary. In addition

$$\chi'_j \approx \chi_j(r_j - r'_{m+1}) \approx \chi_j(-r'_{m+1}) \quad \text{for } j = 1, \ldots, m,$$

whence the signs of real and imaginary parts of the χ_j remain unchanged. This explains also the choice of a (large) real negative number as the new fixed point, since otherwise hard to control rotational factors are generated, which could easily change the types of the original fixed points.

The topological realization of the insertion of a drop ρ_1' is slightly more involved. First, rotate the system in such a way that the sector where the drop has to be inserted covers the position between $\zeta_{2(m-1)}^{m-2}$ and $\zeta_{2(m-1)}^{m-1} = \zeta_2 = \pm\pi$ in the upper half plane. Second, rotate again by $\zeta_{4(m-1)}$ such that

$$\chi'_j = \zeta_4 \chi_j \quad \text{for } j = 1, \ldots, m;$$

then insert $r'_{m+1} \ll 0$, which yields a rescaling

$$\chi''_j = \chi'_j(r_j - r'_{m+1}) \approx \chi'_j(-r'_{m+1}) \quad \text{for } j = 1, \dots, m$$

as in the case of ρ'_2. In this step the new fixed point r'_{m+1} is approximatively a node, because

$$\chi''_{m+1} = \prod_{j=1}^{m} (r'_{m+1} - r_j) \approx (r'_{m+1})^m \quad \text{implies} \quad \Im(\chi''_{m+1}) \approx 0 \ .$$

In the fourth step rotate the system Q'' by $\zeta^{-1}_{4(m-1)}$ such that

$$\chi'''_j = \zeta_4^{-1} \chi''_j \quad \text{for } j = 1, \dots, m+1$$

and

$$\chi'''_j = \zeta_4^{-1} \zeta_4(-r'_{m+1}) \chi_j \quad \text{for } j = 1, \dots, m.$$

Thus r'_{m+1} is approximatively a centre. In summary: $r = (r_1, \dots, r_m)$ is mapped by ρ'_1 to $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_{m+1})$, where

$$\begin{aligned} \tilde{r}_j &= \zeta^{-1}_{4m} \zeta_{4(m-1)} r_j = \zeta_{4m(m-1)} r_j \quad \text{for } j = 1, \dots, m, \text{ and} \\ \tilde{r}_{m+1} &= \zeta^{-1}_{4m} r_{m+1} \ . \end{aligned}$$

It remains to pertube the new vector field in such a way that $\tilde{r}_{m+1}$ becomes actually a centre and the other fixed points $\tilde{r}_1, \dots, \tilde{r}_m$ have the types of $r_1, \dots, r_m$, respectively. Now the drop is inserted in the right position and the topological type of the vector field inside $\Delta(Q)$ remains unchanged (though rotated by a small amount). Note that in case of a insertion of a thorn a small perturbation of the topological type of the vector field inside $\Delta(Q)$ *did* occur, since the splitted orbit was connected to an already existing fixed point.

The realization of the insertion operation ρ'_3 is now easy: first insert a drop, then a second drop into the first drop (— the first drop becomes a belt —), and finally pertube all fixed points in such a way that the topological type of the original system is preserved, but the newly added centre sectors decay into a sector with two thorns and a pure boundary. $\square$

Remark 5.2. The perturbation approach discussed in the proof of Theorem 5.1 shows that the missing vector field for the diagram with signatur $(1, 3, 1)$ can be realized as a perturbation of one, which has signatur $(0, 5, 0)$, but the question, whether this can be done with rational coordinates awaits further investigation.

Whereas the above theorem solves the realisability problem for diagrams, its perturbative approach does not say how to find an associated vector field $Q(z; r)$ of simplest possible form. What is meant by "simplest possible" is explained in:

Problem 11. *Which diagrams are realisable by vector fields $Q(z; r)$ with all fixed point being Gaussian integers, i.e., all $r_j \in \mathbb{Z}[i]$? If realisable over Gaussian integers, what are the realisations with smallest possible coordinates (absolute values)? Are all centre free diagrams realisable over Gaussian integers, and is it possible to realise them in such a way that all fixed points are contained in a lattice square of length $(m-1)$?*

In Tables 4.8-11 we have exhibited the simplest possible realisations we have been able to find. These findings prompted the conjectures about centre free diagrams, which are stated in Problem 11 above as questions. In particular, it is impossible to realise diagrams with signatur $(0, 5, 0)$ over the Gaussian integers: the crucial equations

$$\Re(\chi_j) = 0 \quad \text{for } j = 1, \ldots, \sigma_0 \leq m \tag{5.1}$$

do not have simultaneous rational solutions $r_1, \ldots, r_m$ for $\sigma_0 = m = 5$, because relations like $\Re(r_j) = (-1 \pm \sqrt{2})\Im(r_j)$ are unavoidable. The investigation of this question leads naturally to

Problem 12. *Investigate the properties of the systems of equations* (5.1) *for the possible choices of σ_0 and m. Especially interesting are the "extreme" cases $\sigma_0 = 0$ and $\sigma_0 = m$.*

For $m = 2$ there are only two possibilities: $\sigma_0 = 0$ or $\sigma_0 = 2$. It is easy to see that $\sigma_0 = 2$ if and only if $\Re(r_1) = \Re(r_2)$. Similarly for $m = 3$ one has $\sigma_0 = 3$ if and only if all three fixed point lay on one line parallel to one of the diagonals of the coordinate system; one centre occurs, if the vector field is a translation of $Q(z; 0, p + iq, \lambda(q + ip))$, where of course all three fixed point are different in accordance to our general hypothesis, and where the case of three centres occurs exactly, when $p = q$. Note that all conditions discussed for $m = 2$ and $m = 3$ have a simple geometric interpretation. Is this more generally the case?

Theorem 5.3. *Let $Q(z; r)$ be a non-degenerate vector field of degree m. Then the fixed point r_j is a centre if and only if the sum of arguments of the remaining fixed points with respect to r_j is an integer multiple of $\pi/2$:*

$$\sum_{\substack{\nu=1 \\ \nu \neq j}}^{m} Arg(r_\nu - r_j) \equiv \pi/2 \pmod{\pi}. \tag{5.2}$$

Proof. For the characteristic number χ_j of r_j one has the following explicit formula :

$$\chi_j = \prod_{\substack{\nu=1 \\ \nu \neq j}}^{m} (r_j - r_\nu) = \sum_{\nu=0}^{m-1} (-1)^\nu r_j^{m-1-\nu} e_\nu^{(m,j)}(r) \quad \text{for } j = 1, \ldots, m, \tag{5.3}$$

where $e_\nu^{(m,j)}(r)$ is the elementary symmetric polynomial homogeneous of degree ν in $m - 1$ variables

$$e_\nu^{(m,j)}(r) = \sum_{\substack{1 \leq i_1 < \cdots < i_\nu \leq \nu \\ i_1, \ldots, i_\nu \neq j}} r_{i_1} \cdot \cdots \cdot r_{i_\nu} \ .$$

Since by translation invariance (Lemma 2.1 (a)) one can assume $r_j = 0$, formula (5.3) simplifies to $\chi_m = (-1)^{m-1} r_1 \cdot \cdots \cdot r_{j-1} r_{j+1} \cdot \cdots \cdot r_m$, whence a necessary and sufficient condition for r_j being a centre is

$$\Re(r_1 \cdot \cdots \cdot r_{j-1} r_{j+1} \cdot \cdots \cdot r_m) = 0 \ .$$

Taking the translation into account this is equivalent to the assertion. $\square$

Corollary 5.4. *The fixed points $r_1, \dots, r_m$ of a non-degenerate vector field $Q(z;r)$ are all centres, if they all lay on a line parallel to one of the directions $\zeta_{4(m-1)}^{2k+1}$.*

Proof. Set $k = 0$ and fix some r_j as the origin. In this case all fixed points r_ν $(\nu \neq j)$ lay in the direction of $\zeta_{4(m-1)}$ or $\pi + \zeta_{4(m-1)}$, whence the sum of their arguments equals $\mathrm{Arg}(\zeta_{4(m-1)}^{m-1}) = \pi/2 \pmod{\pi}$. The general case now follows from the translation and rotation properties (Lemma 2.1 (a,d)) of vector fields $Q(z;r)$. $\square$

For every given m the case of no centres $\sigma_0 = 0$ is the generic case, whereas vector fields with greater σ_0 are increasingly more special. Taking any family $Q(z;r(B))$ of non-degenerate vector fields depending continously on one real parameter B it is clear that a transition between two topologically different types occurs though a "switching" of at least one separator from one fixed point to another fixed point, where at the bifurcation value $B = B_0$ the vector field has greater σ_0.

Definition 5.5. *Let $Q(z;r(B))$ be a family of non-degenerate vector fields depending continously on one real parameter B. If for some value $B = B_0$ there exists exactly one $j \in \{1, \dots, m\}$, such that $\Re(\chi_j(B))$ changes its sign at B_0 and all other signs of $\Re(\chi_\nu(B))$ $(\nu \neq j)$ remain unchanged, then one speaks of a* separator switch bifurcation (SSB) *of the topological type of $Q(z;r(B))$ with the bifurcation value B_0.*

Note that in case of a SSB at B_0 the number of centres $\sigma_0(B_0)$ increases by one (compared to $\sigma_0(B)$ for nearby B different from B_0), if $\sigma_0(B) < m-2$, and it increases by two, if $\sigma_0(B) = m-2$.

Closely related to Problems 8 and 12 is the following

Problem 13. *Describe the "unfoldings" of the topological types $\sigma_0 = m$, i.e., how are the topological types related by* SSB*'s beginning from the most degenerate cases $\sigma_0 = m$? In other words: since it is possible to transform any given non-degenerate vector field $Q(z;r)$ of degree m into any other vector field $Q(z;r')$ of degree m in such a way that only* SSB*'s occur, it is important to describe all possible SSB's between topological types and to find a practical algorithm to determine, which transitions occur in a given bifurcation family.*

Example 5.6. Let $r = (-1+2i, 0, 1-i, 2-2i, 1+(2+B)i)$, where B is a real parameter. Since for $B = -3$ the vector field degenerates ($r_5 = r_4$), we restrict to $B > -3$. Computation of the real parts of the characteristic numbers and their real roots gives:

j	$\Re(\chi_j)$	real roots greater -3
1	$5B - 80$	$B_3 = 16$
2	$-4B$	$B_1 = 0$
3	$4B + 12$	none
4	$-12B - 64$	none
5	$B^4 + 9B^3 + 27B^2 + 23B - 12$	$B_2 = W - 2 + 1/W \approx 0.35530$ with $W = (3 + 2\sqrt{2})^{1/3}$

The bifurcation values B_1, B_2, and B_3 are numbered in increasing order. Figure 5 shows the types of $Q(z; r(B))$ for different $B > -3$, where the inner vertex with label j represents the fixed point r_j. Note that despite the fact that only r_5 moves, it need not be involved in the BBS; this is the case, e.g., at B_3. Note further that the increase of B to $B_1 = 0$ and to $B_3 = 16$ results in rather complicated topological changes involving the three fixed point r_2, r_2, r_5 and r_1, r_2, r_3, respectively, and their sectors. This seems to indicate a useful worst case bound for the complexity of a BBS.

Since it is not difficult to determine (approximatively) the bifurcation values by the method of Example 5.6, the hard part of Problem 13 is to control the changes of topological type in a BBS. (An answer to Problem 5 would be helpful in this respect.)

Finally a general problem remains to be mentioned:

Problem 14. *Develop as much as possible of the theory of Sections* 2 - 5 *for degenerate systems in an explicit way, i.e., without recourse to confluence.*

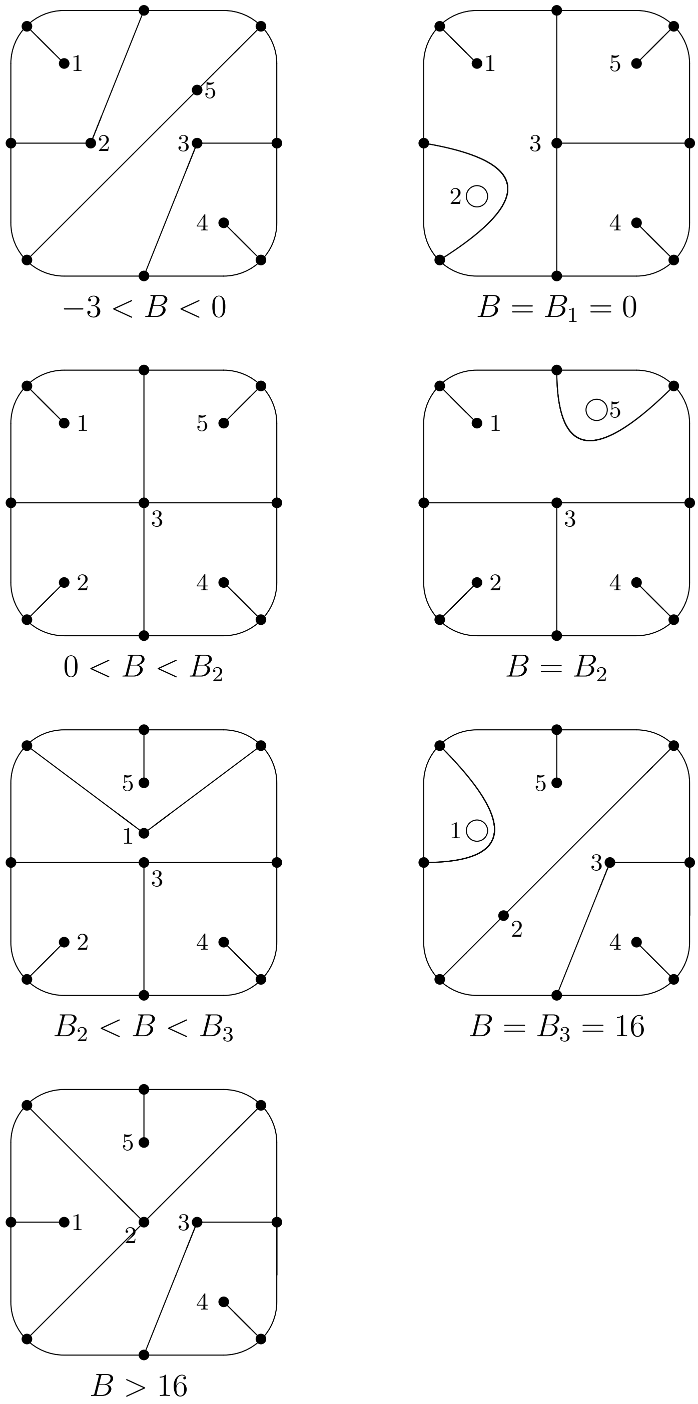


Fig. 5

References

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