GENERALIZED BERNSTEIN POLYNOMIALS AND BÉZIER CURVES: AN APPLICATION OF UMBRAL CALCULUS TO COMPUTER AIDED GEOMETRIC DESIGN

RUDOLF WINKEL

Abstract. The Umbral Calculus is used to generalize Bernstein polynomials and Bézier curves. This adds great geomteric flexibility to these fundamental objects of Computer Aided Geometric Design while retaining their basic properties.

1. Introduction

Bernstein polynomials and Bézier curves are of fundamental importance for Computer Aided Geometric Design (CAGD). They are used for the design of curves and they are the starting point for several generalizations: in particular to higher dimensions and to B-splines. Powerful algorithms are available for both their algebraic construction and their visualization, and their basic theory (explained beautifully in Farin's book [5]) has been examined repeatedly from different new angles: see, e.g., [6,7,8] for the "barycentric" point of view, [15,17,18] for "blossoming", and [3] for the "natural generalization of Bézier curves".

In the present paper we introduce an approach to the generalization of Bernstein polynomials and Bézier curves which seems to be entirely new. It is based on the Umbral Calculus which was first described in its classical form by John Blissard in the 1850's. After a short phase of early success the Umbral Calculus was largely rejected by the mathematics community due to "lack of rigor"; professional jealousy and disdain for its inventor – Blissard was a Land vicar – played an important role, too (cf. [2]). But in the late 1960's the Umbral Calculus was revived, rehabilitated, and put on firm foundations by Gian-Carlo Rota and his co-workers; the basic work is [16], the book [11] gives an extensive and lucid presentation, and a shorter introduction can be found in [12]. The Umbral Calculus allows a unified and algebraically simple treatment of classical polynomials and classical (combinatorial) numbers with respect to generating functions, recursion formulas, characterising differential equations, creation operators, addition theorems, formula for derivatives, reciprocity formulas, expansion theorems, etc. on the basis of viewing formal power series as the "umbra" (latin for: shadow) of linear functionals on polynomials. The Umbral Calculus is thus a mix of linear algebra, the theory of formal power series and classical analysis. An overview with basic definitions and formulas for both Bernstein polynomials and Umbral Calculus will be given in Section 2 and 3.

Date: January 3th, 2001.

What is the connection of the Umbral Calculus with Bernstein polynomials? One of the most basic properties of Bernstein polynomials is that — as a simple consequence of the binomial theorem — Bernstein polynomials of a fixed degree form a partition of unity. On the other hand, Umbral Calculus can be seen as a systematic investigation of all possible "identities of binomial type", whose summands give accordingly all polynomial generalizations of Bernstein polynomials respecting the partition of unity property. This will be elaborated in Section 4.

More precisely, Bernstein polynomials $B_k^n(t)$ are embedded as just one especially simple instance into a multi parameter family of generalized Bernstein polynomials $B_k^n(t;\bar{a})$, where $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ is a sequence of real parameters subject only to a few algebraic restrictions. In fact:

$$B_k^n(t) = B_k^n(t; (1, 0, \ldots, 0)) \ .$$

These parameters $\bar{a}_j$ add a great deal of freedom to the design of generalized Bézier curves while basic features like affine invariance, pseudo local control, and interpolation of first and last control point are (largely) retained. Parameters can be chosen such that the generalized Bézier curves are more *stiff* or more *flexible* than the ordinary Bézier curves; by this we mean that the generalized Bézier curves have the same overall shape as the ordinary Bézier curves, but with less curvature in the stiff case (then the ordinary Bézier curves lay between the generalized Bézier curves and the control polygons) or with more curvature in the flexible case (then the generalized Bézier curves lay between the ordinary Bézier curves and the control polygons). Moreover, the control points can be quasi-interpolated by the generalized Bézier curves or the generalized curves can leave partly or completely the convex hull of the control points. Also the variation diminishing property may get lost for strong deviation from ordinary Bernstein parameters $(1, 0, \ldots, 0)$. But all this might be desirable depending on the application: in design and animation *it is thus possible to deform the appearance of a shape without changing the control structure*. It is clear from the foregoing remarks that our generalized Bézier curves are very different from a generalization relying on weights: our curves may show both a uniform and a non-uniform behaviour in their approximation of the control polygons. All this will be discussed in more detail and with many illustrating figures in Section 5.

Of course the additional freedom is gained at the cost of simplicity: currently, there are no (simple) variants of the de Casteljau and the subdivision algorithms available, and the dependency of the geometric behaviour of the generalized Bézier curves on the parameters $\bar{a}_j$ is not yet clearly understood — though heuristics are available. In Section 5 we will argue that the *simultaneous* change of all parameters with the help of a master control function yields a user friendly set up for further experimentation.

The generalization of Bernstein polynomials and Bézier curves by means of the Umbral Calculus opens up many directions for further research, some of them we hope to cover in future papers:

(1) Extension of generalized Bernstein polynomials and Bézier curves to several dimensions (to generalized Bézier surfaces in particular) by multivariate Umbral Calculus (cf. [13]).
(2) Extension to generalized B-splines.
(3) Exploration of the possibility of a unified theory of approximation and interpolation (cf. Section 5 and [14]).
(4) In the present paper we use only the "associated sequences" of (modern) classical Umbral Calculus. It would be interesting to explore also the possibilities of Appell and Sheffer sequences (cf.[11]), where the property of interpolation of first and last control points for generalized Bézier curves is no longer present.
(5) Other non-classical Umbral Calculi (see [11, Chapter 6]) with non-exponential generating functions and Umbral Calculi for classical special functions (Bessel, Hankel, hypergeometric, Jacobi, Legendre, ...) (cf. [20]) instead of polynomials provide further opportunities for the bold researcher.

Many more problems can be found at the end of Section 4 and scattered throughout Section 5.

2. Basics of Bernstein polynomials and Bézier curves

In this section we collect (mainly for the purpose of reference) some well known definitions and formulas for Bernstein polynomials and Bézier curves (cf. [5]). For every natural number n and the *Bernstein polynomials* are defined by

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k} \text{ for all } k \text{ with } 0 \le k \le n \tag{2.1}$$

and $B_k^n(t) := 0$ otherwise. Clearly, Bernstein polynomials are *symmetric* in the following sense

$$B_k^n(t) = B_{n-k}^n(1-t) \ , \tag{2.2}$$

they satisfy ($\delta_{i,j}$ the Kronecker delta)

$$B_k^n(0) = \delta_{k,0} \quad \text{and} \quad B_k^n(1) = \delta_{k,n} \ , \tag{2.3}$$

and they are nicely bounded on the interval $[0,1]$:

$$0 \le t \le 1 \quad \Longrightarrow \quad 0 \le B_k^n(t) \le 1 \ . \tag{2.4}$$

Since for growing k the order of the zero at 0 grows and the order of the zero at 1 diminishes, the unique maximum of $B_k^n(t)$ (at k/n) shifts from left to right and yields the familiar picture of Figure 1.

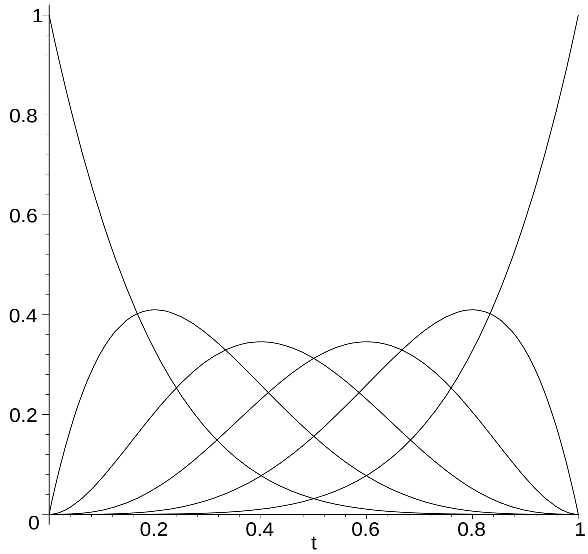


Fig. 1: $B_k^5(t)$ for $k = 0, 1, \ldots, 5$ on $[0, 1]$

Expanding $1 = (t + (1 - t))^n$ by the binomial theorem shows that the Bernstein polynomials of degree n form a *partition of unity*

(2.5)
$$\sum_{k=0}^{n} B_k^n(t) = 1 \; .$$

Simple calculations then yield the *recursion formula*

(2.6)
$$B_k^n(t) = (1 - t)B_k^{n-1}(t) + tB_k^{n-1}(t) \; ,$$

the *formula for the derivative*

(2.7)
$$\frac{d}{dt}B_k^n(t) = n[B_{k-1}^{n-1}(t) - B_k^{n-1}(t)] \; ,$$

the coefficients for the expansion of Bernstein polynomials into powers of t

(2.8)
$$B_k^n(t) = \sum_{j=0}^{n} d_{k,j}^n t^j \quad \Longrightarrow \quad d_{k,j}^n = (-1)^{j-k}\binom{n}{j}\binom{j}{k} \; ,$$

and the representation of the powers of t in terms of Bernstein polynomials

(2.9)
$$t^j = \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_k^n(t) \; .$$

For any sequence of $n + 1$ *control points* or *Bézier points* $\underline{b} = (\underline{b}_0, \ldots, \underline{b}_n)$ in $\mathbb{R}^N$ the associated *Bézier curve* (in N dimensions) is the image of the intervall $[0, 1]$ under the function

(2.10)
$$\underline{x}(t) \equiv \underline{x}(t; \underline{b}) := \sum_{k=0}^{n} \underline{b}_k B_k^n(t) \qquad (0 \leq t \leq 1).$$

The connection of successive control points by straight line segments is called the *control polygon* $C(\underline{b})$.

The symmetry (2.2) of Bernstein polynomials implies the symmetry of Bézier curves: for the graph $\underline{x}([0,1])$ it does not matter whether the control points are labelled in the ordinary or reverse order. (2.3) implies *endpoint interpolation*

$$\underline{x}(0;\underline{b}) = \underline{b}_0 \quad \text{and} \quad \underline{x}(1;\underline{b}) = \underline{b}_n. \tag{2.11}$$

(2.4) implies the *convex hull property*

$$\underline{x}([0,1]) \subset \operatorname{conv}\{\underline{b}_0, \ldots, \underline{b}_n\} . \tag{2.12}$$

(2.5) implies the affine invariance of Bézier curves (i.e., the Bézier curve for affinely transformed control points is identical to the directly transformed Bézier curve) from which it follows that a straight control polygon yields a straight Bézier curve. The unimodality of Bernstein polynomials implies the *pseudo-local control* (i.e., if only one control point is moved, then the dominant change of the associated Bézier curve occurs in the vicinity of this point).

From (2.7) on computes the r-th derivative of a Bézier curve as

$$\underline{x}^{(r)}(t) := \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} \Delta^r \underline{b}_k B_k^{n-r}(t) , \tag{2.13}$$

where Δ is the forward difference operator $\Delta \underline{b}_k = \underline{b}_{k+1} - \underline{b}_k$. (2.13) then shows the *endpoint tangentiality* of Bézier curves and also implies a nice criterion for C^r-connecting two Bézier curves.

Finally, the simple recursion formula (2.6) yields the famous *de Casteljau algorithm* which can be used to evaluate a Bézier curve at some point $t \in [0,1]$ or for the *subdivision* of the control polygon. The latter is a fast method to actually draw a Bézier curve.

3. Basics of Umbral Calculus

Let $f \in \mathbb{R}[[s]]$ be a formal power series (i.e., convergence is not an issue) in exponential form

$$f(s) = \sum_{k=1}^{\infty} \frac{a_k}{k!} s^k . \tag{3.1}$$

If in addition f is a δ*-series*, i.e., $a_1 \neq 0$ and $a_0 = 0$, then f has a compositional inverse

$$\bar{f}(s) = \sum_{k=1}^{\infty} \frac{\bar{a}_k}{k!} s^k \qquad (\bar{a}_1 = 1/a_1) \tag{3.2}$$

such that $f(\bar{f}(s)) = \bar{f}(f(s)) = s$. ($f \in \mathbb{R}[[s]]$ has a multiplicative inverse iff $f(0) \neq 0$.)
Now, every formal power series $f \in \mathbb{R}[[s]]$ can be viewed in three different ways:
first as an *element of the ring* $R[[s]]$;
second as a *linear functional* on $\mathbb{R}[x]$ defined by

$$\langle f(s) | \, \_ \, \rangle : \mathbb{R}[x] \longrightarrow \mathbb{R} , \quad x^n \mapsto a_n \quad (\forall n \geq 0) ; \tag{3.3}$$

and third as a *linear operator* on $\mathbb{R}[x]$ defined by

$$f : \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \ , \quad s^k p(x) \mapsto p^{(k)}(x) \quad (\forall k \geq 0) \ , \tag{3.4}$$

where $p^{(k)}$ is the k-th derivative of p. The interplay of these three perspectives is the reason for the richness and effectiveness of the Umbral Calculus — and the reason for its historical rejection as "non-rigorous", a prejudice, which could not be corrected until Rota clarified the foundations of the Umbral Calculus.

The basic definition and result of Umbral Calculus is the following (for proofs and more details on all assertions in this section see [11]):

Theorem 3.1. *For every δ-series $f \in \mathbb{R}[[s]]$ there exist a uniquely determined sequence of* associated polynomials $p_n \in \mathbb{R}[x]$ *with* $\deg(p_n) = n \quad (\forall n \geq 0)$ *such that*

$$\langle f(s)^k | \ p_n(x) \rangle = n! \ \delta_{k,n} \quad (\forall n, k \geq 0) \ . \tag{3.5}$$

For a δ-series f and its sequence of associated polynomials many useful formulas can be shown. We cite a few simple ones that are used subsequently:

$$e^{x \ \bar{f}(s)} = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} s^n \qquad \text{(generating function);} \tag{3.6}$$

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \qquad \text{(binomial formula);} \tag{3.7}$$

$$p_n'(x) = \sum_{k=0}^{n-1} \binom{n}{k} \langle s | \ p_{n-k}(x) \rangle \ p_k(x) \qquad \text{(derivative);} \tag{3.8}$$

$$x \, p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \langle f'(\bar{f}(s)) | \ x^{n-k} \rangle \ p_{k+1}(x) \ ; \tag{3.9}$$

$$p_n(x) = \sum_{k=1}^{n} \frac{1}{k!} \langle \bar{f}(s)^k | \ x^n \rangle \ x^k \qquad \text{(conjugate representation).} \tag{3.10}$$

Note that always $p_0(x) = 1$ and $p_n(0) = \delta_{n,0}$.

As a *special* example we mention that for the δ-series $f(s) = s = \bar{f}(s)$ the associated sequence is $p_n(x) = x^n$ which yields the well known specializations of the above formulas. As a *general* example we discuss the general δ-series (3.1). In this case the *universal* associated sequence $p_n(x)$ has as its coefficients the *Bell polynomials* [C,R1]: with the notation

$$p_n(x) = \sum_{k=1}^{n} p_{n,k} x^k \tag{3.11}$$

one calculates from the generating function (3.6) or the conjugate representation (3.10)

$$p_{n,k} \equiv p_{n,k}(\bar{a}) = \frac{1}{k!} \sum_{\substack{i_1 + \cdots + i_k = n \\ i_1, \ldots, i_k > 0}} \binom{n}{i_1, \ldots, i_k} \bar{a}_{i_1} \cdot \cdots \cdot \bar{a}_{i_k} \ . \tag{3.12}$$

In particular: $p_{n,1} = \bar{a}_n$, $p_{n,n} = \bar{a}_1^n$, and $p_{n,n-1} = \binom{n}{2}\bar{a}_1^{n-2}\bar{a}_2$. For practical calculations the recursion for Bell polynomials

$$p_{n,k+1} = \frac{1}{k+1}\sum_{j=1}^{n-k}\binom{n}{j}\bar{a}_j p_{n-j,k} \tag{3.13}$$

is better suited.

Let us take a closer look at the binomial formula (3.7), since this is our junction point between Umbral Calculus on one hand and Bernstein polynomials on the other. An alternative approach to associated sequences is to say that a sequence of polynomials $p_n \in \mathbb{R}[x]$ with $deg\ (p_n) = n \quad (\forall n)$ is a *binomial sequence*, if it obeys (3.7). It can be shown that every binomial sequence is in fact the associated sequence of a uniquely determined δ-series f. In other words: the *Scheffer operator* λ_f defined as the linear extension of

$$\lambda_f : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]\ , \quad x^n \mapsto p_n(x) \quad (\forall n)\ , \tag{3.14}$$

is not only an isomorphism of $\mathbb{R}[x]$ as a vector space which maps the monomial basis into the new basis $\{p_n\}$, but it also preserves the additional structure given by the binomial identity. Again it was Rota, who recognised that this additional structure is best understood in the framework of coalgebras and bialgebras [10]. More precisely (see [1,19] for the general theory of co- and bialgebras), for any field K the polynomial ring $K[x]$ is a K-*algebra* with (associative) *multiplication* μ defined by

$$\mu : K[x] \otimes K[x] \longrightarrow K[x]\ , \quad p(x) \otimes q(x) \mapsto p(x)q(x) \quad (\forall p, q \in K[x])\ ,$$

and *unit* η mapping the unit 1_K of K to the unit $1_{K[x]}$ of $K[x]$; in addition — and this is the crucial structural information — $K[x]$ is a K-*coalgebra* with (coassociative) *comultiplication* Δ defined by

$$\Delta : K[x] \longrightarrow K[x] \otimes K[x]\ , \quad x^n \mapsto \sum_{k=0}^{n}\binom{n}{k} x^k \otimes x^{n-k} \quad (\forall n \in \mathbb{N})\ ,$$

and *counit* ε mapping x^n to $\delta_{n,0}1_K$. We remark that for the definition of Δ it is only necessary to set $\Delta(s) := x \otimes 1 + 1 \otimes x$ which implies the above "binomial identity" by coassociativity. Moreover, $(K[x], \mu, \eta, \Delta, \varepsilon)$ is a K-*bialgebra* because Δ and ε are algebra homomorphism. With the isomorphism

$$K[x, y] \cong K[x] \otimes K[x]\ , \quad x \mapsto x \otimes 1\ , \quad y \mapsto 1 \otimes x$$

the binomial identity (3.7) reads

$$p_n(x \otimes 1 + 1 \otimes x) = \sum_{k=0}^{n}\binom{n}{k} p_k(x) \otimes p_{n-k}(x)\ ,$$

which is the last notation needed for the following fundamental

Theorem 3.2. ([10]) *Let $p_1, p_2, p_3, \ldots$ be a sequence of polynomials in $K[x]$ with $deg\ (p_n) = n \quad (\forall n)$ with associated Scheffer operator λ_f. Then λ_f is a coalgebra automorphism of $K[x]$ iff (p_n) is binomial.*

Proof. Since λ_f is obviously an automorphism of the vector space $K[x]$, it is only necessary to check that λ_f is a coalgebra homomorphism, i.e.,

$$\Delta \circ \lambda_f = (\lambda_f \otimes \lambda_f) \circ \Delta$$

iff (p_n) is binomial. But this follows from the following calculations where $(*)$ is valid because Δ is an algebra homomorphism

$$\Delta \circ \lambda_f(x^n) = \Delta(p_n(x)) \stackrel{(*)}{=} p_n(\Delta x) = p_n(x \otimes 1 + 1 \otimes x) \quad \text{and}$$

$$(\lambda_f \otimes \lambda_f) \circ \Delta(x^n) = (\lambda_f \otimes \lambda_f)(\sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) \otimes p_{n-k}(x) \ .$$

□

4. Generalized Bernstein polynomials

Definition 4.1. *For a sequence* $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ *of real numbers with* $\bar{a}_1 \neq 0$ *let* $p_1, \ldots, p_n \in \mathbb{R}[x]$ *be calculated according to* (3.11-13)*. Then for* $k = 0, \ldots, n$ *the polynomials (recall* $p_0 = 1$*)*

$$B_k^n(t; \bar{a}) := \frac{1}{\rho_n} \binom{n}{k} p_k(t) p_{n-k}(1-t) \tag{4.1}$$

are called the generalized Bernstein polynomials of degree n *associated to* $\bar{a}$*, if* $\rho_n := p_n(1) \neq 0$*, i.e., if* 1 *is not a zero of* $p_n(x)$*.*

It is clear that the sequence $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ in Definition 4.1 is the beginning of a sequence of coefficients of a δ-series $\bar{f}$ and that the sequence $p_0, p_1, \ldots, p_n$ is the beginning of a sequence of associated polynomials for f. Therefore the formulas and results from Section 3 can be applied. In particular, Theorem 3.2 says that *all* sequences of polynomials $p_0, p_1, \ldots, p_n$ with deg $(p_k) = k$ which generate generalizations of Bernstein polynomials of the form (4.1) are formed according to the procedure of Definition 4.1

Theorem 4.2. *The generalized Bernstein polynomials of degree* n *associated to* $\bar{a}$ *(from Definition* 4.1*) form a partition of unity*

$$\sum_{k=0}^{n} B_k^n(t; \bar{a}) = 1 \ , \tag{4.2}$$

they are symmetric

$$B_k^n(t; \bar{a}) = B_{n-k}^n(1-t; \bar{a}) \ , \tag{4.3}$$

and they fulfil

$$B_k^n(0; \bar{a}) = \delta_{k,0} \quad \textit{and} \quad B_k^n(1; \bar{a}) = \delta_{k,n} \ . \tag{4.4}$$

Moreover,

$$0 \leq B_k^n(t; \bar{a}) \leq 1 \ \ \textit{for } t \in [0,1], \ \ \textit{if } \bar{a}_j \geq 0 \textit{ for all } j \in \{1, \ldots, n\}. \tag{4.5}$$

Proof. For (4.2) set $x = t$ and $y = 1-t$ in the binomial formula (3.7); (4.3) is immediate from the definition and (4.4) follows from $p_0(x) = 1$ and $p_n(0) = \delta_{n,0}$. For (4.5) one observes that under the assumption $\bar{a}_j \geq 0$ all coefficients of the $p_0, p_1, \ldots, p_n$ are non-negative, whence $p_j(t) \geq 0$ for all $t \in [0,1]$. This proves $0 \leq B_k^n(t;\bar{a})$. But then the other inequality $B_k^n(t;\bar{a}) \leq 1$ follows from (4.2). □

The reverse of (4.5) is not true: to have $0 \leq B_k^n(t;\bar{a}) \leq 1$ on $[0,1]$ it is sufficient that all $p_j(t)$ have non-negative values there; of course it is possible to find parameters $\bar{a}$ with some $\bar{a}_j < 0$, such that this sufficient condition is fulfilled (see Figure 3 and the discussion there). We will derive next a recursion formula for generalized Bernstein polynomials which unfortunately is rather complicated. Therefore it is not possible to derive a de Casteljau-type algorithm from it.

Theorem 4.3. *For generalized Bernstein polynomials associated to the parameter sequence $\bar{a}$ resp. to the δ-series f one has the following recursion formula*

$$\text{(4.6)}\quad B_k^n(t;\bar{a}) = \frac{\bar{a}_1 \rho_{n-1}}{\rho_n}\left[(1-t)B_k^{n-1}(t;\bar{a}) + tB_{k-1}^{n-1}(t;\bar{a})\right] - \frac{\bar{a}_1}{n\rho_n}\left[\sum_{i=1}^{k-1} c_i \rho_{n-i}(k-i)\binom{n}{i} B_{k-i}^{n-i}(t;\bar{a}) + \sum_{i=k+1}^{n-1} c_{n-i}\rho_i(i-k)\binom{n}{i} B_{i-k}^{i}(t;\bar{a})\right],$$

where the numbers c_j are the expansion coefficients of $f'(\bar{f})$:

$$\text{(4.7)}\qquad f'(\bar{f}(s)) = \sum_{j=0}^{\infty} \frac{c_j}{j!} s^j .$$

(The c_j can therefore be calculated from both the a_j and $\bar{a}_j$ or from the $\bar{a}_j$ alone by virtue of $f'(\bar{f}(s)) = (\bar{f}'(s))^{-1}$.)

Proof. From formula (3.9) and the definition (3.3) it follows that

$$xp_n(x) = \sum_{k=0}^{n-1}\binom{n}{k} c_{n-k} p_{k+1}(x) + c_0 p_{n+1}(x) \quad \Longrightarrow \quad (n+1 \mapsto n)$$

$$p_n(x) = \bar{a}_1 x p_{n-1}(x) - \bar{a}_1 \sum_{k=1}^{n-1}\binom{n-1}{k-1} c_{n-k} p_k(x) ,$$

since $c_0 = a_1 = (\bar{a}_1)^{-1}$. We use the latter formula to expand the two summands of

$$B_k^n(t;\bar{a}) = \frac{1}{\rho_n}\binom{n-1}{k} p_k(t)p_{n-k}(1-t) + \frac{1}{\rho_n}\binom{n-1}{k-1} p_k(t)p_{n-k}(1-t) .$$

The first summand gives

$$\frac{\bar{a}_1}{\rho_n}\binom{n-1}{k} p_k(t)\left[(1-t)p_{n-1-k}(1-t) - \sum_{i=1}^{n-k-1}\binom{n-k-1}{i-1} c_{n-k-i} p_i(1-t)\right] =$$

$$\frac{\bar{a}_1 \rho_{n-1}}{\rho_n}(1-t)B_k^{n-1}(t;\bar{a}) - \frac{\bar{a}_1}{\rho_n}\binom{n-1}{k}\sum_{i=1}^{n-k-1}\binom{n-k-1}{i-1} c_{n-k-i} p_k(t) p_i(1-t) ,$$

where

$$p_k(t)p_i(1-t) = \frac{\rho_{k+i}}{\binom{k+i}{i}} B_i^{k+i}(t;\bar{a}) \ .$$

Hence the subtracted sum equals

$$\sum_{i=1}^{n-k-1} \frac{\bar{a}_1 \rho_{k+i} c_{n-k-i}}{\rho_n} \cdot \frac{i}{n} \binom{n}{k+i} B_i^{k+i}(t;\bar{a}) = \sum_{i=k+1}^{n-1} \frac{\bar{a}_1 \rho_i c_{n-i}}{\rho_n} \cdot \frac{i-k}{n} \binom{n}{i} B_{i-k}^{i}(t;\bar{a}) \ .$$

Similarly the second summand equals

$$\frac{\bar{a}_1}{\rho_n} \binom{n-1}{k-1} \left[t p_{k-1}(t) - \sum_{i=1}^{k-1} \binom{k-1}{i-1} c_{k-i} p_i(t) \right] p_{n-k}(1-t) =$$

$$\frac{\bar{a}_1 \rho_{n-1}}{\rho_n} t B_{k-1}^{n-1}(t;\bar{a}) - \frac{\bar{a}_1}{\rho_n} \binom{n-1}{k-1} \sum_{i=1}^{k-1} \binom{k-1}{i-1} c_{k-i} p_i(t) p_{n-k}(1-t) \ ,$$

where

$$p_i(t)p_{n-k}(1-t) = \frac{\rho_{n-k+i}}{\binom{n-k+i}{i}} B_i^{n-k+i}(t;\bar{a}) \ .$$

Hence the subtracted sum equals

$$\sum_{i=1}^{k-1} \frac{\bar{a}_1 \rho_{n-k+i} c_{k-i}}{\rho_n} \cdot \frac{i}{n} \binom{n}{k-i} B_i^{n-k+i}(t;\bar{a}) = \sum_{i=1}^{k-1} \frac{\bar{a}_1 \rho_{n-i} c_i}{\rho_n} \cdot \frac{k-i}{n} \binom{n}{i} B_{k-i}^{n-i}(t;\bar{a}) \ .$$

□

Since for $\bar{a} = (1, 0, \ldots, 0)$ one has $f(s) = s = \bar{f}(s)$, $c_j = \delta_{j,0}$, and $\rho_n = 1$ for all n, the general recursion formula (4.6) reduces to (2.6) in case of ordinary Bernstein polynomials.

Theorem 4.4. *For generalized Bernstein polynomials associated to the parameter sequence $\bar{a}$ one has the formula for the derivative*

(4.8)

$$\frac{d}{dt} B_k^n(t;\bar{a}) = \sum_{j=n-k}^{n-1} \alpha_{n,j} B_{n-k}^{j}(1-t;\bar{a}) - \sum_{j=k}^{n-1} \alpha_{n,j} B_k^j(t;\bar{a}) \ , \ \textit{where } \alpha_{n,j} := \bar{a}_{n-j} \frac{\rho_j}{\rho_n} \binom{n}{j} \ .$$

Proof. The identities

$$\langle s | \ p_{n-k}(x) \rangle \overset{(3.3)}{=} p_{n-k,1} = \bar{a}_{n-k}$$

allow us to rewrite (3.8) as

$$p_n'(x) = \sum_{k=0}^{n-1} \binom{n}{k} \bar{a}_{n-k} \ p_k(x) \ . \qquad (4.9)$$

Using the latter, one computes

$$
\begin{aligned}
\frac{d}{dt}B_k^n(t;\bar a) &\overset{(4.1)}{=} \frac{1}{\rho_n}\binom{n}{k}\,[\dot p_k(t)p_{n-k}(1-t)+p_k(t)\dot p_{n-k}(1-t)]\\
&\overset{(4.9)}{=} \frac{1}{\rho_n}\binom{n}{k}\left[\sum_{j=0}^{k-1}\binom{k}{j}\bar a_{k-j}p_j(t)p_{n-k}(1-t)\right.\\
&\qquad\qquad\left.-\sum_{j=0}^{n-k-1}\binom{n-k}{j}\bar a_{n-k-j}p_k(t)p_j(1-t)\right]\\
&= \frac{1}{\rho_n}\binom{n}{k}\left[\sum_{j=0}^{k-1}\binom{k}{j}\bar a_{k-j}\frac{\rho_{n-k+j}}{\binom{n-k+j}{j}}B_k^{n-k+j}(t;\bar a)\right.\\
&\qquad\qquad\left.-\sum_{j=0}^{n-k-1}\binom{n-k}{j}\bar a_{n-k-j}\frac{\rho_{k+j}}{\binom{k+j}{k}}B_k^{k+j}(t;\bar a)\right]\\
&= \sum_{j=0}^{k-1}\bar a_{k-j}\frac{\rho_{n-k+j}}{\rho_n}\binom{n}{k-j}B_k^{n-k+j}(t;\bar a)-\sum_{j=0}^{n-k-1}\bar a_{n-k-j}\frac{\rho_{k+j}}{\rho_n}\binom{n}{k+j}B_k^{k+j}(t;\bar a)\\
&= \sum_{j=1}^{k}\bar a_j\frac{\rho_{n-j}}{\rho_n}\binom{n}{j}B_{k-j}^{n-j}(t;\bar a)-\sum_{j=k}^{n-1}\bar a_{n-j}\frac{\rho_j}{\rho_n}\binom{n}{j}B_k^j(t;\bar a)\\
&= \sum_{j=n-k}^{n-1}\bar a_{n-j}\frac{\rho_j}{\rho_n}\binom{n}{j}B_{j-n+k}^j(t;\bar a)-\sum_{j=k}^{n-1}\bar a_{n-j}\frac{\rho_j}{\rho_n}\binom{n}{j}B_k^j(t;\bar a)\\
&\overset{(4.3)}{=} \sum_{j=n-k}^{n-1}\alpha_{n,j}B_{n+k}^j(1-t;\bar a)-\sum_{j=k}^{n-1}\alpha_{n,j}B_k^j(t;\bar a)\ .
\end{aligned}
$$

□

Again, for $\bar a=(1,0,\dots,0)$ one sees $\alpha_{n,j}=n\delta_{n-1,j}$, so the general formula for the derivative (4.8) reduces to (2.7) in case of ordinary Bernstein polynomials.

Theorem 4.5. *For generalized Bernstein polynomials associated to the parameter sequence $\bar a$ one has (for $n\ge 2$)*

(4.10)

$$
B_k^n(t;\bar a)=\sum_{j=0}^{n}d_{k,j}^n(\bar a)\,t^j \implies d_{k,j}^n(\bar a)=\frac{1}{\rho_n}\binom{n}{k}\sum_{i=0}^{\min(j,k)}(-1)^{j-i}p_{k,i}\sum_{h=j-i}^{n-k}p_{n-k,h}\binom{h}{j-i}\ .
$$

Proof. One computes

$$
\begin{aligned}
B_k^n(t;\bar a) &\overset{(4.1)}{=} \frac{1}{\rho_n}\binom{n}{k}p_k(t)p_{n-k}(1-t) \overset{(3.11)}{=} \frac{1}{\rho_n}\binom{n}{k}(\sum_{i=1}^{k}p_{k,i}t^i)\ (\sum_{j=1}^{n-k}\rho_{n-k,j}(1-t)^j)\\
&= \frac{1}{\rho_n}\binom{n}{k}(\sum_{i=1}^{k}p_{k,i}t^i)\ (\sum_{j=1}^{n-k}\bar p_{n-k,j}t^j) \text{ with } \bar p_{n-k,j}=(-1)^j\sum_{h=j}^{n-k}\binom{h}{j}p_{n-k,h}\ ,
\end{aligned}
$$

which implies

$$d^n_{k,j}(\bar a) = \frac{1}{\rho_n}\binom{n}{k}\sum_{i=1}^{j} p_{k,i}\,\bar p_{n-k,j-i} = \frac{1}{\rho_n}\binom{n}{k}\sum_{i=0}^{j}(-1)^{j-i}p_{k,i}\sum_{h=j-i}^{n-k} p_{n-k,h}\binom{h}{j-i}\,.$$

Formula (4.10) now follows from $p_{k,i} = 0$ for $i > k$. □

In case of ordinary Bernstein polynomials $\bar a = (1, 0, \ldots, 0)$ the above formula (4.10) reduces to (2.8) by virtue of $p_{n,k} = \delta_{n,k}$.

The explicit matrices of coefficients $\rho_n d^n_{k,j}(\bar a)$ for $n = 1, \ldots, 4$ and arbitrary $\bar a$ are given below, where in case of $n = 4$ the first column $(p_4, 0, 0, 0, 0)^T$ and the last column $(\bar a_1^4, -4\bar a_1^4, 6\bar a_1^4, -4\bar a_1^4, \bar a_1^4)^T$ are omitted for reasons of space:

$$\begin{pmatrix} \bar a_1 & -\bar a_1 \\ 0 & \bar a_1 \end{pmatrix}$$

$$\begin{pmatrix} \rho_2 & -\bar a_2 - 2\bar a_1^2 & \bar a_1^2 \\ 0 & 2\bar a_1^2 & -2\bar a_1^2 \\ 0 & \bar a_2 & \bar a_1^2 \end{pmatrix}$$

$$\begin{pmatrix} \rho_3 & -\bar a_3 - 6\bar a_2\bar a_1 - 3\bar a_1^3 & 3\bar a_2\bar a_1 + 3\bar a_1^3 & -\bar a_1^3 \\ 0 & 3\bar a_2\bar a_1 + 3\bar a_1^3 & -3\bar a_2\bar a_1 - 6\bar a_1^3 & 3\bar a_1^3 \\ 0 & 3\bar a_2\bar a_1 & 3\bar a_1^3 - 3\bar a_2\bar a_1 & -3\bar a_1^3 \\ 0 & \bar a_3 & 3\bar a_2\bar a_1 & \bar a_1^3 \end{pmatrix}$$

$$\begin{pmatrix} -\bar a_4 - 8\bar a_3\bar a_1 - 6\bar a_2^2 - 18\bar a_1^2\bar a_2 - 4\bar a_1^4 & 4\bar a_3\bar a_1 + 3\bar a_2^2 + 18\bar a_1^2\bar a_2 + 6\bar a_1^4 & -6\bar a_1^2\bar a_2 - 4\bar a_1^4 \\ 4\bar a_3\bar a_1 + 12\bar a_1^2\bar a_2 + 4\bar a_1^4 & -4\bar a_3\bar a_1 - 24\bar a_1^2\bar a_2 - 12\bar a_1^4 & 12\bar a_1^2\bar a_2 + 12\bar a_1^4 \\ 6\bar a_2^2 + 6\bar a_1^2\bar a_2 & 6\bar a_1^4 - 6\bar a_1^2\bar a_2 - 6\bar a_2^2 & -12\bar a_1^4 \\ 4\bar a_3\bar a_1 & 12\bar a_1^2\bar a_2 - 4\bar a_3\bar a_1 & 4\bar a_1^4 - 12\bar a_1^2\bar a_2 \\ \bar a_4 & 4\bar a_3\bar a_1 + 3\bar a_2^2 & 6\bar a_1^2\bar a_2 \end{pmatrix}$$

From the representation of the powers of t in terms of Bernstein polynomials (2.9) it is immediate that the Bernstein polynomials of a fixed degree n form a basis of the vector space $\mathbb{R}_{\le n}[t]$ of all polynomials with degree not greater than n. Since the generalized Bernstein polynomials of degree n form a basis for $\mathbb{R}_{\le n}[t]$ exactly when $\det(d^n_{k,j}(\bar a)) \ne 0$, one concludes that the property to be a basis is *generic*, i.e., it may fail only for parameters $\bar a$ on the set of zeros of a certain polynomial Δ_n in the $\bar a_1, \ldots, \bar a_n$.

Calculations with MAPLE reveal that the first few of these *critical polynomials* are

$$
\begin{aligned}
\Delta_1 =& 1\\
\Delta_2 =& 1\\
\Delta_3 =& (2\bar{a}_2 + \bar{a}_1^2)\\
\Delta_4 =& (3\bar{a}_2 + \bar{a}_1^2)\,(\bar{a}_1^4 + 5\bar{a}_1^2\bar{a}_2 + 2\bar{a}_3\bar{a}_1 + 2\bar{a}_2^2)\\
\Delta_5 =& (4\bar{a}_2 + \bar{a}_1^2)(\bar{a}_1^4 + 7\bar{a}_1^2\bar{a}_2 + 3\bar{a}_3\bar{a}_1 + 6\bar{a}_2^2)\cdot\\
& \qquad (\bar{a}_1^5 + 9\bar{a}_1^3\bar{a}_2 + 7\bar{a}_1^2\bar{a}_3 + 12\bar{a}_2^2\bar{a}_1 + 2\bar{a}_4\bar{a}_1 + 6\bar{a}_3\bar{a}_2)\\
\Delta_6 =& (5\bar{a}_2 + \bar{a}_1^2)(\bar{a}_1^4 + 9\bar{a}_1^2\bar{a}_2 + 4\bar{a}_3\bar{a}_1 + 12\bar{a}_2^2)\cdot\\
& \qquad (\bar{a}_1^6 + 12\bar{a}_1^4\bar{a}_2 + 10\bar{a}_1^3\bar{a}_3 + 27\bar{a}_1^2\bar{a}_2^2 + 3\bar{a}_1^2\bar{a}_4 + 18\bar{a}_1\bar{a}_3\bar{a}_2 + 6\bar{a}_2^3)\cdot\\
& (\bar{a}_1^6 + 14\bar{a}_1^4\bar{a}_2 + 16\bar{a}_1^3\bar{a}_3 + 39\bar{a}_1^2\bar{a}_2^2 + 9\bar{a}_1^2\bar{a}_4 + 44\bar{a}_1\bar{a}_3\bar{a}_2 + 2\bar{a}_5\bar{a}_1 + 6\bar{a}_3^2 + 8\bar{a}_4\bar{a}_2 + 12\bar{a}_2^3)
\end{aligned}
$$

where we have neglected factors $\rho_1, \dots, \rho_n$, since they are non-zero by definition.

Finally in this section we mention some open problems concerning generalized Bernstein polynomials:

- Is there an explicit formula for the critical polynomials Δ_n?
- Is there a "geometrical" meaning of the critical polynomials Δ_n in terms of the graphs of the $B_k^n(t; \bar{a})$? For the *existence* of generalized Bézier curves (introduced in the next section) the basis property is not necessary, but does it affect somehow the possible shapes of generalized Bernstein polynomials for fixed control points and varying $\bar{a}$?
- Investigate the expansion of generalized Bernstein polynomials in terms of ordinary Bernstein polynomials instead of powers of t. Then the matrix of coefficients is "pointsymmetric". This may lead to a better understanding of the critical polynomials, too.
- For generalized Bernstein polynomials which form a basis of some $\mathbb{R}_{\leq n}[t]$, investigate the *dual basis* and their properties following the lines of [9].

5. Generalized Bézier curves

Definition 5.1. *For generalized Bernstein polynomials as in Definition* 4.1 *and arbitrary* control points $\underline{b} = (\underline{b}_0, \dots, \underline{b}_n)$ *in* $\mathbb{R}^N$ *the associated* generalized Bézier curve *(in* N *dimensions) is defined as*

$$
\underline{x}(t; \bar{a}) \equiv \underline{x}(t; \bar{a}; \underline{b}) := \sum_{k=0}^{n} \underline{b}_k B_k^n(t; \bar{a}) \qquad (0 \leq t \leq 1). \tag{5.1}
$$

Directly from Theorem 4.2 it follows that

Corollary 5.2. *The generalized Bézier curves formed according to Definition* 5.1 *have the properties of affine invariance, symmetry, and endpoint interpolation. In addition the generalized Bézier curves for a straight control polygon is again straight.*

Subsequently in this section we will present many examples to illustrate the power of the new approach. The principal problem calling for further research is to understand

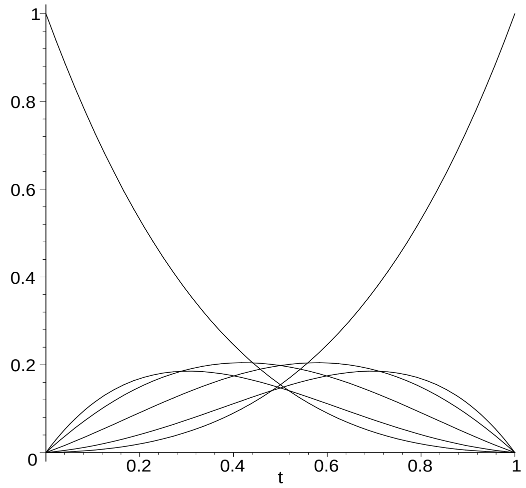
(1) the dependency of the qualitative-geometric features of the generalized Bernstein polynomials on the choice of parameters $\bar{a}$, and
(2) the dependency of the qualitative-geometric features of generalized Bézier curves on the features of their building generalized Bernstein polynomials.

Many special instances will be discussed below. When we speak of an *observation*, we tacitly assume that the reader interprets this as a *problem* in the above sense. We will use the notations $\underline{x}(t)$ for (ordinary) Bézier curves, $\underline{x}(t;\bar{a})$ for generalized Bézier curves, $C(\underline{b})$ for the control polygon associated to the $(n+1)$-tuple $\underline{b}$ of control points, and C_n for the special control polygons which arise from the "n-gonal" control points

$$\underline{b}_k := (\cos(2\pi k/n), \sin(2\pi k/n)) \quad (k = 0, 1, \ldots, n-1) .$$

We have chosen to use C_n as our standard control polygon for all generalized Bézier curves because it allows for a meaningful comparison of the resulting curves even in case of different n. In all subsequent figures involving C_n the curve $\underline{x}(t;\bar{a})$ is depicted in solid style, whereas $\underline{x}(t)$ is dotted.

From (4.5) we know that $\underline{x}(t;\bar{a}) \subset \mathrm{conv}(C(\underline{b}))$ for all choices of $\underline{b}$, if all $\bar{a}_j \geq 0$ (and of course $\bar{a}_1 \neq 0$). A natural example is given in Figure 2.



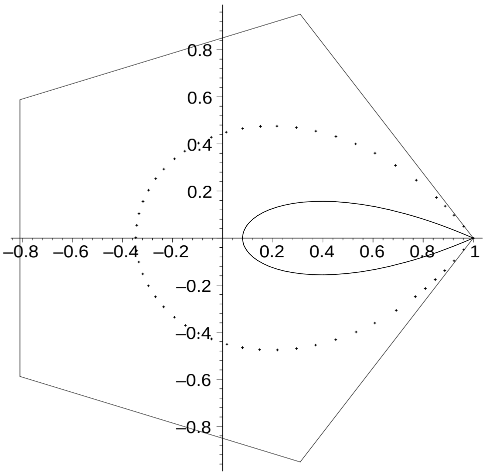
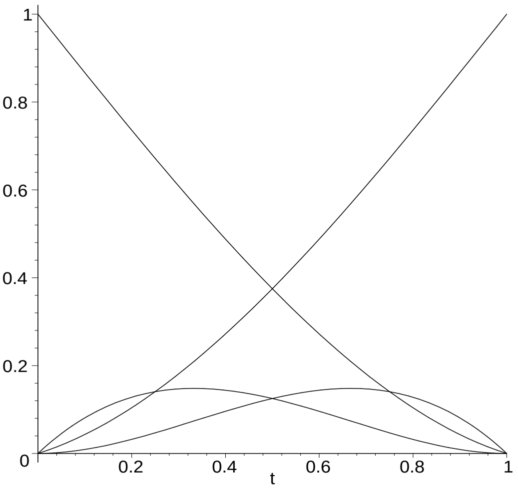


Fig. 2: $\bar{a} = (1,1,1,1,1)$: $B_k^5(t;\bar{a})$ $(k = 0,\ldots,5)$ and $\underline{x}(t;\bar{a})$

One observes that as in the case of ordinary Bernstein polynomials (cf. Figure 1) all $B_k^5(t;\bar{a})$ have unimodal graphs with the unique maximum wandering from left to right for increasing k; but the graphs are less sharply peaked and the maxima are smaller. Accordingly $\underline{x}(t;\bar{a})$ follows the control polygon in a more stiff fashion than $\underline{x}(t)$, and hence $\underline{x}(t)$ lies between $\underline{x}(t;\bar{a})$ and C_5. Examination of many other generalized Bernstein polynomials and generalized Bézier curves with non-negative parameters $\bar{a}_j$ show the same behaviour. We conjecture therefore that $\underline{x}(t)$ is the "best approximation" of any given control polygon $C(\underline{b})$ among all $\underline{x}(t;\bar{a})$ with $\bar{a}_j \geq 0$.

Note that $\underline{x}(t;(1,1,1,1,1))$ does not exhibit endpoint tangentiality for C_5. Most generalized Bézier curves do not do this due to the appearance of the lower order generalized Bernstein polynomials in formula (4.8) for the derivative.

As stated already in Section 4, it can happen that some parameters $\bar{a}_j$ are negative without sacrificing $0 \leq B_k^n(t;\bar{a}) \leq 1$ on $[0,1]$. An example of this is $\bar{a} = (-1,-1,1)$ (see Figure 3), where $\rho_3 B_0^3(t;\bar{a}) = 3 - 4t + t^3$, $\rho_3 B_1^3(t;\bar{a}) = 3t^2 - 3t^3$, etc., but — as it is easily checked — none of the $B_k^3(t;\bar{a})$ has negative values on $[0,1]$.



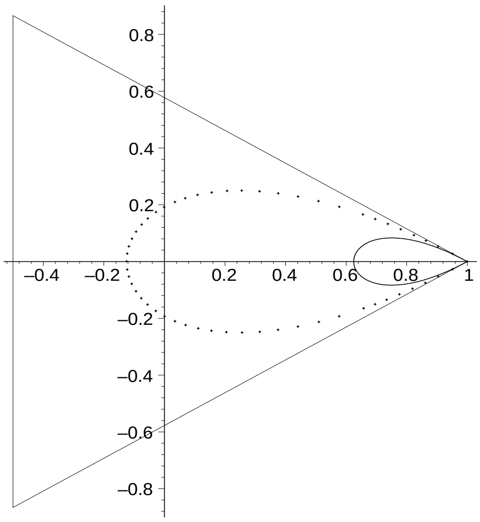


Fig. 3: $\bar{a} = (-1,-1,1)$: $B_k^3(t;\bar{a})$ $(k = 0,\ldots,3)$ and $\underline{x}(t;\bar{a})$

An interesting problem is to investigate systematically the class of all generalized Bernstein polynomials that obey $0 \leq B_k^n(t;\bar{a}) \leq 1$ on $[0,1]$, i.e., have the convex hull property, and have at least one negative parameter $a_j < 0$.

Another problem is to understand more closely the *natural extensions* of generalized Bernstein polynomials where the natural extensions of $B_k^n(t;\bar{a})$ for fixed n and $\bar{a} = (\bar{a}_1,\ldots,\bar{a}_n)$ are the polynomials $B_k^{n'}(t;\bar{a}')$ with $n' > n$ and $\bar{a}' = (\bar{a}_1,\ldots,\bar{a}_n,0,\ldots,0)$. Figures 4 to 7 below show that the natural extension of the parameter sequence $\bar{a}$ leads to — and probably even *generically* leads to — very different qualitative behaviour of the associated generalized Bernstein polynomials and generalized Bézier curves.

So far we have introduced two classes of generalized Bernstein polynomials: the first having non-negative parameters $a_j \geq 0$ (and therefore the convex hull property) and the second having arbitrary parameters, but still preserving the convex hull property. Thus it remains to focus on generalized Bernstein polynomials that do *not* possess the convex hull property (and therefore have some $a_j < 0$).

For every parameter sequence $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ one can define the real numbers

$$\gamma_0 = \gamma_0(\bar{a}) := \min_k \{B_k^n(t;\bar{a}) \mid 0 \le t \le 1\} \ , \quad \gamma_1 = \gamma_1(\bar{a}) := \max_k \{B_k^n(t;\bar{a}) \mid 0 \le t \le 1\} \ .$$

Clearly, $\gamma_0 \le 0$ and $\gamma_1 \ge 1$ by (4.4), hence $\gamma := [\gamma_0, \gamma_1] \supset [0,1]$. We define the γ-*convex set* $\mathrm{conv}_\gamma(\underline{b})$ for any n-tuple $\underline{b}$ of control points $\underline{b}_j \in \mathbb{R}^N$ by

$$\text{(5.2)} \qquad \mathrm{conv}_\gamma(\underline{b}) := \{\underline{x} \in \mathbb{R}^N \mid \underline{x} = \sum_{j=0}^n t_j \underline{b}_j \ , \ \gamma_0 \le t_0, \ldots, t_n \le \gamma_1 \ , \ \sum_{j=0}^n t_j = 1\} \ .$$

Clearly, $\mathrm{conv}_\gamma(\underline{b})$ contains the convex hull of the b_j, and is therefore an enlarging of $\mathrm{conv}(\underline{b})$ depending on how much γ_0 deviates from 0 and how much γ_1 deviates from 1. In addition, $\mathrm{conv}_\gamma(\underline{b})$ is convex.

Proof. Let $\underline{x} = \sum_{j=0}^n t_j \underline{b}_j$ and $\underline{y} = \sum_{j=0}^n s_j \underline{b}_j$ be points in $\mathrm{conv}_\gamma(\underline{b})$ and $\lambda \in (0,1)$. Then

$$\lambda \underline{x} + (1-\lambda)\underline{y} = \sum_{j=0}^n (\lambda t_j + (1-\lambda)s_j)\underline{b}_j$$

is in $\mathrm{conv}_\gamma(\underline{b})$, too, because every coefficient $(\lambda t_j + (1-\lambda)s_j)$ is in γ and the sum of all of them equals 1. □

It is immediate that for any $\bar{a}$ and $\underline{b}$ the generalized Bézier curve $\underline{x}(t;\bar{a})$ lies in $\mathrm{conv}_\gamma(\underline{b})$:

$$\underline{x}([0,1];\bar{a}) \ \subset \ \mathrm{conv}_\gamma(\underline{b}) \ .$$

Since this very coarse characterization — already coarse for the usual convex hull $\mathrm{conv}(\underline{b})$ — depends only on the uniform range of values taken by the generalized Bernstein polynomials on $[0,1]$, future research should investigate and operate with a model that takes into account, for example, the intervals

$$\text{(5.3)} \qquad I_t(\bar{a}) := [\min_k B_k^n(t;\bar{a}), \max_k B_k^n(t;\bar{a})] \quad \text{for all } t \in [0,1]$$

or some other entity depending on t. This would be especially helpful for generalized Bernstein polynomials which do not have the convex hull property and which therefore may not be variation diminishing.

In all subsequent figures we have depicted for reasons of clarity and in view of symmetry only half of the $B_k^n(t;\bar{a})$, if $n \ge 7$:

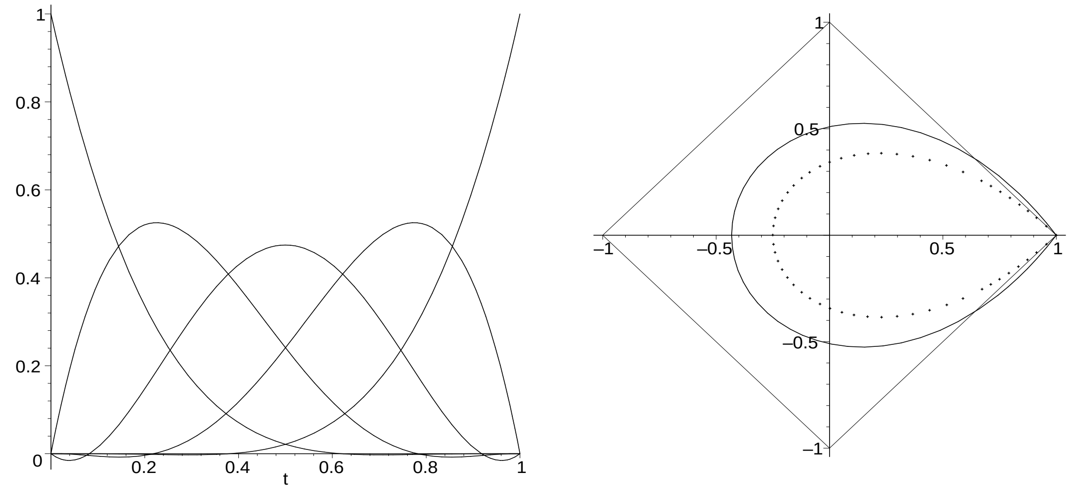


Fig. 4: $\bar{a} = (7, -4, 2, -1)$: $B_k^4(t; \bar{a})$ $(k = 0, \ldots, 4)$ and $\underline{x}(t; \bar{a})$

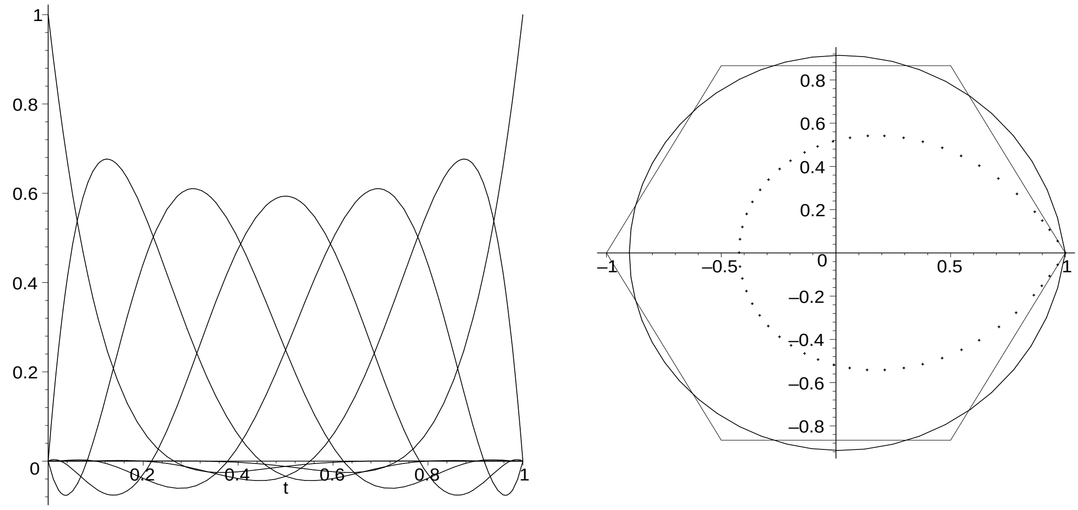


Fig. 5: $\bar{a} = (7, -4, 2, -1, 0, 0)$: $B_k^6(t; \bar{a})$ $(k = 0, \ldots, 6)$ and $\underline{x}(t; \bar{a})$

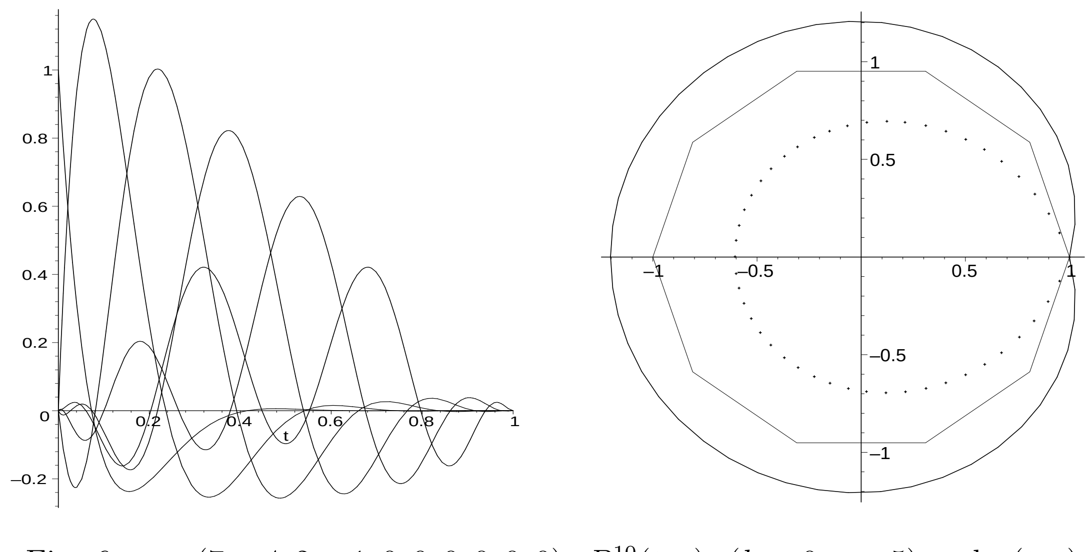


Fig. 6: $\bar{a} = (7, -4, 2, -1, 0, 0, 0, 0, 0, 0)$: $B_k^{10}(t; \bar{a})$ $(k = 0, \ldots, 5)$ and $\underline{x}(t; \bar{a})$

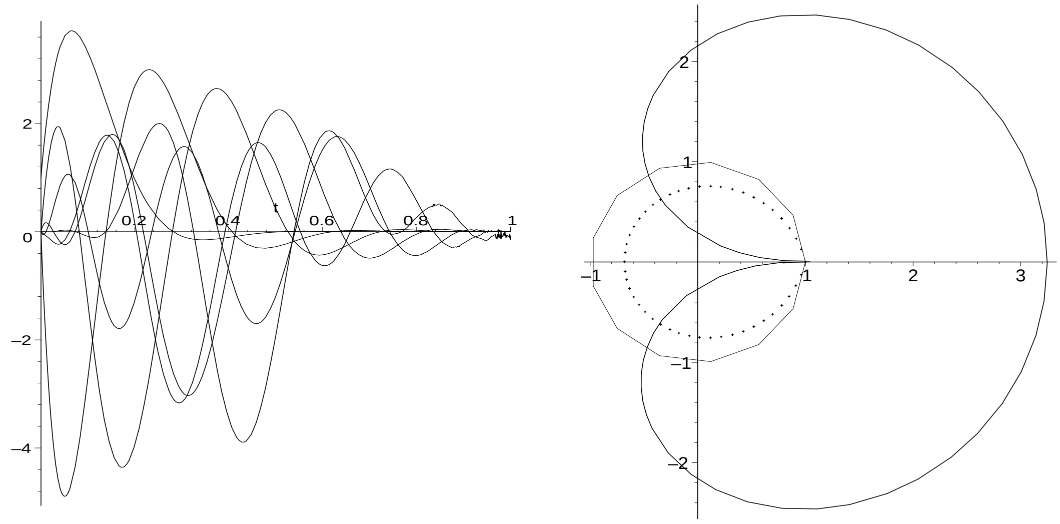


Fig. 7: $\bar{a} = (7, -4, 2, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$: $B_k^{13}(t; \bar{a})$ $(k = 0, \ldots, 7)$ and $\underline{x}(t; \bar{a})$

Figures 4 to 7 are a good example for the rapidly changing geometry of the generalized Bézier curves $\underline{x}(t; \bar{a})$ under natural extensions. But moreover they show an interesting connection between the behaviour of the extrema of the generalized Bernstein polynomials $B_k^n(t; \bar{a})$ and mutual positions of the control polygon $C(\underline{b})$, the generalized Bézier curve $\underline{x}(t; \bar{a})$, the ordinary Bézier curve $\underline{x}(t)$ and conv($\underline{b}$). Namely, we observe and conjecture for generalized Bernstein polynomials that are "essentially unimodal",i.e. which all have a unique predominant extremal value, the following:

- If the unique predominant extremal values of the $B_k^n(t; \bar{a})$ at, say, t_k are positive and smaller [roughly equal or greater but less than one] than the values of the $B_k^n(t)$ at k/n, then $\underline{x}(t; \bar{a})$ is inside conv($\underline{b}$) with $\underline{x}(t)$ between $\underline{x}(t; \bar{a})$ and $C(\underline{b})$ [$\underline{x}(t; \bar{a})$ is close to $\underline{x}(t)$ or $\underline{x}(t; \bar{a})$ is inside conv($\underline{b}$) with $\underline{x}(t; \bar{a})$ between $\underline{x}(t)$ and $C(\underline{b})$].
- If the predominant extremal values of the $B_k^n(t; \bar{a})$ are approximately 1 at the places t_k and the values of the other $B_h^n(t; \bar{a})$ $(h \neq k)$ at t_k are roughly zero, then $\underline{x}(t; \bar{a})$ is roughly an interpolation of the control points $\underline{b}_k$.
- If the predominant extremal values of the $B_k^n(t; \bar{a})$ at the t_k are positive and greater than 1, then $\underline{x}(t; \bar{a})$ is essentially outside of conv($\underline{b}$) with $C(\underline{b})$ between $\underline{x}(t; \bar{a})$ and $\underline{x}(t)$.
- If the predominant extremal values of the $B_k^n(t; \bar{a})$ at the t_k are negative, then $\underline{x}(t; \bar{a})$ is essentially outside of conv($\underline{b}$) with $\underline{x}(t)$ between $\underline{x}(t; \bar{a})$ and $C(\underline{b})$.

To see that these conjectures are plausible assume that all $B_k^n(t; \bar{a})$ have predominant unique extremal values at places t_k. Then the behaviour of a generalized Bézier curves"near" $\underline{b}_k$ is determined mostly by the extremal value of $B_k^n(t; \bar{a})$ at t_k (pseudo-local control). Natural problems are now, to

- determine, which parameter sequences $\bar{a}$ generate essentially unimodal generalized Bernstein polynomials;
- find a measure which indicates "how unimodal" a given polynomial is; and

- investigate what can be said about generalized Bernstein polynomials that are not essentially unimodal.
- In the case of exact interpolation (the “Lagrangian case” $B_k^n(t_h;\bar{a}) = \delta_{h,k}$) compatible with the “binomiality” of the $B_k^n(t;\bar{a})$? If “yes”, find all parameter sequences $\bar{a}$ with associated Lagrangian generalized Bernstein polynomials. If interpolation is only approximately possible (*quasi-interpolation*), find appropriate conditions on the parameters $\bar{a}$.
- For every n find and describe equivalence classes of parameters $\bar{a}$, such that the associated generalized Bernstein polynomials are the same (or “almost” the same).

We add some more examples to illustrate the flexibility of generalized Bernstein polynomials and generalized Bézier curves:

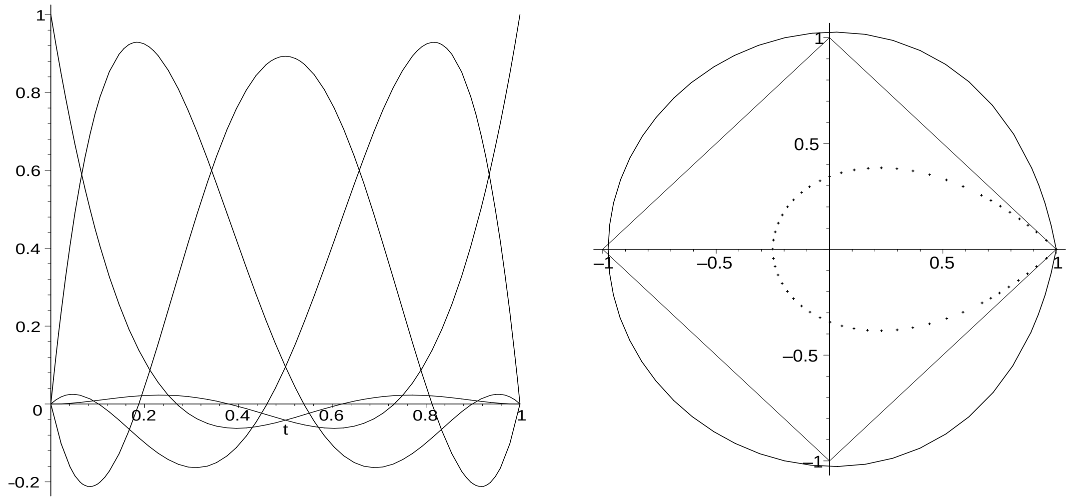


Fig. 8: $\bar{a} = (4, -3, 3, -1)$: $B_k^4(t;\bar{a})$ $(k = 0, \dots, 4)$ and $\underline{x}(t;\bar{a})$

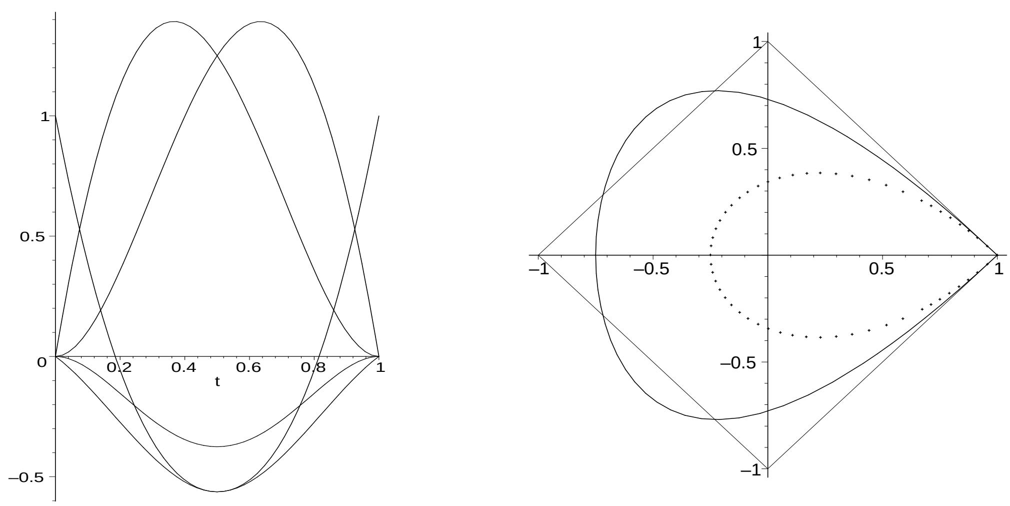


Fig. 9: $\bar{a} = (1, -1, 0, -1)$: $B_k^4(t;\bar{a})$ $(k = 0, \dots, 4)$ and $\underline{x}(t;\bar{a})$

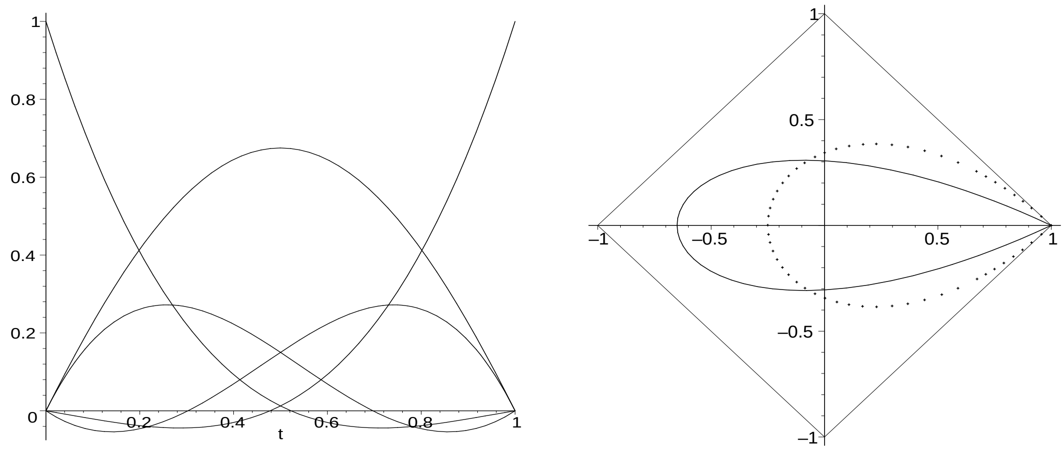


Fig. 10: $\bar{a} = (1, 1, -1, -1)$: $B^4_k(t; \bar{a})$ $(k = 0, \ldots, 4)$ and $\underline{x}(t; \bar{a})$

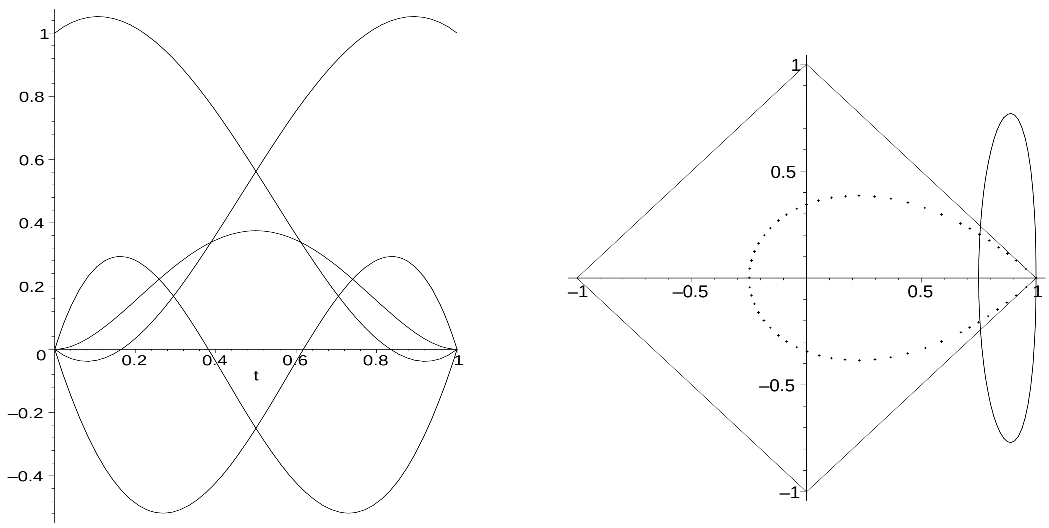


Fig. 11: $\bar{a} = (1, -1, 1, -1)$: $B^4_k(t; \bar{a})$ $(k = 0, \ldots, 4)$ and $\underline{x}(t; \bar{a})$

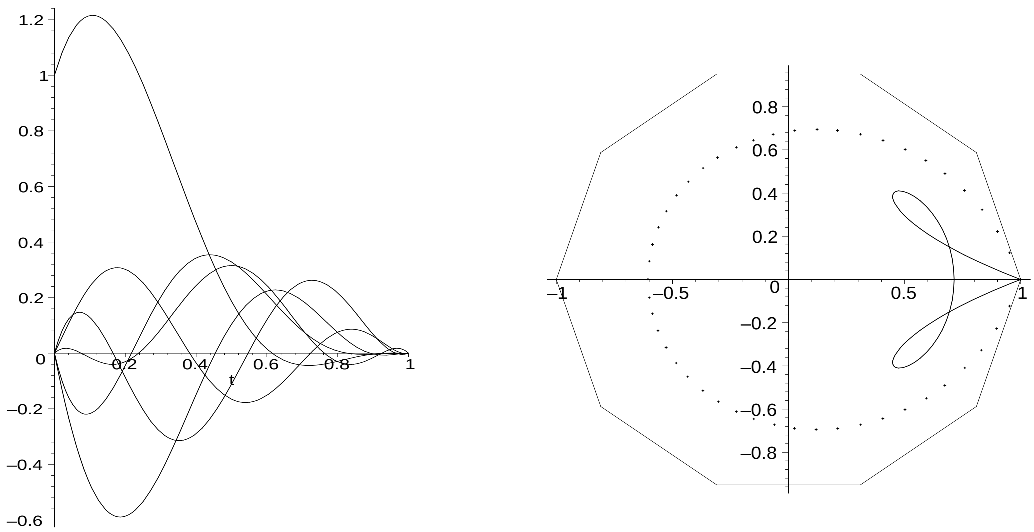


Fig. 12: $\bar{a} = (1, -1, 1, -1, 1, -1, 1, -1, 1, -1)$: $B^{10}_k(t; \bar{a})$ $(k = 0, \ldots, 5)$ and $\underline{x}(t; \bar{a})$

Figure 8 illustrates quasi-interpolation. In Figure 9 the maxima of $B_1^4(t;\bar{a})$ and $B_3^4(t;\bar{a})$ are much greater one and the minimum of $B_2^4(t;\bar{a})$ is less than -.3; accordingly $\underline{x}(t;\bar{a})$ comes close to $\underline{b}_1$ and $\underline{b}_3$ and even leaves conv$(\underline{b})$, but between $\underline{b}_1$ and $\underline{b}_3$ it takes a path between $\underline{b}_2$ and $\underline{x}(t)$. Quite the opposite happens in Figure 10: the maxima of $B_1^4(t;\bar{a})$ and $B_3^4(t;\bar{a})$ are relatively small and the maximum of $B_2^4(t;\bar{a})$ relatively big, but less than 1; accordingly $\underline{x}(t;\bar{a})$ stays inside of $\underline{x}(t)$ near $\underline{b}_1$ and $\underline{b}_3$, but makes a decided move towards $\underline{b}_2$. Figures 11 and 12 supplement the example of Figure 7 in showing how $\underline{x}(t;\bar{a})$ may behave under the influence of strong negative extrema.

Since the variability of different generalized Bernstein polynomials and of the shapes of generalized Bézier curves is far from being well understood at present, we advocate an "experimental" approach that begins with the ordinary Bézier curve of a control polygon for a first orientation and then gradually deforms $\underline{x}(t;(1,0,\ldots,0))$ until the wanted shape is reached. For this deformation approach it is useful to first explore in a quick manner a wide class of possibilities by first tuning a single or a few *master parameters* and then "finetune" using the parameters $\bar{a}_j$ one by one. For quick global tuning one can use a *master function* like

$$\bar{a}_j := f(j-1) \quad \text{with} \quad f(j) = (c_1 j + c_0)\cos(bj\pi)\exp(aj)\ . \tag{5.4}$$

To elucidate this choice of the master function, first set $c_1 = c_0 = b = 0$ and $a \ll 0$; then $\bar{a}$ is close to $1, 0, \ldots, 0$ and $\underline{x}(t;\bar{a})$ close to $\underline{x}(t)$. Increasing just the a of the exponential factor then yields various stiff modifications of $\underline{x}(t)$, but simultaneously changing b in the oscillation factor introduces negative parameters a_j and great flexibility in curve design. Finally, the parameters c_0, c_1 of the linear (polynomial) factor add even more flexibility. On a computational level it is useful to store the universal generalized Bernstein polynomials of a certain degree, evaluate their coefficients for different sequences of parameters $\bar{a}$, and finally evaluate the generalized Bézier curves using a Horner scheme. (More effective approaches to the computation of generalized Bézier curves will be the subject of a subsequent paper.) It may also be useful to depict curves $\underline{x}(t_k;\bar{a}(f))$ for several fixed $t_k \in (0,1)$ and some continuously varying master parameter of f.

Below we depict $\underline{x}(t;\bar{a})$ for C_9 and different values of a in $f(j) = \cos(0.4\,j\pi)\exp(aj)$. This will put the isolated pictures of Figures 2 to 12 into perspective and at the same time explain the geometric meaning of the condition $\rho_n \neq 0$ in the definition of generalized Bernstein polynomials.

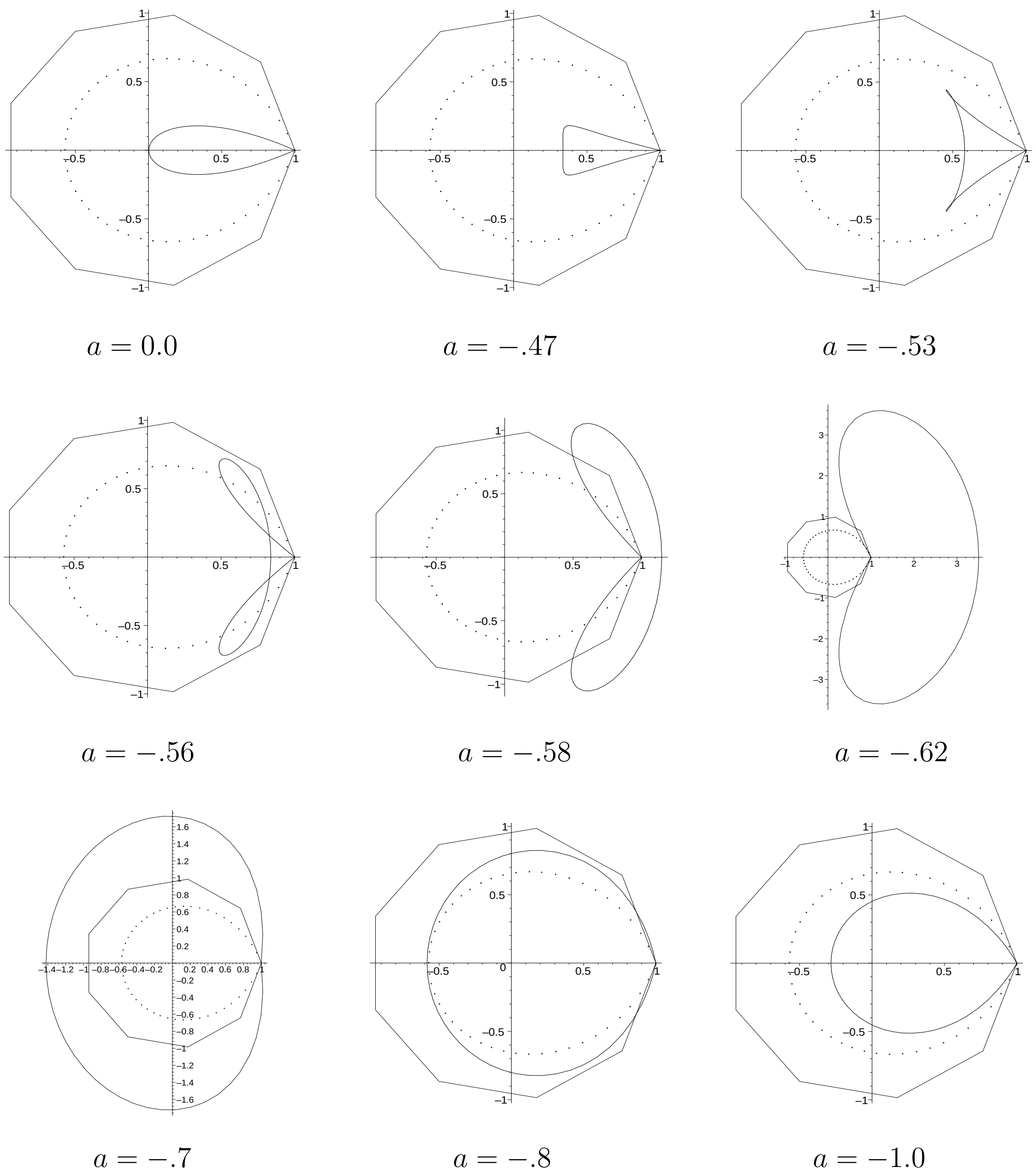


Fig. 13: $\underline{x}(t;\bar{a})$ for C_9 and different $\bar{a}$ given by $\bar{a}_{j+1} = \cos(0.4\,j\pi)\exp(aj)$ $(j = 0, \ldots, 8)$.

While changing the master parameter a from 0.0 to -1.0 in f, the generalized Bézier curve undergoes a considerable change as indicated by the "snapshots" for different values of a. The connection with Figures 2 to 12 is obvious, but the change occurring between $a = -.62$ and -.7 needs a further comment: at $a = -.62$ the curve is outside of $\mathrm{conv}(C_9)$ on the right and its size is growing infinitely with decreasing a; at $a \sim -.641361\ldots$ a flip of the curve to the left of $\mathrm{conv}(C_9)$ occurs; then the curve decreases in size again. The value of a, where the flipping occurs, is exactly the point, where $\rho_9(\bar{a}) = 0$. Thus the values $\rho_n = 0$ which have to be excluded in the definition of generalized Bernstein polynomials are points where a qualitative change in the behaviour of these polynomials occurs. However, this does not mean that the shape of the generalized Bernstein polynomials undergoes a visible change except for the scaling of the ordinate, but instead the qualitative change is one that takes place on the level of the dominance of positive and negative parts of polynomials (see Figure 14). It would be nice to have a pictorial representation of generalized Bernstein polynomials that makes obvious the predominance of either the positive or the negative parts.

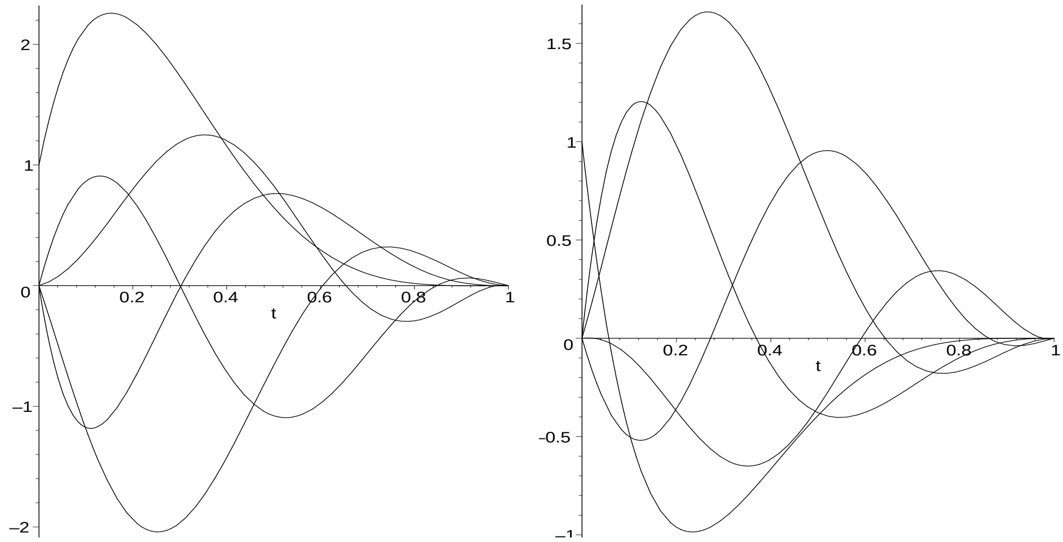


Fig. 14: $B_k^9(t;\bar{a})$ $(k = 0, \ldots, 5)$ for $a = -.62$ and $a = -.7$ from Figure 13.

6. Composite generalized Bézier curves

Let a generalized Bézier curve $\underline{x}(t;\bar{a})$ with control points $\underline{b} = (\underline{b}_0, \ldots, \underline{b}_n)$ be given as in Definition 4.1 and a second curve $\underline{X}(t;\bar{A})$ with control points $\underline{B} = (\underline{B}_0, \ldots, \underline{B}_N)$ by

$$\underline{X}(t;\bar{A}) := \sum_{k=0}^{N} \underline{B}_k B_k^n(t;\bar{A}) \qquad (0 \le t \le 1).$$

When is the second a "smooth" continuation of the first? Clearly, for the *composite curve* to be continuous (C^0), it is necessary and sufficient that $\underline{b}_n = \underline{B}_0$, for the composite curve to be visibly C^1 and C^2, it is necessary that $\dot{\underline{x}}(1;\bar{a}) = \dot{\underline{X}}(0;\bar{A})$, and in addition $\ddot{\underline{x}}(1;\bar{a}) = \ddot{\underline{X}}(0;\bar{A})$ *and* that the control points are sufficiently distinct. For ordinary Bézier curves it is well known that the composite curve is C^1, if $\underline{B}_1$ lies on the line through $\underline{b}_{n-1}$ and $\underline{b}_n$, and that it is C^2, if in addition a certain condition on $\underline{B}_2$ is fulfilled. The theorem below shows that also for generalized Bézier curves it is only necessary to adjust $\underline{B}_1$ and $\underline{B}_2$ (independently of $n, N, \bar{a}$, and $\bar{A}$) in a certain way to get C^1 and C^2 composite generalized Bézier curves. However, in general — and here lies the difference to the ordinary case — it is necessary for the computation of the correct "local" information on $\underline{B}_1$ and $\underline{B}_2$ to use global knowledge about the entire curves $\underline{x}(t;\bar{a})$ and $\underline{X}(t;\bar{A})$.

Theorem 6.1. *Let $n, N \geq 3$, $\bar{a}$ and $\bar{A}$, $\underline{b}_1, \ldots, \underline{b}_n = \underline{B}_0, \underline{B}_3, \ldots, \underline{B}_N$, and $\underline{x}(t;\bar{a})$ and $\underline{X}(t;\bar{A})$ be given as above. Then the composite generalized Bézier curve of $\underline{x}(t;\bar{a})$ and $\underline{X}(t;\bar{A})$ is C^1, if*

$$(6.1)\quad \underline{B}_1 = \frac{1}{\omega_{N,N-1}}\left(\dot{\underline{x}}(1;\bar{a}) - \sum_{j=2}^{N-1}(\omega_{N,N-j}\underline{B}_j - \omega_{N,j}\underline{b}_n) - \frac{\bar{A}_N}{\chi_N}(\underline{B}_N - \underline{b}_n) + \omega_{N,1}\underline{b}_n\right) ,$$

where $\omega_{N,N-1}$ has to be non-zero,

$$\alpha_{i,j} := \bar{a}_{i-j}\frac{\rho_j}{\rho_i}\binom{i}{j} \quad \text{and} \quad \omega_{i,j} := \bar{A}_{i-j}\frac{\chi_j}{\chi_i}\binom{i}{j}$$

as in (4.8) (P_j associated to $\bar{A}$, $\chi_j := P_j(1)$), and

$$(6.2)\qquad \dot{\underline{x}}(1;\bar{a}) = \sum_{j=1}^{n-1}(\alpha_{n,n-j}\underline{b}_n - \alpha_{n,j}\underline{b}_j) + \frac{\bar{a}_n}{\rho_n}(\underline{b}_n - \underline{b}_0) .$$

The composite generalized Bézier curve is C^2, if it is C^1 and in addition

$$(6.3)\qquad \underline{B}_2 = \frac{1}{\ddot{B}_2^N(0;\bar{A})}\left(\ddot{\underline{x}}(1;\bar{a}) - \sum_{\substack{k=0\\k\neq 2}}^{N}\underline{B}_k\ddot{B}_k^N(0;\bar{A})\right) ,$$

where $\ddot{B}_2^N(0;\bar{A})$ has to be non-zero,

$$(6.4)\quad \ddot{\underline{x}}(1;\bar{a}) = \sum_{k=0}^{n}\underline{b}_k\left(\sum_{i=n-k}^{n-1}\alpha_{n,i}\left(\sum_{j=0}^{i-1}\alpha_{i,j} - \alpha_{i,i+k-n}\right) - \sum_{i=k}^{n-1}\alpha_{n,i}\left(\sum_{j=0}^{i-1}\alpha_{i,j} - \alpha_{i,k}\right)\right) ,$$

and

$$(6.5)\quad \ddot{B}_k^N(0;\bar{A}) = \sum_{i=N-k}^{N-1}\omega_{N,i}\left(\omega_{i,N-k} - \sum_{j=0}^{i-1}\omega_{i,i+k-N}\right) - \sum_{i=k}^{N-1}\omega_{N,i}\left(\omega_{i,i-k} - \sum_{j=0}^{i-1}\omega_{i,j}\right) .$$

Proof. All the formulas result from straightforward computations that start from Definition 4.1 of generalized Bernstein polynomials and use the formulas for the derivative (4.8) and the end points (4.4). As intermediary results we note

$$\begin{aligned}
\dot{\underline{x}}(t;\bar{a}) =& \sum_{j=1}^{n-1}\left(\alpha_{n,n-j}\sum_{k=0}^{n-j}\underline{b}_{k+j}B_k^{n-j}(t;\bar{a}) - \alpha_{n,j}\sum_{k=0}^{j}\underline{b}_kB_k^j(t;\bar{a})\right) + \frac{\bar{a}_n}{\rho_n}(\underline{b}_n - \underline{b}_0) ,\\
\dot{B}_k^n(0;\bar{a}) =& \alpha_{n,n-k} - \delta_{k,0}\sum_{j=0}^{n-1}\alpha_{n,j} ,\\
\dot{B}_k^n(1;\bar{a}) =& \delta_{k,n}\sum_{j=0}^{n-1}\alpha_{n,j} - \alpha_{n,k} ,
\end{aligned}$$

and

$$\ddot{B}^n_k(t;\bar{a}) = \sum_{i=n-k}^{n-1} \alpha_{n,i} \left(\sum_{j=n-k}^{n-1} \alpha_{i,j} B^j_{n-k}(1-t;\bar{a}) - \sum_{j=i+k-n}^{i-1} \alpha_{i,j} B^j_{i+k-n}(t;\bar{a}) \right) - \sum_{i=k}^{n-1} \alpha_{n,i} \left(\sum_{j=i-k}^{i-1} \alpha_{i,j} B^j_{i-k}(1-t;\bar{a}) - \sum_{j=k}^{i-1} \alpha_{i,j} B^j_k(t;\bar{a}) \right) .$$

□

References

[1] E. Abe, "Hopf Algebras", Cambridge University Press, Cambridge, 1977.

[2] E.T. Bell, The history of Blissard's symbolic method, with a sketch of its inventor's life, *Amer. Math. Monthly* **XLV** (1938), 414 - 421.

[3] P.J. Barry, R.N. Goldman, What is the Natural Generalization of a Bézier curve?, *in*: T. Lyche, L.L. Schumaker (eds.), "Mathematical Methods inComputer Aided Geometric Design", Academic Press, Boston, 1989.

[4] L. Comtet, "Advanced Combinatorics", Reidel, Dordrecht, 1974.

[5] G. Farin, "Curves and Surfaces for Computer Aided Geometric Design. A Practical Guide", Academic Press, Boston, 1988.

[6] R.T. Farouki, Computing with Barycentric Polynomials, *Math. Intelligencer* **13** (1991), 61 - 69.

[7] R.T. Farouki, On the stability of transformations between power and Bernstein polynomial forms, *CAGD* **8** (1991), 29 - 36.

[8] R.T. Farouki, V.T. Rajan, Algorithms for polynomials in Bernstein form, *CAGD* **5** (1988), 1 - 26.

[9] R.N. Goldman, Dual Polynomial Basis, *J. Approx. Theory* **79** (1994), 311 - 346.

[10] S.A. Joni, G.-C. Rota, Coalgebras and Bialgebras in Combinatorics, *Studies in Appl. Math.* **61** (1979), 93 - 139, *reprinted in*: J.S. Kung (ed.), "Gian-Carlo Rota on Combinatorics", Birkhäuser, Boston, 1995.

[11] S. Roman, "The Umbral Calculus", Academic Press, Boston, 1984.

[12] S. Roman, "Advanced Linear Algebra", *Graduate Texts in Mathematics* **135**, Springer, New York, 1992.

[13] S. Roman, The Algebra of Formal Series III: Several Variables, *J. Approx. Theory* **26** (1979), 340 - 381.

[14] S. Roman, Polynomials, Power Series and Interpolation, *J. Math. Anal. Appl.* **80** (1991), 333 - 371.

[15] L. Ramshaw, Blossoms are polar forms, *CAGD* **6** (1989), 323 - 258.

[16] S. Roman, G.-C. Rota, The umbral calculus, *Adv. Math.* **27** (1978), 95 - 188.

[17] H.-P. Seidel, A new multiaffine approach to B-splines, *CAGD* **6** (1989), 23 - 32.

[18] Y. Stephanus, R.N. Goldman, Blossoming Marsden's identity, *CAGD* **9** (1992), 73 - 84.

[19] M.E. Sweedler, "Hopf Algebras", Benjamin, New York, 1969.

[20] K. Ueno, Umbral Calculus and Special Functions, *Adv. Math.* **67** (1988), 174 - 229.

Institut für Reine und Angewandte Mathematik, RWTH Aachen, D-52056 Aachen, Germany, http://www.iram.rwth-aachen.de/ ~winkel/

E-mail address: winkel@iram.rwth-aachen.de