ON A GENERALIZATION OF BERNSTEIN POLYNOMIALS AND BÉZIER CURVES BASED ON UMBRAL CALCULUS

RUDOLF WINKEL

Abstract. In [20] a generalization of Bernstein polynomials and Bézier curves based on umbral calculus has been introduced. In the present paper we describe new geometric and algorithmic properties of this generalization including: (1) families of polynomials introduced by Stancu [19] and Goldman [12], i.e., families that include both Bernstein and Lagrange polynomial, are generalized in a new way, (2) a generalized de Casteljau algorithm is discussed, (3) an efficient evaluation of generalized Bézier curves through a linear transformation of the control polygon is described, (4) a simple criterion for endpoint tangentiality is established.

1. Introduction

For the vector spaces $\mathbb{R}_{\leq n}[t]$ of polynomials in t of degree $\leq n$ many different bases are known and in use where every basis has its advantages and disadvantages.

The monomials t^k $(0 \leq k \leq n\ )$ are popular because of easy elementary arithmetic operations. However, coefficients do not carry much obvious geometric meaning. When geometrically intuitive approximation properties and numeric stability are required, the *Bernstein polynomials* of degree n

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k} \text{ for } 0 \leq k \leq n \tag{1.1}$$

are a much better choice due to the classical work of Bernstein, Bézier and de Casteljau (see [4, 8] for general information, [6, 10] for algorithms, and [9, 7, 11] for numerical stability).

A *control polygon* $C(\mathbf{b})$ is described by a sequence of $n+1$ *control points* $(\mathbf{b}_0, \ldots, \mathbf{b}_n)$ in $\mathbb{R}^N$ such that successive control points are connected by straight line segments. Then a "good" polynomial approximation of this control polygon is given by the *Bézier curve*

$$\mathbf{x}(t) \equiv \mathbf{x}(t;\mathbf{b}) := \sum_{k=0}^{n} \mathbf{b}_k B_k^n(t) \qquad (0 \leq t \leq 1)\ , \tag{1.2}$$

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"Good" means that the Bézier curve $\mathbf{x}(t)$ starts at $\mathbf{b}_0$ and ends at $\mathbf{b}_n$ in a tangential fashion and follows the course of the control polygon in variation diminishing way. Since the Bézier curve can be intuitively controlled by moving the control points and since there are several efficient evaluation algorithms – de Casteljau, subdivision, blossoming – available, Bernstein polynomials and Bézier curves are the cornerstone of Computer Aided Geometric Design (CAGD) [4].

However, if the control points are assumed to be fixed for some reason, e.g., in data smoothing, the geometric connection between the control polygon and its Bézier curve is also fixed. Therefore several generalizations that allow a more flexible response of the resulting curve on the control points have been introduced. Most prominent are certainly *rational Bézier curves* that change the response of the curve to the fixed control points by weighting the control points. Since this approach does not permit changing the resulting curve in an uniform way – one point can only gain attraction at the cost of others – families of polynomials that interpolate between Bernstein and Lagrange polynomials have been invented: the polynomials by Stancu [19] and Goldman [12] reappearing as as h-Bernstein bases in [18]. These families are further generalized by Farin and Barry [5] and Barry and Goldman [2]. A generalization of the Bernstein-Lagrange family in a completely different direction is achieved by the generalized Bernstein polynomials introduced in [20] (see Thm 2.1 below).

But first of all let us recall the definition of *generalized Bernstein polynomials* $B_k^n(t;\bar{a})$ of degree n:

$$B_k^n(t;\bar{a}) := \frac{1}{\rho_n}\binom{n}{k} p_k(t)p_{n-k}(1-t) \ , \tag{1.3}$$

where p_k and p_{n-k} in $\mathbb{R}[t]$ are polynomials of degree k and $n-k$, respectively, that replace the monomials t^k and $(1-t)^{n-k}$ in the definition of Bernstein polynomials, $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ $(\bar{a}_1 \neq 0)$ are real parameters which determine the shape of p_k and p_{n-k}, and $1/\rho_n$ is a normalizing factor. Of course, p_k and p_{n-k} not only have the right degree, but are chosen such that the crucial property of *partition of unity*

$$1 = \sum_{k=0}^{n} B_k^n(t) \tag{1.4}$$

of Bernstein polynomials is preserved which guarantees *affine invariance*. Since (1.4) for Bernstein polynomials follows immediately from the binomial formula

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k} x^k y^{n-k}$$

by setting $x = t$ and $y = 1-t$, the following generalized binomial formula

$$p_n(x+y) = \sum_{k=0}^{n}\binom{n}{k} p_k(x)p_{n-k}(y) \tag{1.5}$$

for a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ with $\deg(p_n) = n$ would yield the desired generalization of Bernstein polynomials: To get the generalized Bernstein polynomials of (1.3) just take the summands of (1.5), set $x = t$ and $y = 1 - t$, and divide by $\rho_n := p_n(1)$ (if $\neq 0$).

Do polynomials exist that fulfill (1.5)? The study of *binomial polynomials* that fulfill (1.5) is at the center of the Umbral Calculus founded by John Blissard in the 1850's [1] and revived by Gian-Carlo Rota and Steven Roman in the 1970's [16, 14, 15]. In fact, one can find such polynomials for every sequence $\bar{a}$ of real parameters. In order to make the present paper self-contained some crucial formulas are repeated here.

For the n-tuple $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ of real numbers one defines the *Bell polynomials* [3, 16] by

$$p_n(t) \equiv p_n(t; \bar{a}) = \sum_{k=1}^{n} p_{n,k} t^k \tag{1.6}$$

with

$$p_{n,k} \equiv p_{n,k}(\bar{a}) = \frac{1}{k!} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k>0}} \binom{n}{i_1, \dots, i_k} \bar{a}_{i_1} \cdot \cdots \cdot \bar{a}_{i_k} \ . \tag{1.7}$$

The recursion

$$p_{n,k+1} = \frac{1}{k+1} \sum_{j=1}^{n-k} \binom{n}{j} \bar{a}_j p_{n-j,k} \tag{1.8}$$

allows a fast calculation.

The notation $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ with the bars is chosen to conform with the established notation in umbral calculus. There one starts with a real formal power series

$$f(t) = \sum_{n=1}^{\infty} \frac{a_n}{n!} t^n \ ,$$

where $a_1 \neq 0$, i.e., with a *delta series*. For every delta series there exists a second delta series

$$\bar{f}(t) = \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n!} t^n \tag{1.9}$$

that is the compositional inverse of f with $(\bar{f} \circ f)(t) = (f \circ \bar{f})(t) = t$ and that is used as a generating function for a sequence of *associated polynomials* $p_n(x)$

$$e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} t^n \ . \tag{1.10}$$

Clearly, the generalized Bernstein polynomials contain the Bernstein polynomials as a special case:

$$B_k^n(t) = B_k^n(t; (1, 0, \dots, 0)) \ .$$

Given a feasible parameter sequence $\bar{a} \in \mathbb{R}^n$ (such that $\rho_n = p_n(1) \neq 1$) and a sequence of control points $\mathbf{b} = (\mathbf{b}_0, \ldots, \mathbf{b}_n)$ the evaluation of generalized Bézier curves

$$\mathbf{x}(t; \bar{a}) \equiv \mathbf{x}(t; \bar{a}; \mathbf{b}) := \sum_{k=0}^{n} \mathbf{b}_k B_k^n(t; \bar{a}) \quad (0 \leq t \leq 1) \ , \tag{1.11}$$

via formulas (1.8), (1.6) and (1.3) (in this order) is however quite cumbersome, in particular, the final evaluation of (1.11). At first sight the fast algorithms available for the evaluation of Bernstein polynomials do not seem to have a counterpart in the generalized case. However, in Section 4 we show that the generalized Bézier curve can be computed as an (ordinary) Bézier curve of a certain linear $\bar{a}$-transformation of the control polygon $C(\mathbf{b})$. The examples from [20] are revisited in the light of this new computational approach.

We emphasize once more that our generalized Bernstein polynomials and generalized Bézier curves are completely in line with the direction of development that began with the rational case, namely, fix the control points and change the polynomials to get a different response of the resulting curve to the control polygon. The transformation of the control polygon $C(\mathbf{b})$ is not visible and relevant for the designer or data analyst who manipulates the responsiveness of the curve via parameters. In Section 2 we show that the generalized Bernstein polynomials include as special cases the Lagrange and the h-Bernstein polynomials. There we introduce the "responsiveness" c that controls as a master parameter all single parameters of $\bar{a}$ simultaneously (Thm. 2.1). Moreover, we show by example of the "Bernstein-Lagrange family" how the parameter sequence $\bar{a}$ controls the geometry of the generalized Bézier curves , thereby pointing out a direction in which to proceed with further investigations on the geometric meaning of different parameter sequences. As a starting point for this investigation we give a summary of simple properties of generalized Bernstein polynomials :

- $B_k^n(0) = \delta_{k,0}$ and $B_k^n(1) = \delta_{n,0}$ imply endpoint interpolation,
- symmetry: $B_k^n(t) = B_{n-k}^n(1-t)$,
- affine invariance by the partition of unity property,
- non-negativity of parameters or more generally non-negativity of the generalized Bernstein polynomials on the interval $[0, 1]$ implies the convex hull property,
- if interpolation is desired, the convex hull property cannot be fulfilled and therefore at least one parameter must be negative,
- variants of generalized Bernstein polynomials that are "stiffer" than the ordinary Bernstein polynomials, e.g., when $c < 0$ in Thm. 2.1, or simply when all parameters are non-negative (by the optimality of Bernstein polynomials), have the variation-diminishing property,
- for every fixed degree $n \geq 4$ there exist parameter sequences different from $(1, 0, \ldots, 0)$ that imply tangency at endpoints (Thm. 5.5).

In Section 3 we generalize the de Casteljau algorithm along the lines of umbral calculus and in Section 5 we explore some algebraic and geometric properties of the parameter sequences $\bar{a}$ and the transformation matrix $M(\bar{a})$.

We finally remark that the different q-generalization of Bernstein polynomials fit well into the umbral calculus based framework as displayed here, but we defer the investigation of this topic to another paper.

2. Between Bernstein and Lagrange

If the parameter sequence $\bar{a}$ is chosen suitably one gets the Lagrange polynomials. With an additional "master" parameter c one gets a whole family of generalized Bernstein polynomials that interpolates between Bernstein and Lagrange polynomials. This family is identical with the ones described by Stancu [19] and Goldman [12] and reappears in [18] as h-Bernstein bases. The further generalizations of these families by Farin and Barry [5] and Barry and Goldman [2] are completely different from our approach. In the following theorem one sees that the "master parameter" c controls the parameter sequence $\bar{a}(c)$ as a whole and the cases $c = 0$, 1 and $-\infty$ suggest to calling it the *responsiveness* of the generalized Bézier curves.

Theorem 2.1. *For $\bar{a}(c) := (\bar{a}_1(c), \dots, \bar{a}_n(c))$ with*

$$\bar{a}_i(c) := (-1)^{i-1}(\frac{c}{n})^{i-1}(i-1)!$$

one gets for

- $c = 0$ *the Bernstein basis* $B_k^n(t; \bar{a}(0)) = B_k^n(t)$
- $c = 1$ *the Lagrange polynomials* $B_k^n(t; \bar{a}(1)) = L_k^n(t)$ *of order* n *with equidistant nodes* $t_i = i/n$ *for* $i = 0, \dots, n$
- $0 < c < 1$ *a continuous transition between Bernstein and Lagrange case*
- $c < 0$ *and* $c \to -\infty$ *the Bézier curve settles at the straight line from* $\mathbf{b}_0$ *to* $\mathbf{b}_n$
- $1 < c < \frac{n}{n-1}$ *"blow up" of the generalized Bézier curve from the interpolating curve to the straight line from* $\mathbf{b}_n$ *to* ∞ *to* $\mathbf{b}_0$
- $\frac{n}{n-1} < c < \frac{n}{n-2}$, $\frac{n}{n-2} < c < \frac{n}{n-3}$, $\dots$, $\frac{n}{3} < c < \frac{n}{2}$ *the curves cycles between "blow up's"*
- $\frac{n}{2} < c$ *and* $c \to \infty$ *the curve settles at the straight line from* $\mathbf{b}_0$ *to* $\mathbf{b}_n$

Proof. Umbral calculus [14] shows the "classical" fact that

$$\bar{f}(t) = \alpha^{-1}\log(1+t) = \frac{1}{\alpha}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}t^i =: \sum_{i=1}^{\infty}\frac{\bar{a}_i}{i!}t^i$$

is the generating function of the lower factorial polynomials

$$p_n(x) = \frac{x}{\alpha}(\frac{x}{\alpha} - 1)\cdots(\frac{x}{\alpha} - n + 1) = \prod_{k=0}^{n-1}(\frac{x}{\alpha} - k)$$

whose coefficients in the monomial basis are the Stirling numbers of the first kind. The substitution $t = \alpha s$ rescales the lower factorial polynomials such that

$$e^{x\bar{f}(t)} = e^{x\bar{f}(\alpha s)} = \sum_{n=0}^{\infty} \frac{\alpha^n \tilde{p}_n(x)}{n!} s^n$$

with

$$p_n(x) = \alpha^n \tilde{p}_n(x) = \prod_{k=0}^{n-1}(x - \alpha k) \ .$$

Setting $\alpha = \frac{1}{n}$ for some fixed $n \in \mathbb{N}$ now gives the polynomials

$$p_n(x) = \prod_{k=0}^{n-1}(x - \frac{k}{n})$$

for the parameter sequence $\bar{a} = (\bar{a}_1, \bar{a}_2, \dots)$ with

$$\bar{a}_i := (-1)^{i-1}\frac{1}{n^{i-1}}(i-1)! \ .$$

Now for this $\bar{a}$ one computes with $\rho_n = p_n(1) = \frac{n!}{n^n}$ the generalized Bernstein polynomials

$$\begin{aligned} B_k^n(t;\bar{a}) &= \frac{1}{\rho_n}\binom{n}{k} p_k(t) p_{n-k}(1-t) \\ &= \frac{n^n}{k!(n-k)!} \prod_{i=0}^{k-1}(t - \frac{i}{n}) \prod_{j=0}^{n-k-1}(1 - \frac{j}{n} - t) \\ &= \frac{n^n(-1)^{n-k}}{k!(n-k)!} \prod_{i=0}^{k-1}(t - \frac{i}{n}) \prod_{j=0}^{n-k-1}(t - (1 - \frac{j}{n})) \\ &= \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{t - \frac{i}{n}}{\frac{k}{n} - \frac{i}{n}} = L_k^n(t) \end{aligned}$$

If $h(c)$ is any continuous function on $[0,1] \subset \mathbb{R}$ with $h(0) = 0$ and $h(1) = 1$, then the substitution $\alpha = \frac{h(c)}{n}$ yields parameters $\bar{a}_i(h(c)) := (-1)^{i-1}(\frac{h(c)}{n})^{i-1}(i-1)!$ such that for $c = 0$ one gets Bernstein polynomials and for $c = 1$ Lagrange polynomials.

To keep things as simple as possible we make the choice $h(c) := c$. Then the substitution $\alpha = \frac{c}{n}$ yields the parameter sequence $\bar{a}(c) = (\bar{a}_1(c), \bar{a}_2(c), \dots)$ with

$$\bar{a}_i(c) := (-1)^{i-1}(\frac{c}{n})^{i-1}(i-1)! \ .$$

The sequence of associated polynomial is

$$p_n(x;c) := \prod_{k=0}^{n-1}(x - k\frac{c}{n})$$

with

$$\rho_n(c) := p_n(1;c) = \prod_{k=1}^{n-1}(1 - k\frac{c}{n}) = (n-1)!(\frac{-1}{n})^{n-1}\prod_{k=1}^{n-1}(c - \frac{n}{k}) \ .$$

The generalized Bernstein polynomials are then given by

$$B_k^n(t;\bar{a}(c)) = \frac{1}{\rho_n(c)}\binom{n}{k}\prod_{i=0}^{k-1}(t - i\frac{c}{n})\prod_{j=0}^{n-k-1}(1 - j\frac{c}{n} - t)) \ .$$

With

$$B_0^n(t;\bar{a}(c)) = \frac{1}{\rho_n(c)}\prod_{j=0}^{n-1}(1 - j\frac{c}{n} - t)) = \prod_{j=0}^{n-1}\frac{1 - j\frac{c}{n} - t}{1 - j\frac{c}{n}} = (1-t)\prod_{j=1}^{n-1}(1 - \frac{t}{1 - j\frac{c}{n}})$$

one sees that $\lim_{c\to\pm\infty} B_0^n(t;\bar{a}(c)) = 1 - t$ and similarly $\lim_{c\to\pm\infty} B_n^n(t;\bar{a}(c)) = t$. By the partition of unity property of generalized Bernstein polynomials it follows that $\lim_{c\to\pm\infty} B_k^n(t;\bar{a}(c)) = 0$ for $k = 1,\dots,n-1$ and hence the continuous transition of the generalized Bézier curve to the straight line from $\mathbf{b}_0$ to $\mathbf{b}_n$ for $c = \pm\infty$.

Since $\rho_n(c)$ has the zeros $\frac{n}{n-1} < \frac{n}{n-2} < \dots \frac{n}{2} < n$ the $B_k^n(t;\bar{a}(c))$ "blow up" to ∞ for these c's.

In each of the intervals $\frac{n}{n-1} < c < \frac{n}{n-2}$, $\frac{n}{n-2} < c < \frac{n}{n-3}$, $\dots$, $\frac{n}{3} < c < \frac{n}{2}$ between the zeros of $\rho_n(c)$ the generalized Bernstein polynomials behave differently. The key to understanding this behavior are the zeros of the generalized Bernstein polynomials, which we will investigate next.

Let

$$R(\bar{a}) := \bigcup_{k=0}^{n}\{r \in \mathbb{R} \mid B_k^n(r;\bar{a}) = 0\} \tag{2.1}$$

be the set of zeros of the generalized Bernstein polynomials $B_k^n(t;\bar{a})$ for given $\bar{a}$. In general $|R(\bar{a}(c))| = 2n$ except for certain values of c where different $B_k^n(t;\bar{a}(c))$ share zeros. This happens of course in the Bernstein case where $R(\bar{a}(0)) = \{0,1\}$ and in the Lagrange case where $R(\bar{a}(1)) = \{i/n \mid i = 0,\dots,n\}$, but also for other values of c (see Fig.1 (A)).

Note that numbering the zeros for some $c < 0$ by $1, 2, \dots, 2n$ in decreasing order as in Fig. 1 (A) implies that $B_k^n(t;\bar{a}(c))$ has exactly the zeros $k+1,\dots,k+n$ for $k = 0,\dots,n$ — in Fig.1 (B) one sees the zero-contours $3,4,5,6,7$ for $B_2^5(t;\bar{a}(c))$).

Now for $c = 1 = \frac{n}{n}$ a fixed zero is shared by all but one polynomial, for $c = \frac{n}{n-1}$ a fixed zero is shared by all but two polynomials, for $c = \frac{n}{n-2}$ a fixed zero is shared by all but three polynomials, etc., and at every new zero of $\rho_n(c)$ two more zeros of the $B_k^n(t;\bar{a}(c))$ leave the interval $[0,1]$.

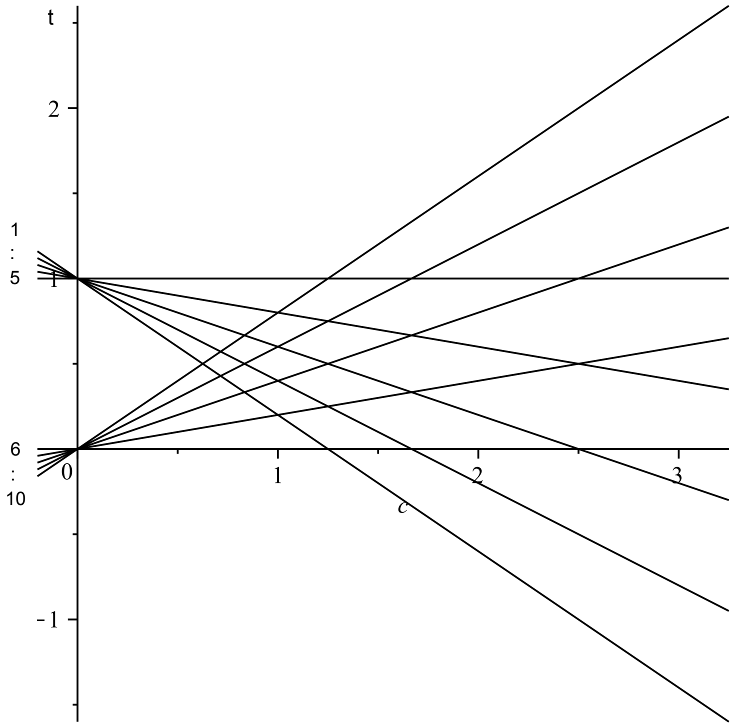
For fixed c the zeros of a single $B_k^n(t;\bar{a}(c))$ determine the position, height and width of its global maximum (and minimum) and therefore the way the Bézier curve is built by the $B_k^n(t;\bar{a}(c))$ (see the subsequent example). $\square$

Example 2.2. Fig.2 shows a sequence of pairs of pictures of the generalized Bernstein polynomials $B_k^3(t;\bar{a}(c))$ and their generalized Bézier curves for different values of c. The control polygon is always the positively oriented regular triangle with start and end point $(1,0) \in \mathbb{R}^2$. The blow-up values are $c = 1.5$, where the Bézier curve leaves the origin in positive vertical direction and returns to the origin from the negative vertical direction, and $c = 3$, where the Bézier curve leaves the origin in negative horizontal direction and returns to the origin from the positive horizontal direction.

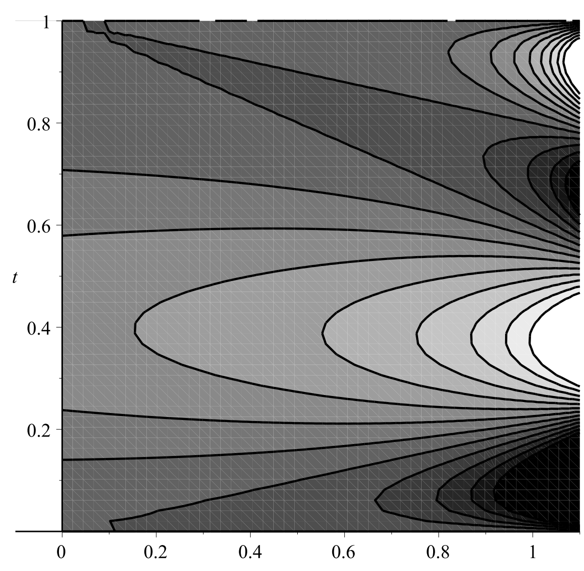
In fact, the Bézier curve stretches in the vertical direction for $c = 1.3$ due to the coupling of a big maximum of B_1^3 with a big minimum of B_2^3 and vice versa. Similarly, the Bézier curve stretches in negative horizontal direction for $c = 2.3$ due to the coupling of a big maximum of B_1^3 with a big minimum of B_3^3 and a big maximum of B_2^3 with a big minimum of B_0^3.

For $c = 3.8$ the Bézier curve flips over into the positive horizontal direction due to the coupling of a big maximum of B_0^3 with a big minimum of B_1^3 and a big maximum of B_3^3 with a big minimum of B_2^3.

Remark 2.3. The proof of Thm. 2.1 showed that the Lagrange polynomials are a "convolution" of the falling factorials. In an analogous way other classical sequences of associated polynomials like exponential polynomials, Gould polynomials, and Abel polynomials (see [14]) could be used to define corresponding generalized Bernstein polynomials. That this does not seem to lead to geometrically interesting families of polynomials is not surprising, because these polynomials have not been designed with a geometric but rather with combinatorial and other applications in mind. But the choice of the parameter sequence $\bar{a}$ of course leaves much room for geometric exploration.

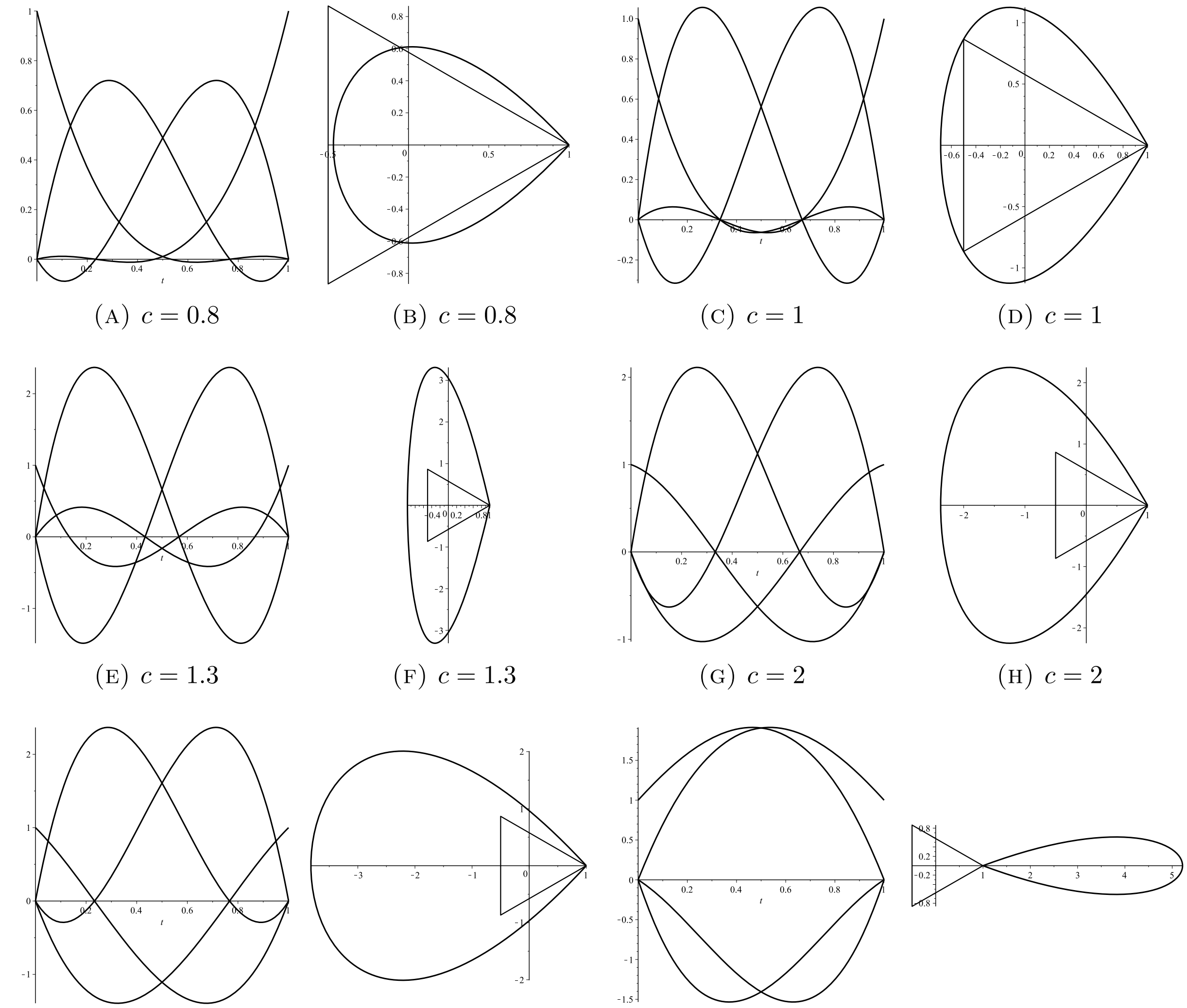


(A) $R(\bar{a}(c))$ for different c



(B) $B_2^5(t;\bar{a}(c))$ with contours from -0.5 (black) to 1.5 (white) and steps of 0.125

FIGURE 1. Zeros and contours for $B_k^5(t;\bar{a}(c))$



(A) $c = 0.8$ (B) $c = 0.8$ (C) $c = 1$ (D) $c = 1$

(E) $c = 1.3$ (F) $c = 1.3$ (G) $c = 2$ (H) $c = 2$

(I) $c = 2.3$ (J) $c = 2.3$ (K) $c = 3.8$ (L) $c = 3.8$

FIGURE 2. $B^3_k(t; \bar{a}(c))$ and their generalized Bézier curves for different c

3. Generalized de Casteljau algorithm

The de Casteljau algorithm

$$\begin{aligned} \mathbf{b}^0_k(t) &:= \mathbf{b}_k && (k = 0, \dots, n) \\ \mathbf{b}^r_k(t) &:= (1-t)\ \mathbf{b}^{r-1}_k(t) + t\ \mathbf{b}^{r-1}_{k+1}(t) && (r = 1 \dots, n; k = 0, \dots, n-r) \end{aligned}$$

generates the Bézier curve $\mathbf{x}(t; \mathbf{b}) = \mathbf{b}^n_0(t)$ by repeated linear interpolation. It can be interpreted as an application of the recursion of Bernstein polynomials

$$B^r_k(t) = (1-t) B^{r-1}_k(t) + t B^{r-1}_{k-1}(t)\ . \tag{3.1}$$

Rewriting the recursion (3.1) with $x = t$, $y = 1 - t$, $p_k(t) = t^k$ and $B_k^n(y) = B_{n-k}^n(x)$ one sees that the essential underlying identities are

$$p_k(x)p_{n-k}(y) = (x\ p_{k-1}(x))\ p_{n-k}(y) = p_k(x)\ (y\ p_{n-k}(y))$$

or independently of the variable

$$p_n(x) = x\ p_{n-1}(x) \quad \text{for } n \in \mathbb{N}\ .$$

This observation shows how to set up the recursion formula for generalized Bernstein polynomials and therefore how to generalize the de Casteljau algorithm appropriately:

For a given parameter sequence $\bar{a}$ with generating function $\bar{f}(t)$ the *umbral shift operator* θ_f generates the sequence of *associated polynomials* of $p_n(x)$ according to

$$p_{n+1}(x) = \theta_f\ p_n(x) \quad \text{for } n \in \mathbb{N} \tag{3.2}$$

where f is the compositional inverse of $\bar{f}$. The general formula for the umbral shift is ([14], Cor. 3.6.6)

$$\theta_f^x\ p_n(x) = x\ \bar{f}'(f(t))p_n(x) = x\ [f'(t)]^{-1}p_n(x)$$

with t acting as differentiation according to $tp(x) := p'(x)$ on any polynomial $p(x)$.

Now with

$$p_k(x)p_{r-k}(y) = (\theta_f p_{k-1}(x))\ p_{r-k-1}(y) = p_k(x)\ (\theta_f p_{r-k}(y))$$

and the linearity of θ_f the recursion for generalized Bernstein polynomials $B_k^r(t;a)$ and therefore a generalized de Casteljau algorithm can be derived from

$$\begin{aligned}\binom{r}{k}p_k(x)p_{n-k}(y) &= \binom{r-1}{k}p_k(x)\ (\theta_f p_{r-1-k}(y)) + \binom{r-1}{k-1}(\theta_f p_{k-1}(x))\ p_{r-k}(y) \\ &= \theta_f^y\ \binom{r-1}{k}p_k(x)\ p_{r-1-k}(y) + \theta_f^x\ \binom{r-1}{k-1}p_{k-1}(x)\ p_{r-k}(y)\end{aligned}$$

where θ_f^x and θ_f^y operate on the variables x and y, respectively.

In fact, in the case of ordinary Bernstein polynomials $f(t) = t = \bar{f}(t)$, $\bar{a} = (1, 0, 0\dots)$, $p_n(x) = x^n$ and $f'(\bar{f}(t)) = 1$. Therefore we arrive at the recursion formula (3.1).

For $f(t) = a^{\alpha t} - 1$ one has $\bar{f}(t) = \alpha^{-1}\log(1+t)$ and the associated polynomials $p_n(x)$ are the lower factorials as discussed in Section 2. In this case one computes

$$\theta_f^x\ p_n(x) = x\ \frac{1}{\alpha}e^{-\alpha t}\prod_{k=0}^{n-1}(\frac{x}{\alpha} - k) = \frac{x}{\alpha}\prod_{k=0}^{n-1}(\frac{x-\alpha}{\alpha} - k) = \frac{x}{\alpha}\prod_{k=1}^{n}(\frac{x}{\alpha} - k) = \prod_{k=0}^{n}(\frac{x}{\alpha} - k)$$

since $e^{-\alpha t}$ acts as shift operator on the variable x according to $e^{-\alpha t}p(x) = p(x-\alpha)$. This gives – in a completely different way – the usual Aitken algorithm for the interpolation polynomial with equidistant nodes.

All in all the approach to generalized Bernstein polynomials and generalized Bézier curves via umbral shifts leads to new theoretical insights and formulas, but concerning efficient evaluation it is in general too involved: executing θ_f, substituting $x = t$ and

$y = 1 - t$, and normalizing by multiplication with ρ_r^{-1}. For applications in CAGD we propose therefore the $\bar{a}$-transformation of the control polygon as discussed in the next section.

4. Computation of generalized Bézier curves

Subsequently we use for fixed $n \in \mathbb{N}$ the vector notation

$$\mathbf{B}(t;\bar{a}) := \begin{pmatrix} B_0^n(t;\bar{a}) \\ B_1^n(t;\bar{a}) \\ \dots \\ B_n^n(t;\bar{a}) \end{pmatrix}, \quad \mathbf{B}(t) := \begin{pmatrix} B_0^n(t) \\ B_1^n(t) \\ \dots \\ B_n^n(t) \end{pmatrix}, \quad \mathbf{t} := \begin{pmatrix} 1 \\ t \\ \dots \\ t^n \end{pmatrix}.$$

Then the representation of the generalized Bernstein polynomials in powers of t is

$$\mathbf{B}(t;\bar{a}) = D(\bar{a})\ \mathbf{t} \tag{4.1}$$

$$D(\bar{a}) = (d_{k,j}(\bar{a})) \in \mathbb{R}^{(n+1)\times(n+1)} \tag{4.2}$$

with the rather unpleasant coefficients [20], (4.10)

$$d_{k,j}(\bar{a}) = \frac{1}{\rho_n}\binom{n}{k}\sum_{i=0}^{\min(j,k)}(-1)^{j-i}p_{k,i}\sum_{h=j-i}^{n-k}p_{n-k,h}\binom{h}{j-i}\ . \tag{4.3}$$

One expects that expressing the generalized Bernstein polynomials in terms of Bernstein polynomials should lead to nicer coefficients, since they share the symmetry

$$B_k^n(t;\bar{a}) = B_{n-k}^n(1-t;\bar{a})\ . \tag{4.4}$$

The representation of powers of t in Bernstein polynomials is

$$\mathbf{t} = T\ \mathbf{B} \tag{4.5}$$

$$T = (t_{k,j}) \in \mathbb{R}^{(n+1)\times(n+1)} \tag{4.6}$$

with the well known coefficients [4], (5.28)

$$t_{k,j} \equiv t_{k,j}^{[n]} = \frac{\binom{j}{k}}{\binom{n}{k}}\ . \tag{4.7}$$

Therefore the representation of generalized Bernstein polynomials in terms of Bernstein polynomials is

$$\mathbf{B}(t;\bar{a}) = M(\bar{a})\ \mathbf{B}(t) \tag{4.8}$$

$$M(\bar{a}) := D(\bar{a})\ T \tag{4.9}$$

$$M(\bar{a}) = (m_{k,j}(\bar{a})) \in \mathbb{R}^{(n+1)\times(n+1)} \tag{4.10}$$

where a "nice" formula for the coefficient $m_{k,j}(\bar{a})$ of $B_j^n(t)$ in $B_k^n(t;\bar{a})$ is to be determined – a direct evaluation of the matrix product $D(\bar{a})\ T$ does not look promising.

Before doing this we make however a crucial observation on the impact of the last formula for generalized Bézier curves. Interpreting the sequence of control points as a matrix

$$\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathbb{R}^{N\times(n+1)}$$

the definition (1.11) of generalized Bézier curves can be rewritten

$$\mathbf{x}(t;\bar{a};\mathbf{b}) = \mathbf{b}\ \mathbf{B}(t;\bar{a}) \tag{4.11}$$

$$= \mathbf{b}\ M(\bar{a})\ \mathbf{B}(t)\ . \tag{4.12}$$

The following definitions are now natural.

Definition 4.1. *For given $n \in \mathbb{N}$ a sequence of real parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ is called* feasible, *if $\bar{a}_1 \neq 0$ and $\rho_n = p_n(1) \neq 0$ where p_n is computed via* (1.6-8)*.*

Definition 4.2. *For a given sequence of control points $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ and a feasible sequence of parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ the sequence of control points*

$$\mathbf{b}(\bar{a}) = (\mathbf{b}(\bar{a})_0, \mathbf{b}(\bar{a})_1, \ldots, \mathbf{b}(\bar{a})_n)$$

computed by

$$\mathbf{b}(\bar{a}) := \mathbf{b}\ M(\bar{a}) \in \mathbb{R}^{N\times(n+1)} \tag{4.13}$$

is called an $\bar{a}$-transformation *of* $\mathbf{b}$*. $C(\mathbf{b}(\bar{a}))$ is called the* $\bar{a}$-transformation *of the control polygon $C(\mathbf{b})$.*

Theorem 4.3. *The generalized Bézier curve for a given sequence of control points $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n)$ and a feasible sequence of parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ can be computed as (ordinary) Bézier curve of the $\bar{a}$-transformed sequence $\mathbf{b}(\bar{a})$ of control points:*

$$\mathbf{x}(t;\bar{a};\mathbf{b}) = \mathbf{x}(t;\mathbf{b}(\bar{a}))\ . \tag{4.14}$$

Proof. Immediate from (4.11-13) and the rewritten formula (1.1)

$$\mathbf{x}(t;\mathbf{b}(\bar{a})) = \mathbf{b}(\bar{a})\ \mathbf{B}(t)\ .$$

□

Now the fast algorithms for the evaluation of Bézier curves can be put to use as soon as the matrix $M(\bar{a})$ is computed. This can be done directly – without using (4.9) – with the formula given by the corollary of the following more general theorem whose proof is deferred to the appendix:

Theorem 4.4. *Given polynomials* $p_k(t) = \sum_{\nu=0}^{k} p_{k,\nu} t^\nu$ *and* $q_k(t) = \sum_{\mu=0}^{k} q_{k,\mu} t^\mu$ *for* $k = 0, \ldots, n$, *respectively. Then the "convoluted" polynomials* $s_k^n(t)$ *below can be expressed as linear combination of Bernstein polynomials*

$$s_k^n(t) := \binom{n}{k} p_k(t) q_{n-k}(1-t) = \sum_{l=0}^{n} m_{k,l} B_l^n(t) \tag{4.15}$$

with

$$m_{k,l} = \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{j=0}^{k} \pi_{k,j}\ \pi'_{n-k,n-k-l+j} \tag{4.16}$$

$$= \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{i=0}^{n-k} \pi_{k,k+i+l-n}\ \pi'_{n-k,i} \tag{4.17}$$

where

$$\pi_{n,k} := \sum_{\nu=0}^{k} \binom{n-\nu}{n-k} p_{n,\nu} \quad \text{and} \quad \pi'_{n,k} := \sum_{\nu=0}^{k} \binom{n-\nu}{n-k} q_{n,\nu}\ . \tag{4.18}$$

Corollary 4.5. *For a given feasible sequence of parameters* $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ *the matrix* $M(\bar{a}) = (m_{k,l}(\bar{a})) \in \mathbb{R}^{(n+1)\times(n+1)}$ *from* (4.10) *for the* $\bar{a}$*-transformation of control points can be computed by:*

$$\rho_n\ m_{k,l}(\bar{a}) = \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{j=0}^{k} \pi_{k,j}\ \pi_{n-k,n-k-l+j} \tag{4.19}$$

$$= \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{i=0}^{n-k} \pi_{k,k+i+l-n}\ \pi_{n-k,i} \tag{4.20}$$

where

$$\pi_{n,k} := \sum_{\nu=0}^{k} \binom{n-\nu}{n-k} p_{n,\nu} \text{ with } p_{n,0} = \delta_{n,0} \text{ (using the Kronecker symbol)}\ . \tag{4.21}$$

Remark 4.6. The matrix product $M(\bar{a}')(M(\bar{a}))^{-1}$ transforms the control points $\mathbf{b}(\bar{a})$ for $\bar{a}$ directly to the control points $\mathbf{b}(\bar{a}')$ for some different $\bar{a}'$.
Also – since the inverse of T is well known – the matrix product $T^{-1}(M(\bar{a}))^{-1}$ gives an explicit representation for the ordinary powers of t in terms of the generalized Bernstein polynomials $B_k^n(t; \bar{a})$.
It would therefore be helpful to have a formula for the inverse matrix $(M(\bar{a}))^{-1}$ (with unevaluated variables $\bar{a}_1, \ldots, \bar{a}_n$). However, there does not seem to exist such a simple formula.

We revisit now some of the generalized Bézier curves from [20] in the light of this new computational approach. Every picture below for a given parameter sequence shows four curves

- the control polygon $C(\mathbf{b})$ a regular n-gon starting at point (1,0), running counterclockwise, and ending at point (1,0) (thin line)
- the drop-like Bézier curve $\mathbf{x}(t;\mathbf{b})$ following $C(\mathbf{b})$ (dotted line)
- the generalized Bézier curve $\mathbf{x}(t;\bar{a};\mathbf{b})$ for $\bar{a}$ and $C(\mathbf{b})$ (thick line)
- the $\bar{a}$-transformed control polygon $C(\mathbf{b}(\bar{a}))$ for which $\mathbf{x}(t;\bar{a};\mathbf{b})$ is nothing but the ordinary Bézier curve $\mathbf{x}(t;\mathbf{b}(\bar{a}))$ (thick line)

Note that $C(\mathbf{b}(\bar{a}))$ is a computational device behind the scenes and is not displayed in practice.

Figs. 3 (A), (B) show that the generalized Bézier curves can be stiffer than ordinary Bézier curves , whereas (C) shows a very responsive almost interpolating generalized Bézier curve.

Figs. 3 (D), (E), (F) show generalized Bézier curves that are attracted or repelled by different parts of the control polygon.

Fig. 4 shows two well possible but odd variants of generalized Bézier curves: The first is an example that generalized Bézier curves need not be variation diminishing and the second is an example for the curve flipping over when a zero of ρ_n is passed.

As expected in each case the generalized Bézier curve mimics closely the $\bar{a}$-transformed control polygon, since the generalized Bézier curve is the Bézier curve for the $\bar{a}$-transformed control polygon.

5. Some properties of $M(\bar{a})$ and the parameter sequence $\bar{a}$

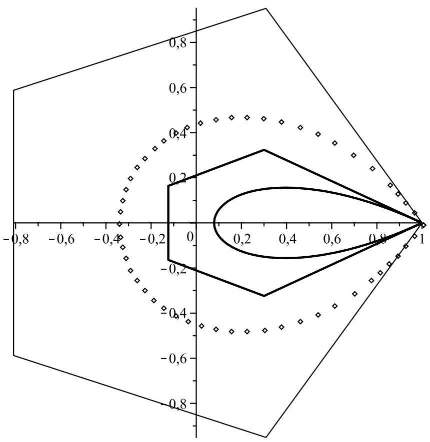
Since the entries of $M(\bar{a})$ are calculated as a convolution of the $\pi_{n,k}$'s that are binomially weighted sums of the $p_{n,k} \in \mathbb{R}[\bar{a}] = \mathbb{R}[\bar{a}_1, \ldots, \bar{a}_n]$, we collect first some properties of the Bell polynomials $p_{n,k}$ and the derived $\pi_{n,k}$, before proceeding to the matrix entries $m_{n,k}$.

For the convenience of the reader we show a table of the $p_{n,k}$ for $n, k = 0, \ldots, 5$

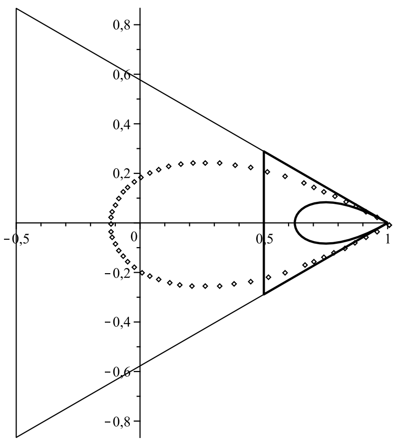
1	0	0	0	0	0
0	$\bar{a}_1$	0	0	0	0
0	$\bar{a}_2$	$\bar{a}_1^2$	0	0	0
0	$\bar{a}_3$	$3\,\bar{a}_2\bar{a}_1$	$\bar{a}_1^3$	0	0
0	$\bar{a}_4$	$4\,\bar{a}_3\bar{a}_1 + 3\,\bar{a}_2^2$	$6\,\bar{a}_2\bar{a}_1^2$	$\bar{a}_1^4$	0
0	$\bar{a}_5$	$5\,\bar{a}_4\bar{a}_1 + 10\,\bar{a}_3\bar{a}_2$	$10\,\bar{a}_1^2\bar{a}_3 + 15\,\bar{a}_2^2\bar{a}_1$	$10\,\bar{a}_2\bar{a}_1^3$	$\bar{a}_1^5$

Obviously by definition (and (1.8)) one has $p_{n,0} = \delta_{n,0}$, $p_{n,1} = \bar{a}_n$, $\quad p_{n,n} = \bar{a}_1^n$, and $p_{n,n-1} = \binom{n+1}{2}\bar{a}_1^{n-1}\bar{a}_2$.

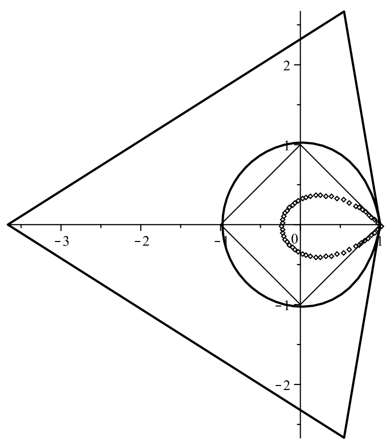
The table of the $\pi_{n,k}$ for $n, k = 0, \ldots, 4$ is



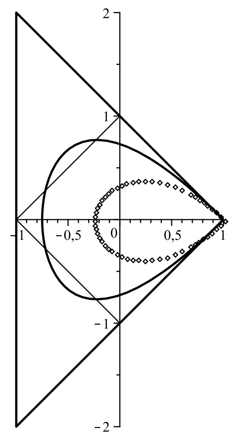
(A) $\bar{a} = (1, 1, 1, 1, 1)$



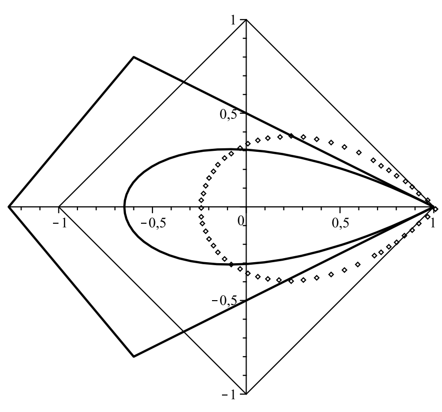
(B) $\bar{a} = (-1, -1, 1)$



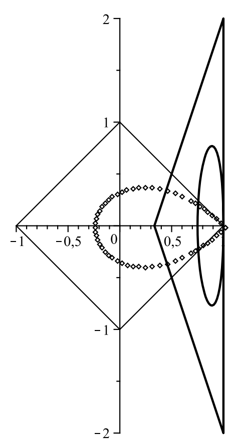
(C) $\bar{a} = (4, -3, 3, -1)$



(D) $\bar{a} = (1, -1, 0, 1)$

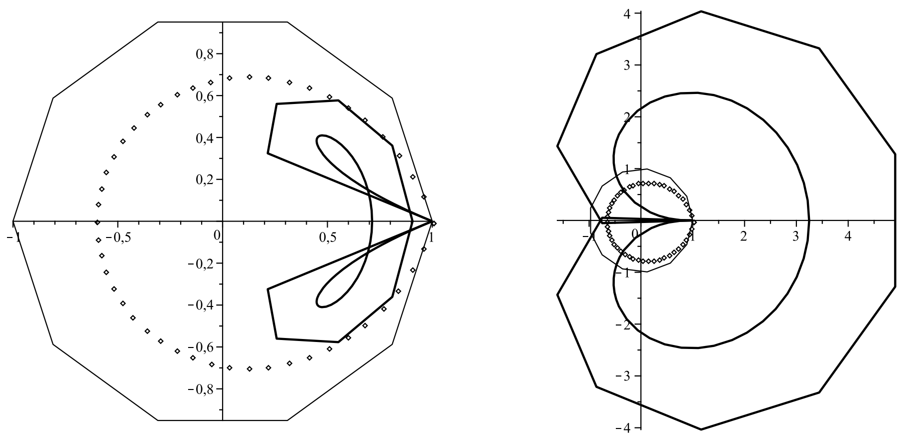


(E) $\bar{a} = (1, 1, -1, -1)$



(F) $\bar{a} = (1, -1, 1, -1)$

FIGURE 3. Bézier curves, generalized Bézier curves, and their control polygonials for different parameter sequences $\bar{a}$



(A) $\bar{a} = (1, -1, 1, -1, 1, -1, 1, -1, 1, -1)$

(B) $\bar{a} = (7, -4, 2, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$

FIGURE 4. Two odd variants

$$\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
0 & \bar{a}_1 & 0 & 0 & 0 \\
0 & \bar{a}_2 & \rho_2 & 0 & 0 \\
0 & \bar{a}_3 & 2\,\bar{a}_3 + 3\,\bar{a}_2\bar{a}_1 & \rho_3 & 0 \\
0 & \bar{a}_4 & 3\,\bar{a}_4 + 4\,\bar{a}_3\bar{a}_1 + 3\,\bar{a}_2^2 & 3\,\bar{a}_4 + 8\,\bar{a}_3\bar{a}_1 + 6\,\bar{a}_2^2 + 6\,\bar{a}_2\bar{a}_1^2 & \rho_4
\end{array}$$

where $\rho_0 = 1$, $\rho_1 = \bar{a}_1$, $\rho_2 = \bar{a}_2 + \bar{a}_1^2$, $\rho_3 = \bar{a}_3 + 3\,\bar{a}_2\bar{a}_1 + \bar{a}_1^3$ and $\rho_4 = \bar{a}_4 + 4\,\bar{a}_3\bar{a}_1 + 3\,\bar{a}_2^2 + 6\,\bar{a}_2\bar{a}_1^2 + \bar{a}_1^4$.

By the definition (4.21) of $\pi_{n,k}$, and $\rho_n = p_n(1) = \sum_{k=1}^{n} p_{n,k}$ one sees $\pi_{n,0} = p_{n,0} = \delta_{n,0}$, $\pi_{n,1} = p_{n,1} = \bar{a}_n$ and $\pi_{n,n} = \rho_n$.

We are now prepared for a closer examination of the properties of the matrix $M(\bar{a})$.

Theorem 5.1. *For a feasible sequence of parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ the $\bar{a}$-transformation matrix $M(\bar{a})$ is* point-symmetric *that means*

$$m_{k,l}(\bar{a}) = m_{n-k,n-l}(\bar{a}) \quad \textit{for } k, l = 0, \ldots, n. \tag{5.1}$$

Proof. Replacing k by $n-k$ and l by $n-l$ in (4.19) (and j by i) gives (4.20) which equals (4.19).

Alternatively, recall that $m_{k,l}(\bar{a})$ is the coefficient of $B_l^n(t)$ in $B_k^n(t; \bar{a})$, hence $m_{n-k,n-l}(\bar{a})$ is the coefficient of $B_{n-l}^n(t)$ in $B_{n-k}^n(t; \bar{a})$. Now the symmetry (4.4) of generalized Bernstein polynomials implies the point-symmetry of $M(\bar{a})$. □

Directly from the definitions and the previous special results for $\pi_{n,k}$ and $p_{n,k}$ one computes

$$m_{k,0}(\bar{a}) = \delta_{k,0} \tag{5.2}$$

$$\rho_n \; m_{0,l}(\bar{a}) = \frac{1}{\binom{n}{l}} \; \pi_{n,n-l} \text{ for } l > 0 \tag{5.3}$$

$$\rho_n \; m_{k,1}(\bar{a}) = \frac{\binom{n}{k}}{n} \; \bar{a}_k \; \rho_{n-k} \text{ for } k > 0 \tag{5.4}$$

For $n = 2$ and 3 the matrix $\rho_n \; M(\bar{a})$ is

$$\begin{bmatrix} \bar{a}_2 + \bar{a}_1^2 & 1/2\,\bar{a}_2 & 0 \\ 0 & \bar{a}_1^2 & 0 \\ 0 & 1/2\,\bar{a}_2 & \bar{a}_2 + \bar{a}_1^2 \end{bmatrix}$$

$$\begin{bmatrix} \bar{a}_3 + 3\,\bar{a}_2\bar{a}_1 + \bar{a}_1^3 & 2/3\,\bar{a}_3 + \bar{a}_2\bar{a}_1 & 1/3\,\bar{a}_3 & 0 \\ 0 & \bar{a}_2\bar{a}_1 + \bar{a}_1^3 & \bar{a}_2\bar{a}_1 & 0 \\ 0 & \bar{a}_2\bar{a}_1 & \bar{a}_2\bar{a}_1 + \bar{a}_1^3 & 0 \\ 0 & 1/3\,\bar{a}_3 & 2/3\,\bar{a}_3 + \bar{a}_2\bar{a}_1 & \bar{a}_3 + 3\,\bar{a}_2\bar{a}_1 + \bar{a}_1^3 \end{bmatrix}$$

and for $n = 4$ (only columns 1 and 2 for reasons of space)

$$\begin{bmatrix} 3/4\,\bar{a}_4 + 2\,\bar{a}_3\bar{a}_1 + 3/2\,\bar{a}_2^2 + 3/2\,\bar{a}_2\bar{a}_1^2 & 1/2\,\bar{a}_4 + 2/3\,\bar{a}_3\bar{a}_1 + 1/2\,\bar{a}_2^2 \\ \bar{a}_3\bar{a}_1 + 3\,\bar{a}_2\bar{a}_1^2 + \bar{a}_1^4 & 4/3\,\bar{a}_3\bar{a}_1 + 2\,\bar{a}_2\bar{a}_1^2 \\ 3/2\,\bar{a}_2^2 + 3/2\,\bar{a}_2\bar{a}_1^2 & 2\,\bar{a}_2^2 + 2\,\bar{a}_2\bar{a}_1^2 + \bar{a}_1^4 \\ \bar{a}_3\bar{a}_1 & 4/3\,\bar{a}_3\bar{a}_1 + 2\,\bar{a}_2\bar{a}_1^2 \\ 1/4\,\bar{a}_4 & 1/2\,\bar{a}_4 + 2/3\,\bar{a}_3\bar{a}_1 + 1/2\,\bar{a}_2^2 \end{bmatrix}$$

A further elementary property of the matrix $M(\bar{a})$ is the following

Theorem 5.2. *For every feasible $\bar{a}$ the sum of entries of any column of $M(\bar{a})$ is 1:*

$$\sum_{k=0}^{n} m_{k,l}(\bar{a}) = 1 \quad \text{for } l = 0 \ldots n. \tag{5.5}$$

Proof. The partition of unity property $\sum_{k=0}^{n} B_k^n(t;\bar{a}) = 1$ implies

$$\sum_{k=0}^{n} d_{k,j}(\bar{a}) = \delta_{0,j} \ ,$$

since $d_{k,j}(\bar{a})$ is the coefficient of t^j in $B_k^n(t;\bar{a})$ according to (4.2-3). Then

$$\sum_{k=0}^{n} m_{k,l}(\bar{a}) \overset{(2.9)}{=} \sum_{k=0}^{n}\sum_{j=0}^{n} d_{k,j}(\bar{a}) t_{j,l} = \sum_{j=0}^{n} t_{j,l} \sum_{k=0}^{n} d_{k,j}(\bar{a}) = \sum_{j=0}^{n} t_{j,l}\ \delta_{0,j} = t_{0,l} \overset{(2.7)}{=} 1 \ .$$

□

A more in depth investigation of the combinatorial properties of the coefficients, eigenvalues and eigenvectors of the matrix $M(\bar{a})$ will be the subject of a different paper.

Next we address a point that facilitates greatly the computation of the matrices $M(\bar{a})$. Namely that of equivalence classes of parameter sequences $\bar{a}$ which have identical generalized Bernstein polynomials.

Definition 5.3. *Two feasible parameter sequences $\bar{a}$ and $\bar{a}'$ of the same length n are called* equivalent*, $\bar{a} \sim \bar{a}'$, if and only if $B^n_k(t;\bar{a}) = B^n_k(t;\bar{a}')$ for $k = 0, \ldots, n$.*
A parameter sequence $\bar{a} = (1, \bar{a}_2, \ldots, \bar{a}_n)$ is called normalized.

The following theorem yields a practical way to describe and represent the equivalence classes of parameter sequences.

Theorem 5.4. *Let $\bar{a}$ be a feasible parameter sequence. Then*

$$\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n) \sim \bar{a}' = (1, \frac{\bar{a}_2}{\bar{a}_1^2}, \ldots, \frac{\bar{a}_n}{\bar{a}_1^n}) \tag{5.6}$$

$$\bar{a} = (1, \bar{a}_2, \ldots, \bar{a}_n) \sim \bar{a}' = (1, \bar{a}'_2, \ldots, \bar{a}'_n) \implies \bar{a}_k = \bar{a}'_k \quad \text{for } k = 2, \ldots, n, \tag{5.7}$$

In words: For every feasible parameter sequence there exists an equivalent normalized parameter sequence, and two equivalent normalized feasible parameter sequences are identical.

Proof. For (5.6) one computes

$$p_{n,k}(\bar{a}') \overset{(1.7)}{=} \frac{1}{k!} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \binom{n}{i_1, \ldots, i_k} \bar{a}'_{i_1} \cdot \cdots \cdot \bar{a}'_{i_k}$$

$$= \frac{1}{k!} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \binom{n}{i_1, \ldots, i_k} \frac{1}{\bar{a}_1^n} \bar{a}_{i_1} \cdot \cdots \cdot \bar{a}_{i_k} = \frac{1}{\bar{a}_1^n} p_{n,k}(\bar{a})$$

$$p_n(t;\bar{a}') \overset{(1.6)}{=} \sum_{k=1}^{n} p_{n,k}(\bar{a}')\, t^k = \frac{1}{\bar{a}_1^n} p_n(t;\bar{a})$$

$$\rho_n(\bar{a}') = \sum_{k=1}^{n} p_{n,k}(\bar{a}') = \frac{1}{\bar{a}_1^n} \sum_{k=1}^{n} p_{n,k}(\bar{a}) = \frac{1}{\bar{a}_1^n} \rho_n(\bar{a})$$

which implies $B^n_k(t;\bar{a}) = B^n_k(t;\bar{a}')$ for $k = 0, \ldots, n$ by the definition (1.3) of generalized Bernstein polynomials.

For the proof of (5.7) we assume that $\bar{a}_1 = 1$. Clearly, $B^n_k(t;\bar{a}) = B^n_k(t;\bar{a}')$ for $k = 0, \ldots, n$ iff the generalized Bernstein polynomials have the same coefficients. Since the coefficients are polynomials in $\mathbb{Z}[\bar{a}_2, \ldots, \bar{a}_n]$ it is possible in principle that different normalized parameter sequences yield the same coefficients. To demonstrate that this

is not the case assume first that $\rho_n(\bar{a}) = \rho_n(\bar{a}')$.
From (1.7) one concludes that $p_{n,n}(\bar{a}) = \bar{a}_1^n = 1$ and for $k < n$:

$$\begin{aligned} p_{n,k}(\bar{a}) =& \frac{1}{(k-1)!}\frac{n!}{1!\dots 1!(n-(k-1)!)}\bar{a}_1\dots\bar{a}_1\bar{a}_{n-(k-1)} + q_k^n \\ =& \binom{n}{k-1}\bar{a}_{n-k+1} + q_k^n \text{ with } q_k^n \in \mathbb{Z}[\bar{a}_2,\dots,\bar{a}_{n-k}] \end{aligned}$$

Then

$$\begin{aligned} \rho_n(\bar{a})\ B_{n,k}(t;\bar{a}) =& p_0(t;\bar{a})\ p_n(1-t;\bar{a}) \\ =& p_n(1-t;\bar{a}) = \sum_{k=1}^{n} p_{n,k}(\bar{a})(1-t)^k \\ =& \sum_{k=1}^{n}\left[(-1)^k p_{n,k}(\bar{a})t^k + p_{n,k}(\bar{a})\ q_k(t)\right] \quad \text{with } \deg q_k < k \end{aligned}$$

Therefore the coefficient of t^k in $\rho_n B_{n,k}(t;\bar{a})$ is

$$\begin{aligned} &(-1)^k p_{n,k}(\bar{a}) + \overline{q}_k \text{ with } \overline{q}_k \in \mathbb{Z}[p_{n,k+1}(\bar{a}),\dots,p_{n,n}(\bar{a})] \\ =& (-1)^k\binom{n}{k-1}\bar{a}_{n-k+1} + \overline{\overline{q}}_k \text{ with } \overline{\overline{q}}_k \in \mathbb{Z}[\bar{a}_2,\dots,\bar{a}_{n-k}] \end{aligned}$$

If now k_0 is the smallest k such that $\bar{a}_k \neq \bar{a}'_k$, then the coefficients of t^{n-k_0+1} in $\rho_n(\bar{a})\ B_{n,k}(t;\bar{a})$ and $\rho_n(\bar{a}')\ B_{n,k}(t;\bar{a}')$ are different.
If $\rho_n(\bar{a}) \neq \rho_n(\bar{a}')$, then the coefficients of t^n in $B_{n,k}(t;\bar{a})$ and $B_{n,k}(t;\bar{a}')$ are different. □

We close with results about endpoint tangentiality of generalized Bézier curves .

Theorem 5.5. *The generalized Bernstein polynomials $B_{n,k}(t;\bar{a})$ of degree n possess endpoint tangentiality iff the the parameter sequence $\bar{a}$ satisfies*

$$\bar{a}_k\rho'_{n-k} = 0 \quad \text{for } k = 2,\dots,n, \tag{5.8}$$

where ρ'_{n-k} results from ρ_{n-k} by setting $\bar{a}_1 = 1$. For every $n \geq 4$ there exists such a parameter sequence different from $\bar{a} = (1,0,\dots,0)$.

Proof. By symmetry we need to investigate endpoint tangentiality only at $t = 0$ which means that the vector of control points $\mathbf{b}_1 - \mathbf{b}_0$ ($\neq$ null vector) and the vector of $\bar{a}$-transformed control points $\mathbf{b}(\bar{a})_1 - \mathbf{b}(\bar{a})_0$ are linearly dependent. In other words

$$\exists\alpha \in \mathbb{R}\setminus\{0\} : \mathbf{b}(\bar{a})_1 - \mathbf{b}(\bar{a})_0 = \alpha(\mathbf{b}_1 - \mathbf{b}_0)\ .$$

Now

$$\mathbf{b}(\bar{a})_1 - \mathbf{b}(\bar{a})_0 \overset{(4.13)}{=} \sum_{k=0}^{n} \mathbf{b}_k \, (m_{k,l}(\bar{a}) - m_{k,0}(\bar{a}))$$
$$\overset{(5.2)}{=} \mathbf{b}_0 \, (m_{k,l} - 1) + \sum_{k=1}^{n} \mathbf{b}_k m_{k,l}(\bar{a}) \; .$$

Independence from the choice of the $\mathbf{b}_k$ implies first $m_{k,l}(\bar{a}) = 0$ for $k = 2, \ldots, n$ which by (5.4) and $\bar{a}_1 = 1$ is (5.8) and second $m_{1,1}(\bar{a})\mathbf{b}_1 - (1 - m_{0,1}(\bar{a}))\, \mathbf{b}_0 = \alpha(\mathbf{b}_1 - \mathbf{b}_0)$ or $m_{1,l}(\bar{a}) = 1 - m_{0,l}$. Since the latter is already implied by (5.3), we do not have any further restriction on the parameters beyond (5.8).

Note that $\rho'_0 = \rho'_1 = 1$ implies $\bar{a}_n = \bar{a}_{n-1} = 0$.

We finish the proof by showing that the alternating factorial parameter sequence $\bar{a} = (1, -1, 2, -6, \ldots, (-1)^n(n-3)!, 0, 0)$ satisfies (5.8) for every $n \geq 4$. The infinite parameter sequence $\bar{a} = (1, -1, 2, -6, \ldots, (-1)^n(n-1)!, \ldots)$ is associated to $\bar{f}(t) = \ln(1+t)$ according to (1.9). Therefore the sequence of associated polynomials vanishes for $n \geq 2$ at $x = 1$ according to (1.10). But then by definition $\rho'_{n-k} = 0$ for $k = 2, \ldots, n-2$ in (5.8) and for $k = n-1$ and n we noted already $\bar{a}_n = \bar{a}_{n-1} = 0$. □

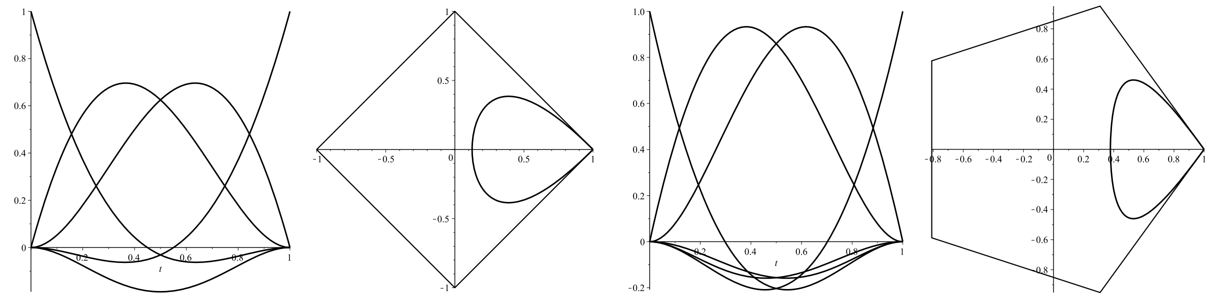
For $n \leq 3$ Eqs. (5.8) do not have non-trivial solutions (different from the Bernstein case $\bar{a} = (1, 0, \ldots, 0)$), but for $n = 4$ we get $\bar{a} = (1, -1, 0, 0)$ (see Figs. 5 (A),(B)) and for $n = 5$ the two sequences $\bar{a} = (1, -1, 2, 0, 0)$ (see Figs. 5 (C),(D)) and $\bar{a} = (1, -1/3, 0, 0, 0)$ (see Figs. 5 (E),(F)).

For $n = 6$ the following non-trivial parameter sequences exist:

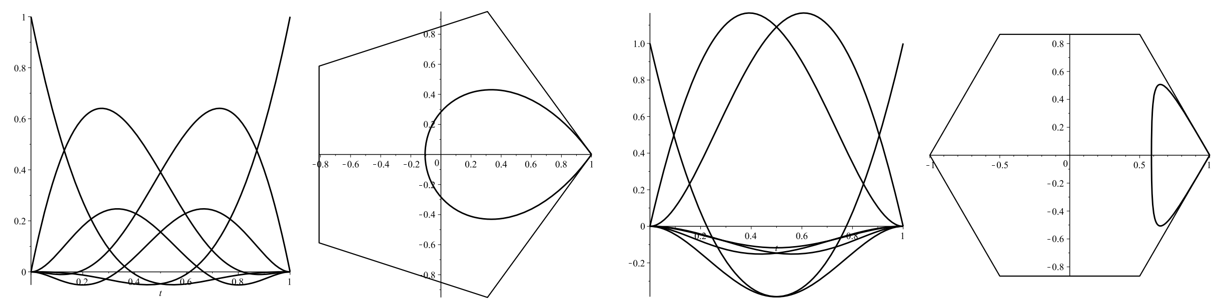
(1) $(1, -1 + \sqrt{6}/3, 0, 0, 0, 0)$ see Figs. 5 (I),(J)
(2) $(1, -1 - \sqrt{6}/3, 0, 0, 0, 0)$ see Figs. 5 (K),(L)
(3) $(1, 1 - \sqrt{2}, -3(1 - \sqrt{2}) - 1, 0, 0, 0)$
(4) $(1, 1 + \sqrt{2}, -3(1 + \sqrt{2}) - 1, 0, 0, 0)$
(5) $(1, 0, -1, 0, 0, 0)$
(6) $(1, -1, 0, 2, 0, 0)$
(7) $(1, -1, 2, -6, 0, 0)$ see Figs. 5 (G),(H)

and for $n = 7$ the following ones:

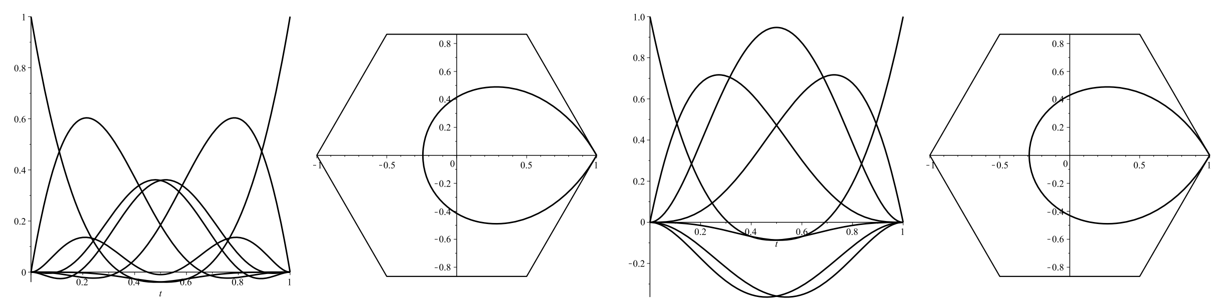
(1) $(1, -1/3 - \sqrt{10}/15, 0, 0, 0, 0, 0)$
(2) $(1, -1/3 + \sqrt{10}/15, 0, 0, 0, 0, 0)$
(3) $(1, -.2454621693, 0.07300449660, 0, 0, 0, 0)$ (approximate numbers)
(4) $(1, 0, -1/4, 0, 0, 0, 0)$
(5) $(1, -1/\sqrt{5}, -1 + 3/\sqrt{5}, 12/5 - 6/\sqrt{5}, 0, 0, 0)$
(6) $(1, 1/\sqrt{5}, -1 - 3/\sqrt{5}, 12/5 + 6/\sqrt{5}, 0, 0, 0)$
(7) $(1, 0, -1, 3, 0, 0, 0)$
(8) $(1, -1/3, 0, 2/15, 0, 0, 0)$



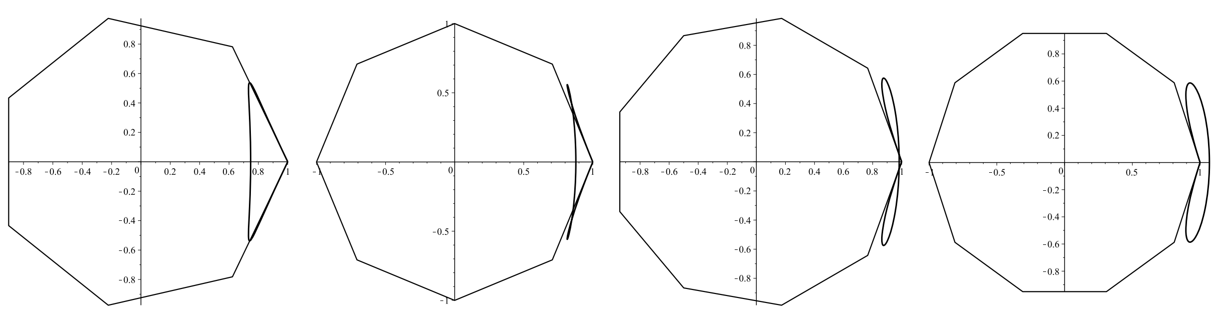
(A) $(1, -1, 0, 0)$ (B) $(1, -1, 0, 0)$ (C) $(1, -1, 2, 0, 0)$ (D) $(1, -1, 2, 0, 0)$



(E) $(1, -1/3, 0, 0, 0)$ (F) $(1, -1/3, 0, 0, 0)$ (G) $(1, -1, 2, -6, 0, 0)$ (H) $(1, -1, 2, -6, 0, 0)$



(I) $(1, -1 + \sqrt{6}/3, 0, 0, 0, 0)$ (J) $(1, -1 + \sqrt{6}/3, 0, 0, 0, 0)$ (K) $(1, -1 - \sqrt{6}/3, 0, 0, 0, 0)$ (L) $(1, -1 - \sqrt{6}/3, 0, 0, 0, 0)$



(M) $(1, -1, 2, -6, 24, 0, 0)$ (N) $(1, -1, \ldots, -120, 0, 0)$ (O) $(1, -1, \ldots, 720, 0, 0)$ (P) $(1, -1, \ldots, -5040, 0, 0)$

Figure 5. Some generalized Bernstein polynomials with endpoint tangentiality and their generalized Bézier curves

(9) $(1,-1,0,0,-6,0,0)$
(10) $(1,-1,1/2,0,-6,0,0)$
(11) $(1,-1,2,-6,24,0,0)$ see Fig. 5 (M)

For $n=8,9$ and 10 MAPLE delivers 25, 57 and 98 non-trivial solutions, respectively, of (5.8).

Note how the generalized Bézier curves of the alternating factorial parameter sequence develop for growing n: Figs. 5 (D),(H),(M),(N),(O),(P) for $n=5,\ldots,10$.

6. Appendix: Two proofs of Theorem 4.4

Direct proof (conceptually simple but tedious):

Proof. Essentially one drills down from $s_k^n(t)$ to monomials in t back to Bernstein polynomials .

$$
\begin{aligned}
s_k^n(t) &= \binom{n}{k} p_k(t) q_{n-k}(1-t) \\
&= \binom{n}{k} \left[\sum_{\nu=0}^{k} p_{k,\nu} t^\nu\right] \cdot \left[\sum_{\mu=0}^{n-k} q_{n-k,\mu}(1-t)^\mu\right] \\
&\overset{(4.5)}{=} \binom{n}{k} \left[\sum_{\nu=0}^{k} p_{k,\nu} \left(\sum_{j=\nu}^{k} t_{\nu,j}^{[k]} B_j^k(t)\right)\right] \cdot \left[\sum_{\mu=0}^{n-k} q_{n-k,\mu} \left(\sum_{i=\mu}^{n-k} t_{\mu,i}^{[n-k]} B_i^{n-k}(1-t)\right)\right] \\
&\overset{(4.4)}{=} \binom{n}{k} \sum_{\nu=0}^{k} \sum_{\mu=0}^{n-k} p_{k,\nu} q_{n-k,\mu} \left(\sum_{j=\nu}^{k} t_{\nu,j}^{[k]} B_j^k(t)\right) \cdot \left(\sum_{i=\mu}^{n-k} t_{\mu,i}^{[n-k]} B_{n-k-i}^{n-k}(t)\right) \\
&= \binom{n}{k} \sum_{\nu=0}^{k} \sum_{\mu=0}^{n-k} p_{k,\nu} q_{n-k,\mu} \left(\sum_{j=\nu}^{k} \sum_{i=\mu}^{n-k} t_{\nu,j}^{[k]}\, t_{\mu,i}^{[n-k]} B_j^k(t) B_{n-k-i}^{n-k}(t)\right)
\end{aligned}
$$

From the definition (1.2) of Bernstein polynomials one calculates

$$
\begin{aligned}
t_{\nu,j}^{[k]}\, t_{\mu,i}^{[n-k]}\, B_j^k(t) B_{n-k-i}^{n-k}(t) &\overset{(4.7)}{=} \frac{\binom{j}{\nu}}{\binom{k}{\nu}} \cdot \frac{\binom{i}{\mu}}{\binom{n-k}{\mu}} \cdot \binom{k}{j} \binom{n-k}{n-k-i} \cdot t^{n-(k+i-j)} (1-t)^{k+i-j} \\
&= \binom{k-\nu}{j-\nu} \binom{n-k-\mu}{i-\mu} \cdot t^{n-(k+i-j)} (1-t)^{k+i-j} \\
&\overset{(4.4)}{=} \binom{k-\nu}{k-j} \binom{n-k-\mu}{n-k-i} \frac{1}{\binom{n}{n-(k+i-j)}}\, B_{n-(k+i-j)}^n(t)
\end{aligned}
$$

and concludes with $l := n - (k + i - j)$

$$s_k^n(t) = \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{\nu=0}^{k} \sum_{\mu=1}^{n-k} p_{k,\nu} q_{n-k,\mu} \sum_{j=\nu}^{k} \sum_{i=\mu}^{n-k} \binom{k-\nu}{k-j} \binom{n-k-\mu}{n-k-i} B_l^n(t)$$

$$= \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{\nu=0}^{k} \sum_{j=\nu}^{k} \binom{k-\nu}{k-j} p_{k,\nu} \sum_{\mu=0}^{n-k} \sum_{i=\mu}^{n-k} \binom{n-k-\mu}{n-k-i} q_{n-k,\mu} B_l^n(t)$$

$$= \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{j=0}^{k} \sum_{\nu=0}^{j} \binom{k-\nu}{k-j} p_{k,\nu} \sum_{i=0}^{n-k} \sum_{\mu=0}^{i} \binom{n-k-\mu}{n-k-i} q_{n-k,\mu} B_l^n(t)$$

Using the abbreviations (4.18) one computes

$$s_k^n(t) = \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{j=0}^{k} \sum_{i=0}^{n-k} \pi_{k,j} \pi'_{n-k,i} B_l^n(t)$$

Since n, k, l are fixed and $l := n - (k + i - j)$ either the choice of i determines j and the sum over j collapses or the choice of j determines i and the sum over i collapses. This gives (4.16-17). □

Second proof with blossoming [13, 17]:

Proof. For a polynomial $p(t)$ of degree n the *blossom* $f_p(u_1, \dots, u_n)$ can be generated by replacing the monomials t^ν of $p(t)$ by the polynomials $e_\nu(u_1, \cdots, u_n)/\binom{n}{\nu}$ where e_ν is the elementary symmetric polynomial of degree ν in the n variables $u_1, \cdots, u_n$. Clearly, f_p is then the unique multi-affine, symmetric polynomial that coincides with p on the diagonal: $f_p(t, \dots, t) = p(t)$.

It is well-known ([13, 17, 4]) that the n control points of the Bézier curve $p([0, 1])$ are given by $f_p(0^{(n-r)}, 1^{(r)})$ for $r = 0, \dots, n$ where the arguments of f_p consist of $n - r$ zeros and r ones. With

$$\frac{1}{\binom{n}{\nu}} e_\nu(0^{(n-r)}, 1^{(r)}) = \frac{\binom{r}{\nu}}{\binom{n}{\nu}} = \frac{\binom{n-\nu}{n-r}}{\binom{n}{r}}$$

one computes with the abbreviation (4.18)

$$f_p(0^{(n-r)}, 1^{(r)}) = \sum_{\nu=0}^{n} p_{n,\nu} \frac{1}{\binom{n}{\nu}} e_\nu(0^{(n-r)}, 1^{(r)}) = \sum_{\nu=0}^{n} p_{n,\nu} \frac{\binom{n-\nu}{n-r}}{\binom{n}{r}}$$

$$= \frac{1}{\binom{n}{r}} \sum_{\nu=0}^{n} \binom{n-\nu}{n-r} p_{n,\nu} = \frac{1}{\binom{n}{r}} \pi_{n,r} =: \xi_{n,r} \ .$$

Similarly

$$f_q(0^{(n-r)}, 1^{(r)}) = \frac{1}{\binom{n}{r}} \pi'_{n,r} =: \xi'_{n,r} \ .$$

For the computation of the blossom and the control points of $s_k^n(t)$ we need some more notations. Set $[n] := \{1, \dots, n\}$ and $C(n, k) := \{I \mid I \subset [n], \ |I| = k\}$.

For $U := \{u_1, \ldots, u_n\}$ and $I = \{i_1, \ldots, i_k\} \in C(n,k)$ set $U_I := \{u_{i_1}, \ldots, u_{i_k}\}$ and $1 - U_I := \{1 - u_{i_1}, \ldots, 1 - u_{i_k}\}$. Then the blossom of $s_k^n(t)$ is

$$f_{s_k^n}(u_1, \ldots, u_n) = \sum_{I \in C(n,k)} f_{p_k}(U_I) f_{q_{n-k}}(1 - U_{[n]\setminus I}) \ . \tag{6.1}$$

Note that the l-th control point of $s_k^n(t)$ is $f_{s_k^n}(0^{(n-l)}, 1^{(l)}) = m_{k,l}$.

To compute this expression let j be an integer such that $f_{p_k}(0^{(k-j)}, 1^{(j)}) = \xi_{k,j}$ makes a non-vanishing contribution to $f_{s_k^n}(0^{(n-l)}, 1^{(l)})$ in (6.1). In other words, we choose j ones from the l available ones as arguments for f_{p_k} and take the remaining $l - j$ ones for $f_{q_{n-k}}(0^{(l-j)}, 1^{(n-k-(l-j))}) = \xi'_{n-k,n-k-l+j}$.

We also choose $k - j$ zeros from the $n - l$ available zeros for $f_{p_k}(0^{(k-j)}, 1^{(j)})$ so for fixed j we get an all in all contribution to $m_{k,l}$ of

$$\begin{aligned} \binom{n-l}{k-j}\binom{l}{j} \ \xi_{k,j}\xi'_{n-k,n-k-l+j} &= \frac{\binom{n-l}{k-j}\binom{l}{j}}{\binom{k}{j}\binom{n-k}{n-k-l+j}} \pi_{k,j}\pi'_{n-k,n-k-l+j} \\ &= \frac{\binom{n}{k}}{\binom{n}{l}} \ \pi_{k,j}\pi'_{n-k,n-k-l+j} \end{aligned}$$

where surprisingly the binomial coefficients are independent of j.

Summing up for $j = 0, \ldots, l$ gives the result. □

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University of Applied Sciences Bingen, Berlinstr. 109, D-55411 Bingen on the Rhine, Germany, http://www.fh-bingen.de/lehrende/winkel-rudolf.html
E-mail address: winkel@fh-bingen.de