ON A GENERALIZATION OF BERNSTEIN POLYNOMIALS AND BÉZIER CURVES BASED ON UMBRAL CALCULUS (III): BLOSSOMING

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ABSTRACT. The investigation of $\bar{a}$-Bernstein polynomials and $\bar{a}$-Bézier curves is continued in this paper. It is shown that convolution of the parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ is fundamental for (1) the definition of $\bar{a}$-Bernstein polynomials, (2) a simplified derivation of the $\bar{a}$-de Casteljau algorithm, (3) the recurrences that give the blossoming of $\bar{a}$-Bernstein polynomials and $\bar{a}$-Bézier curves, (4) the dual functional property and the $\bar{a}$-dual functional property for an $\bar{a}$-Bézier curve – it is necessary to make this distinction – and (5) the $\bar{a}$-degree elevation.

1. INTRODUCTION

Blossoming is the Swiss army knife for the understanding of geometric design of curves and surfaces. With blossoming it is easy to derive algorithms and formulas for control points, recursive evaluation, subdivision, degree elevation, basis transformations and more. Blossoming has been pioneered by Ramshaw ([4, 5, 6]) and brought to perfection in [3]. The easiest way to understand blossoming – and the standard approach since its introduction by Ramshaw – is to label the points in the de Casteljau algorithm appropriately and to replace the variable t by different variables u_j on different levels j of the algorithm. A triangular scheme that connects functions or geometric points by labeled arrows gives a clean picture of the algebraic structure of the de Casteljau algorithm and its blossoming.

However, as we will see in this paper, the blossoming of the de Casteljau algorithm as described above works only for Bernstein polynomials and can not be adapted in a straightforward manner to generalized Bernstein polynomials, although triangular schemes remain useful. We will see that the common basis of the generalized de Casteljau algorithm, generalized Bernstein polynomials, and generalized blossoming is the convolution of the constitutive parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$. In addition this approach allows us to derive essential results of the previous papers with a minimum of algebra and combinatorics. Since the generalization investigated is based on the parameters $\bar{a}$, we will speak subsequently of $\bar{a}$-X, whenever we generalize a classical concept X

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with the help of these parameters, e.g., $\bar{a}$-Bernstein polynomials , $\bar{a}$-Bézier curves, $\bar{a}$-de Casteljau algorithm. Note that we use $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ instead of $a = (a_1, \ldots, a_n)$ in compliance with notational conventions in umbral calculus (cf. [7]).

For a sequence $\mathbf{P} = (P_0, \ldots, P_n)$ of control points in $\mathbb{R}^N$ and an (almost) arbitrary sequence of real parameters $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$, $\bar{a}_1 \neq 0$ with associated $\bar{a}$-Bernstein polynomials $B^n_k(t; \bar{a})$ — for the precise definition of these polynomials see the beginning of section 2 up to formula (2.7) — we established in [13] a geometrically meaningful *up recurrence* or $\bar{a}$-de Casteljau algorithm

$$\begin{aligned} &\widetilde{\mathbf{X}}^0_k = P_k \quad \text{for } k = 0, \ldots, n, \\ &\widetilde{\mathbf{X}}^r_k = \sum_{i=1}^{r} \binom{r-1}{i-1} \bar{a}_i \left((1-t)\, \widetilde{\mathbf{X}}^{r-i}_k + t\, \widetilde{\mathbf{X}}^{r-i}_{k+i} \right) \\ &\qquad \text{for } r = 1, \ldots, n \text{ and } k = 0, \ldots, n-r. \end{aligned} \tag{1.1}$$

Then the top element $\widetilde{\mathbf{X}}^n_0(t; \bar{a})$ of this up recurrence is – up to a normalizing factor $\rho_n(\bar{a})$ defined after formula (2.7) – the $\bar{a}$-Bézier curve

$$\mathbf{X}(t; \bar{a}) = \sum_{k=0}^{n} P_k B^n_k(t; \bar{a}). \tag{1.2}$$

For $\bar{a} = (1, 0, \ldots, 0)$ this clearly specializes to the ordinary de Casteljau algorithm.

A straightforward calculation (see the proof of Thm 4.6 in [13]) shows that (1.1) is equivalent to the *down recurrence* for the (non-normalized) bivariate $\bar{a}$-Bernstein polynomials $\widetilde{B}^r_k(x, y; \bar{a}) = \rho_r(\bar{a}) B^r_k(x, y; \bar{a})$

$$\begin{aligned} &\widetilde{B}^0_0(x, y; \bar{a}) = 1 \text{ and for } r = 1, \ldots, n \text{ and } k = 0, \ldots, r: \\ &\widetilde{B}^r_k(x, y; \bar{a}) = y \sum_{i=1}^{r-k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}^{r-i}_k(x, y; \bar{a}) + x \sum_{i=1}^{k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}^{r-i}_{k-i}(x, y; \bar{a}) \ . \end{aligned} \tag{1.3}$$

However, the derivation of (1.1) or (1.3) directly from the original definition of $\bar{a}$-Bernstein polynomials turned out to be less than straightforward. The blossoming of (1.1) or (1.3) that is the subject of the present paper can be derived from convolutional powers of the the parameter sequence $\bar{a}$. In section 2 we recast and simplify essential parts of the previous papers in terms of the convolutional approach. The convolutional approach will also turn out to be crucial for the quite involved derivation of the $\bar{a}$-blossoming formulas in section 3.

In the remaining parts of the paper we investigate properties of $\bar{a}$-Bernstein polynomials and $\bar{a}$-Bézier curves that can be derived easily from blossoming in the ordinary case, namely, the dual functional property (in section 4) and degree elevation (in section 5). However, it turns out again that in the generalized case things are not that easy and that we have to deploy additional arguments.

More precisely, in section 4 it will turn out that a distinction between the dual functional property and the $\bar{a}$-dual functional property is necessary, because for an $\bar{a}$-Bézier curve there are two relevant kinds of control points: the ordinary control points $\mathbf{P}$ that represents the $\bar{a}$-Bézier curve as combination of $\bar{a}$-Bernstein polynomials according to (1.2) and the $\bar{a}$-control points $\mathbf{P}(\bar{a})$ that represents the same $\bar{a}$-Bézier curve as a combination of ordinary Bernstein polynomials $B_k^n(t)$ according to

$$\mathbf{X}(t;\bar{a}) = \sum_{k=0}^{n} P_k(\bar{a}) B_k^n(t). \tag{1.4}$$

We will see that the $\bar{a}$-control points $\mathbf{P}(\bar{a})$ can be derived from the blossom of $\mathbf{X}(t;\bar{a})$ in the usual (ordinary) fashion, whereas the recovery of ordinary control points $\mathbf{P}$ from the $\bar{a}$-Bézier curve $\mathbf{X}(t;\bar{a})$ need a modified blossoming. We demonstrate in Theorem 4.3 that such a modified blossoming for $\bar{a}$-Bézier curves is available only if the $\bar{a}$-Bernstein polynomials are specialized to the h-Bernstein functions of [8].

In the final section 5 $\bar{a}$-degree elevation is investigated, and in particular the problem of how to choose parameters $\bar{b} = (\bar{b}_1, \dots, \bar{b}_n, \bar{b}_{n+1})$ for given parameters $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ such that the $\bar{b}$-control points $\mathbf{Q}(\bar{b})$ for the degree elevated control points $\mathbf{Q}$ of $\mathbf{P}$ are matched by the degree elevation of the $\bar{a}$-control points $\mathbf{P}(\bar{a})$. In other words: in Theorem 5.1 we give a pleasing answer to the natural question of how to find parameters $\bar{b}$ for given parameters $\bar{a}$ such that the control polygon for the $\bar{b}$-Bézier curve is the (ordinarily) degree elevated control polygon of the $\bar{a}$-Bézier curve.

2. $\bar{a}$-Bernstein polynomial explained by convolution of parameters

2.1. **Generating functions.** Let $\mathcal{P} = \mathbb{R}[[t]]$ denote the $\mathbb{R}$-algebra of formal power series in t with coefficients in $\mathbb{R}$ and let $\mathcal{P}^+ = \{f \in \mathbb{R}[[t]] \mid f(0) = 0, f'(0) \neq 0\}$ be the subalgebra of *delta series* – these are exactly the formal power series $f(t) = \sum_{n\geq 1} \frac{a_n}{n!} t^n$ that have a compositional inverse. Let $\mathbb{R}_{\leq n}[t] = \{p \in \mathbb{R}[t] \mid \deg p \leq n\}$ and $\mathbb{R}_n[x,y] = \{p \in \mathbb{R}[x,y] \mid \deg p = n\}$ denote the $\mathbb{R}$-vector spaces of polynomials of degree $\leq n$ in t and of bivariate homogeneous polynomials of degree n in x and y.

A delta series can be written in *exponential form* with coefficients $\bar{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3, \dots)$ or in *ordinary form* with coefficients $\widehat{a} = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3, \dots)$ as

$$\bar{f}(t) = \sum_{n\geq 1} \frac{\bar{a}_n}{n!} t^n = \sum_{n\geq 1} \widehat{a}_n t^n \in \mathcal{P}^+, \tag{2.1}$$

i.e., $\widehat{a}_n := \bar{a}_n / n!$. Then $\bar{f}(t)$ (with compositional inverse $f(t) = \sum_{n\geq 1} \frac{a_n}{n!} t^n$) is the exponential generating function for the *associated polynomials* $p_n(x;\bar{a})$ in the following sense

$$e^{x\bar{f}(t)} = \sum_{n\geq 0} \frac{p_n(x;\bar{a})}{n!} t^n \quad \text{with } p_n(x;\bar{a}) = \textstyle\sum_{k=0}^{n} p_{n,k}(\bar{a}) x^k. \tag{2.2}$$

In other words: Formal exponentiation of $x\bar{f}(t)$ and collection of terms generates the $p_n(x;\bar{a})$ as coefficients of the t^n (see [3] or the more recent [10] for more details about generating functions).

Thus the coefficients $p_{n,k}(\bar{a})$ of the associated polynomials are given as k-fold Cauchy product of power series or convolution of the $\widehat{a}_n$ (for $\bar{a}_0 = \widehat{a}_0 = 0$) by

$$\begin{aligned} e^{x\bar{f}(t)} &= \sum_{k\geq 0} \frac{\bar{f}(t)^k}{k!} x^k = \sum_{k\geq 0} \left(\sum_{n\geq 1} \left(\sum_{i_1+\cdots+i_k=n} \frac{\bar{a}_{i_1}}{i_1!} \cdots \frac{\bar{a}_{i_k}}{i_k!} \right) t^n \right) \frac{x^k}{k!} \\ &= \sum_{n\geq 1} \left(\sum_{k\geq 0} \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_k} \bar{a}_{i_1} \cdots \bar{a}_{i_k} \frac{x^k}{k!} \right) \frac{t^n}{n!} = \sum_{n\geq 0} p_n(x;\bar{a}) \frac{t^n}{n!} \end{aligned}$$

with the *Bell polynomials* [1, 7]

$$p_{n,k}(\bar{a}) = \frac{1}{k!} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k>0}} \binom{n}{i_1,\ldots,i_k} \bar{a}_{i_1} \cdots \bar{a}_{i_k} \, . \tag{2.3}$$

The recursion

$$p_{n,k+1} = \frac{1}{k+1} \sum_{j=1}^{n-k} \binom{n}{j} \bar{a}_j p_{n-j,k} \tag{2.4}$$

allows a fast calculation. To see that (2.4) is correct, plug in (2.3) on both sides in (2.4) and set $j = i_{k+1}$.

Moreover, (2.2) applied to

$$e^{(x+y)\bar{f}(t)} = e^{x\bar{f}(t)} e^{y\bar{f}(t)}$$

yields

$$p_n(x+y;\bar{a}) = \sum_{k=0}^{n} \binom{n}{k} p_k(x;\bar{a}) p_{n-k}(y;\bar{a}). \tag{2.5}$$

Since (2.5) is a generalization of the binomial theorem, the associated polynomials $p_k(x;\bar{a})$ are said to be of *binomial type*. Now the homogeneous $\bar{a}$-Bernstein polynomials of order n defined by

$$B_k^n(x,y;\bar{a}) = \frac{1}{\rho_n(\bar{a})} \binom{n}{k} p_k(x;\bar{a}) p_{n-k}(y;\bar{a}), \quad 0 \leq k \leq n, \tag{2.6}$$

with normalizing factor $\rho_n(\bar{a}) = p_n(x+y;\bar{a})$ are a partition of unity. The *canonical dehomogenization* $(x,y) = (t, 1-t)$ of (2.6) gives the $\bar{a}$-Bernstein polynomials of order n

$$B_k^n(t;\bar{a}) = \frac{1}{\rho_n(\bar{a})} \binom{n}{k} p_k(t;\bar{a}) p_{n-k}(1-t;\bar{a}), \quad 0 \leq k \leq n, \tag{2.7}$$

with normalizing factor $\rho_n(\bar{a}) = p_n(1;\bar{a})$ as long as $\rho_n(\bar{a}) \neq 0$, i.e., the parameter sequence $\bar{a}$ is *feasible*. Subsequently we write $\widetilde{B}^n_k(x,y;\bar{a})$ and $\widetilde{B}^n_k(t;\bar{a})$ for the non-normalized $\bar{a}$-Bernstein polynomials and $p_n(t)$, $p_{n,k}$, and ρ_n if $\bar{a}$ is clear from the context.

Since formula (2.3) is symmetric in the summation indices $i_1,\dots,i_k$, but we can make progress in the understanding of $\bar{a}$-Bernstein polynomials only if we rewrite (2.3) in a form that collects all summands with identical products $\bar{a}_{i_1}\cdots\bar{a}_{i_k}$.

Let $\Lambda(n,k)$ be the set of all *(number) partitions* of n into k parts, i.e., $\lambda \in \Lambda(n,k)$ means that $\lambda = (\lambda_1,\dots,\lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k > 0$, $\lambda_1 + \cdots + \lambda_k = n$ is a partition of length $l(\lambda) = k$. The multiplicity $j_i = j_i(\lambda) = |\{\nu \mid \lambda_\nu = i\}|$ of a part of size i of λ allows for the convenient notation $\lambda = \dots 2^{j_2}1^{j_1}$, e.g., $\lambda = (4,2,2,2,1,1) = 4\,2^3 1^2 \in \Lambda(12,6)$. Note that $j(\lambda) = (j_1,j_2,\dots)$ is a *composition* or ordered number partition of k.

Now by ordering the summation indices $i_1,\dots,i_k$ for every summand of (2.3) in a decreasing fashion one gets with the above notation

$$p_{n,k}(\bar{a}) = \sum_{\substack{j_1+j_2+\cdots=k\\ j_1+2j_2+\cdots=n}} \frac{n!}{j_1!j_2!\dots(1!)^{j_1}(2!)^{j_2}\dots}\,\bar{a}_1^{j_1}\bar{a}_2^{j_2}\cdots = \sum_{\lambda\in\Lambda(n,k)} c_\lambda\,\bar{a}_\lambda\ , \tag{2.8}$$

where $\bar{a}_\lambda = \bar{a}_{\lambda_1}\bar{a}_{\lambda_2}\cdots = \bar{a}_1^{j_1}\bar{a}_2^{j_2}\cdots = \bar{a}^{j(\lambda)}$. Note that

$$p_{n,k}(\bar{a}) = \frac{n!}{k!} \sum_{\substack{j_1+j_2+\cdots=k\\ j_1+2j_2+\cdots=n}} \binom{k}{j(\lambda)} \widehat{a}_1^{j_1}\widehat{a}_2^{j_2}\dots \quad \text{with } \tbinom{k}{j(\lambda)} = \tfrac{k!}{j_1!j_2!\dots}\ . \tag{2.9}$$

2.2. $\bar{a}$**-Recursions.** To understand why $p_{n,k}(\bar{a})$ can be expressed by (2.3) or (2.9) and to proof (2.11) below it is necessary to observe that for $\lambda \in \Lambda(n,k)$ the coefficient c_λ counts the number of set partitions of $[n] := \{1,\dots,n\}$ of type λ, i.e., of set partitions of $[n]$ into k blocks such that $j_i(\lambda)$ of them have cardinality i (see [1]). In (2.3) the partitions of $[n]$ into k blocks are formed by taking any ordering of the numbers $1,\dots,n$ and subdividing this permutation of numbers into consecutive blocks of size $i_1,\dots,i_k$. Since ordering of blocks and ordering of elements inside blocks does not matter for set partitions, one has to divide by $k!$ and the i_ν's. In (2.9) one subdivides any ordering of $1,\dots,n$ into consecutive blocks of sizes $\lambda_1,\dots,\lambda_k$. Since ordering of blocks of equal size and ordering of elements inside blocks cannot matter, one has to divide by the j_i's, and the $(i!)^{j_i}$'s.

For example, there are $c_{3\,1^2} = 10$ set partitions of $[5]$ into three blocks of sizes $3,1,1$. Hence 10 is the coefficient of $\bar{a}_1^2\bar{a}_3$ in the following table of the $p_{n,k}$ for $n,k = 0,\dots,5$:

1	0	0	0	0	0
0	$\bar{a}_1$	0	0	0	0
0	$\bar{a}_2$	$\bar{a}_1^2$	0	0	0
0	$\bar{a}_3$	$3\,\bar{a}_2\bar{a}_1$	$\bar{a}_1^3$	0	0
0	$\bar{a}_4$	$4\,\bar{a}_3\bar{a}_1 + 3\,\bar{a}_2^2$	$6\,\bar{a}_2\bar{a}_1^2$	$\bar{a}_1^4$	0
0	$\bar{a}_5$	$5\,\bar{a}_4\bar{a}_1 + 10\,\bar{a}_3\bar{a}_2$	$10\,\bar{a}_1^2\bar{a}_3 + 15\,\bar{a}_2^2\bar{a}_1$	$10\,\bar{a}_2\bar{a}_1^3$	$\bar{a}_1^5$

We derive now the crucial recursions (2.11-12) and some other formulas from (2.9) with only a little algebra and combinatorics.

Theorem 2.1. *For the numbers c_λ, the coefficients $p_{n,k}(\bar{a})$, and the polynomials $p_n(x;\bar{a})$ defined by* (2.8), (2.3), *and* (2.2), *one has the following recursions*

$$\sum_{\lambda\in\Lambda(n,k+1)} c_\lambda = \sum_{\nu=1}^{n-k} \binom{n-1}{n-\nu} \Big(\sum_{\lambda'\in\Lambda(n-\nu,k)} c_{\lambda'} \Big) \ , \tag{2.10}$$

$$p_{n,k+1}(\bar{a}) = \sum_{\nu=1}^{n-k} \binom{n-1}{n-\nu} \bar{a}_\nu \, p_{n-\nu,k}(\bar{a}) \ , \tag{2.11}$$

$$p_n(x;\bar{a}) = x \ \sum_{i=1}^{n} \binom{n-1}{n-i} \bar{a}_i \, p_{n-i}(x;\bar{a}) \ . \tag{2.12}$$

The (non-normalized) homogeneous and dehomogenized $\bar{a}$-Bernstein polynomial

$$\widetilde{B}_k^r = \widetilde{B}_k^r(x,y;\bar{a}) = \rho_n(\bar{a}) B_k^r(x,y;\bar{a})$$

and

$$\widetilde{B}_k^r = \widetilde{B}_k^r(t;\bar{a}) = \rho_n(\bar{a}) \widetilde{B}_k^r(t;\bar{a})$$

can be computed recursively for $r = 1,\dots,n$ and $k = 0,\dots,r$ from $\widetilde{B}_0^0 = 1$ by the down recurrences

$$\widetilde{B}_k^r = y \sum_{i=1}^{r-k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_k^{r-i} + x \sum_{i=1}^{k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_{k-i}^{r-i} \ \text{and} \tag{2.13}$$

$$\widetilde{B}_k^r = (1-t) \sum_{i=1}^{r-k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_k^{r-i} + t \sum_{i=1}^{k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_{k-i}^{r-i} \ . \tag{2.14}$$

Proof. The l.h.s. of (2.10) is the number of partitions of the set $[n]$ into $k+1$ parts. Take any partition π of $[n]$ into $k+1$ blocks. Then the number n must be contained in some block $\beta(\pi)$ of π with size ν, $1 \le \nu \le n-k$. The other blocks not containing n then form a partition π' of $[n] \setminus \beta(\pi)$ with block sizes $\lambda' \in \Lambda(n-\nu,k)$. Since there are $\binom{n-1}{n-\nu}$ possible ways to choose the other elements of $\beta(\pi)$ and there are $c_{\lambda'}$ partitions of type λ' formula (2.10) follows. But this implies (2.11) with $\bar{a}_\lambda = \bar{a}_\nu \bar{a}_{\lambda'}$ and (2.8). For (2.12) one computes

$$\begin{aligned}
p_n(x;\bar{a}) &\overset{(2.3)}{=} \sum_{k=1}^{n} p_{n,k}(\bar{a}) x^k = \sum_{k=0}^{n-1} p_{n,k+1}(\bar{a}) x^{k+1} \\
&\overset{(2.11)}{=} \sum_{k=0}^{n-1} \left(\sum_{i=1}^{n-k} \binom{n-1}{n-i} \bar{a}_i \, p_{n-i,k}(\bar{a}) \right) x^{k+1} \\
&= x \ \sum_{i=1}^{n} \binom{n-1}{n-i} \bar{a}_i \left(\sum_{k=0}^{n-i} p_{n-i,k}(\bar{a}) x^k \right) \overset{(2.3)}{=} x \ \sum_{i=1}^{n-1} \binom{n-1}{n-i} \bar{a}_i \, p_{n-i}(x;\bar{a}) \ .
\end{aligned}$$

For (2.12) one computes with $\widetilde{B}_k^r = \widetilde{B}_k^r(x,y;\bar{a})$ and (2.12)

$$\begin{aligned}
\widetilde{B}_k^r &= \binom{r}{k} p_k(x)p_{r-k}(y) = \binom{r-1}{k} p_k(x)p_{r-k}(y) + \binom{r-1}{k-1} p_k(x)p_{r-k}(y) \\
&= \binom{r-1}{k} p_k(x) y \sum_{i=1}^{r-k} \binom{r-k-1}{i-1} \bar{a}_i p_{r-k-i}(y) + \binom{r-1}{k-1} p_{r-k}(y) x \sum_{i=1}^{k} \binom{k-1}{i-1} \bar{a}_i p_{k-i}(x) \\
&= y \sum_{i=1}^{r-k} \binom{r-1}{i-1} \bar{a}_i \binom{r-i}{k} p_k(x) p_{r-i-k}(y) + x \sum_{i=1}^{k} \binom{r-1}{i-1} \bar{a}_i \binom{r-i}{k-i} p_{k-i}(x) p_{r-k}(y) \\
&= y \sum_{i=1}^{r-k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_k^{r-i} + x \sum_{i=1}^{k} \binom{r-1}{i-1} \bar{a}_i \widetilde{B}_{k-i}^{r-i} \ .
\end{aligned}$$

(2.13) follows from (2.12) by the canonical dehomogenization. □

Remark 2.2. Formulas (2.10) and (2.11) are probably new. Formula (2.12) is proved in section 4,1.8 of [7] in a non-elementary fashion that uses Cor.3.6.6 of [7] and the umbral operator $\lambda_f : x^n \mapsto p_n(x;\bar{a})$. Formula (2.12) is derived in a non-elementary fashion in [13] from the definition of $\bar{a}$-Bernstein polynomials using the umbral shift operator θ_f.

2.3. **Convolutions.** For $\bar{a}$-blossoming it will be crucial to return to the convolutional representation (2.3) of the Bell polynomials $p_{n,k}(\bar{a})$ and to recast formula (2.3) in terms of convolution of sequences. Let $l(\mathbb{R})$ be the set of all real sequences $a = (a_0, a_1, a_2, \dots) = (a_n)_{n\geq 0}$. With the usual component wise addition and scalar multiplication and the (ordinary) convolution

$$a \otimes b = (a_0, a_1, a_2, \dots) \otimes (b_0, b_1, b_2, \dots) := \left(\sum_{k=0}^{n} a_k b_{n-k} \right)_{n\geq 0}$$

$l(\mathbb{R})$ becomes a commutative $\mathbb{R}$-algebra that is isomorphic to $\mathcal{P}$ with Cauchy multiplication of ordinary power series (cf. (2.)). Let $a^{k\otimes}$ denote the k-fold convolutional power of $a \in l(\mathbb{R})$ with n-th component $a_n^{k\otimes} = (a^{k\otimes})_n$. Then

$$\widehat{a}_n^{k\otimes} = \sum_{i_1+\cdots+i_k=n} \widehat{a}_{i_1} \cdots \widehat{a}_{i_k} \ .$$

For sequences written in exponential form, e.g., $\widehat{a}_i = \bar{a}_i / i!$, this gives

$$\widehat{a}_n^{k\otimes} = \sum_{i_1+\cdots+i_k=n} \frac{1}{i_1! \cdots i_k!} \bar{a}_{i_1} \cdots \bar{a}_{i_k} \overset{!}{=} \frac{c_n}{n!}$$

or

$$c_n = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1, \dots, i_k} \bar{a}_{i_1} \cdots \bar{a}_{i_k} =: \bar{a}_n^{k*}$$

for the exponential convolutional multiplication

$$a * b = (a_0, a_1, a_2, \dots) * (b_0, b_1, b_2, \dots) := \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right)_{n \geq 0}$$

Therefore

$$p_{n,k}(\bar{a}) = \frac{1}{k!}\, \bar{a}_n^{k*} = \frac{n!}{k!}\, \widehat{a}_n^{k\otimes} \;. \tag{2.15}$$

Note that in $l(\mathbb{R})$ convolution distributes through addition and that the neutral element of both convolutions is $\mathbf{1} = (1, 0, 0, \dots)$, e.g., $\mathbf{1} \otimes a = a \otimes \mathbf{1} = a$ for every $a \in l(\mathbb{R})$. We also define $a^0 = \mathbf{1}$. Since by definition $\bar{a}_0 = 0$ for the parameter sequence $\bar{a}$ as well as the delta series $\bar{f}$ in (2.1) and since the order n of some $\bar{a}$-Bernstein polynomial is fixed, we subsequently use the short form $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$ for $\bar{a} \in l(\mathbb{R})$ instead of $\bar{a} = (0, \bar{a}_1, \dots, \bar{a}_n, 0, 0, \dots)$.

3. $\bar{a}$-BLOSSOMING

3.1. **Elementary symmetric polynomials and blossoms.** For $n, r \in \mathbb{N}_0$ and $0 \leq r \leq n$ let $\mathcal{C}(n, r) = \{I \mid I \subset [n], |I| = r\}$ again with $[n] := \{1, \dots, n\}$. Of course, $|\mathcal{C}(n, r)| = \binom{n}{r}$. We also use the notation

$$\begin{aligned} u_I &= u_{i_1} u_{i_2} \cdots u_{i_r} \text{ (product) for } I = \{i_1, i_2, \dots, i_r\} \in \mathcal{C}(n, r), \\ u_{(r)} &= u_1, u_2, \dots, u_r \text{ (sequence)}, \\ x^{(r)} &= \underbrace{x, \dots, x}_{r\text{-fold}}, \text{ (sequence) where } x \text{ is any symbol or number.} \end{aligned}$$

Then the *elementary symmetric polynomials* of order n are defined by

$$e_r^n(u) = e_r(u_{(n)}) := \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} u_{i_1} u_{i_2} \cdots u_{i_r} = \sum_{I \in \mathcal{C}(n,r)} u_I \;. \tag{3.1}$$

For $r > n$ and $n, r < 0$ we set $e_r^n(u) = 0$. In particular, $e_0^n(u) = 1$, $e_1^n(u) = u_1 + \cdots + u_n$, and $e_n^n(u) = u_1 \cdots u_n$. The elementary symmetric polynomials of order n have the generating function

$$E_n(t) = \prod_{i=1}^{n} (1 + u_i t) = \sum_{r=0}^{n} e_r^n(u)\, t^r \;. \tag{3.2}$$

Symmetric means $e_r(u_1, \dots, u_n) = e_r(u_{\pi(1)}, \dots, u_{\pi(n)})$ for all permutations $\pi \in S_n$, which is clear by (3.1) or by (3.2). Repeated application of (3.2) gives the recursion

$$e_r^n(u) = e_r^{n-1}(u) + u_n\, e_{r-1}^{n-1}(u) \;, \tag{3.3}$$

namely,

$$\sum_{r=0}^{n} e_r^n(u)\, t^r = (1+u_n t)\prod_{i=1}^{n-1}(1+u_i t) = (1+u_n t)\sum_{r=0}^{n-1} e_r^{n-1}(u)\, t^r$$
$$= \sum_{r=0}^{n-1} e_r^{n-1}(u)\, t^r + \sum_{r=1}^{n} u_n\, e_{r-1}^{n-1}(u)\, t^r \ .$$

Now the *blossom* of any univariate polynomial $P(t) = \sum_{r=0}^{n} c_r t^r$ of degree n is given by the unique multivariate polynomial

$$p(u) = p(u_{(n)}) = \sum_{r=0}^{n} \frac{c_r}{\binom{n}{r}} e_r^n(u) \ .$$

The blossom $p(u)$ of $P(t)$ is symmetric and has the diagonal property $P(t) = p(t^{(n)})$, because the division by $|\mathcal{C}(n,r)| = \binom{n}{r}$ normalizes $e_r^n(u)$. Moreover, the blossom is multiaffine, i.e.,

$$p(\ldots,(1-\alpha)u_i + \alpha u_i', \ldots) = (1-\alpha)p(\ldots,u_i,\ldots) + \alpha p(\ldots,u_i',\ldots)$$

for all i and all $\alpha \in \mathbb{R}$. An easy to check criterion for multiaffinity of a multivariate polynomial $p \in \mathbb{R}[u_1,\ldots,u_n]$ is that in every term of p every variable u_i appears at most in the first power ([3], Lemma 6.2). Clearly, $P(t)$ can be blossomed also in a number of variables $j \neq n = \deg(P)$ by

$$p(u_{(j)}) = \sum_{r=0}^{\min(n,j)} \frac{c_r}{\binom{j}{r}} e_r^j(u) \ . \tag{3.4}$$

We call this expression the *j-blossom* of $P(t)$. Note that for $j < n$ the j-blossom is the blossom of the *j-truncation* of P:

$$P^{[j]} = \sum_{r=0}^{j} c_r t^r \ . \tag{3.5}$$

and that for $j > n$ one does not have $p(u_{(n)}) = p(u_{(j)})|_{u_{n+1}=\cdots=u_j=0}$ because of the normalizing factors.

The power of blossoming for Bernstein polynomials and Bézier curves is based to a great extent in the availability of a recursive (triangular) scheme for its computation. Since $p_n(x;\bar{a}) = \widetilde{B}_n^n(x,y;\bar{a})$ we first describe such a recursive scheme for this special case before proceeding to the general case of $\widetilde{B}_k^n(x,y;\bar{a})$, $0 \le k \le n$. For $p_r = p_r(x;\bar{a})$ the down recurrence (2.12) reads

$$p_r = x \sum_{i=1}^{r} \binom{r-1}{i-1} \bar{a}_i\, p_{r-i} \quad \text{for } r = 1,\ldots,n \text{ with } p_0 = 1. \tag{3.6}$$

Simply setting $x = u_r$ will not lead to a symmetric $p_n(u)$. Instead one has to run n times through the levels $r = 1,\ldots,n$:

Theorem 3.1. *For $r = 1, \dots, n$ the blossoms $p_r^r(u;\bar a) = p_r(u_{(r)};\bar a)$ of the polynomials $p_r(x;\bar a)$ can be computed recursively from $p_r^0(s) := \delta_{0,r}$ and $p_0^r(s) := 1$ by*

$$p_r^j(s) = p_r^{j-1}(s) + s u_j \sum_{i=1}^{r} \binom{r-1}{i-1} \bar a_i \, p_{r-i}^{j-1}(s) \tag{3.7}$$

for $j = 1, \dots, n$ and $r = 1, \dots, n$ for fixed j, where $p_r^j(s) = p_r(s, u_{(j)};\bar a) = p_r(s, u_1, \dots, u_j;\bar a)$ is the j-blossom of the j-truncation $p_r^{[j]}(x;\bar a) = \sum_{k=1}^{j} p_{r,k}(\bar a)x^k$ of $p_r(x;\bar a)$ in the variables $u_{(j)}$, where $p_{r,k}(\bar a) = 0$ for $k > r$ by (2.3) or (2.8).

Then the $p_r^j(u;\bar a)$ result from substituting in $p_r^j(s)$ the expressions $\binom{j}{\nu}^{-1}$ for the powers s^ν.

Proof. The proof of (3.7) proceeds by induction on j. For $j = 1$ the recursion (3.7) gives

$$p_r^1(s) = \delta_{0,r} + s u_1 \sum_{i=1}^{r-1} \binom{r-1}{i-1} \bar a_i \, \delta_{0,r-i} = s u_1 \bar a_r$$

and therefore $p_r^1(u;\bar a) = p_r(u_1;\bar a) = u_1 \bar a_r$.

From (3.6) and (2.2) one computes

$$\sum_{i=1}^{r} \binom{r-1}{i-1} \bar a_i \, p_{r-i}^{[j-1]}(x;\bar a) = \frac{1}{x} p_r^{[j-1]}(x;\bar a) = \sum_{k=1}^{j-1} p_{n,k}(\bar a) x^{k-1} \ .$$

Assume that the $p_{r-i}^{j-1}(s)$ for $i = 0, 1, \dots, r$ are the non-normalized $(j-1)$-blossoms of the $p_{r-i}^{[j]}(x;\bar a)$ and note that the powers of s in any term count the number of factors of type u_j. Then with the recursion (3.3) for elementary symmetric polynomials

$$\begin{aligned} p_r^{j-1}(s) &+ s u_j \sum_{i=1}^{r} \binom{r-1}{i-1} \bar a_i \, p_{r-i}^{j-1}(s) \\ &= \sum_{k=1}^{j-1} p_{n,k}(\bar a) s^k e_k^{j-1}(u) + s u_j \sum_{k=1}^{j-1} p_{n,k}(\bar a) s^{k-1} e_{k-1}^{j-1}(u) \\ &= \sum_{k=1}^{j-1} p_{n,k}(\bar a) s^k \left(e_k^{j-1}(u) + u_j e_{k-1}^{j-1}(u) \right) = \sum_{k=1}^{j-1} p_{n,k}(\bar a) s^k e_k^j(u) = p_r^j(s) \ . \end{aligned}$$

□

3.2. **Bivariate elementary symmetric polynomials and blossoms.** To blossom the non-normalized homogeneous $\bar a$-Bernstein polynomials $\widetilde B_k^n(x, y;\bar a)$ we define the *bivariate elementary symmetric polynomials* of order n by

$$e_{r,k}^n(u, v) = e_{r,k}(u_{(n)}, v_{(n)}) := \sum_{I \in \mathcal{C}(n,r)} \sum_{J \in \mathcal{C}(I,k)} u_J v_{I \setminus J}, \tag{3.8}$$

where $\mathcal{C}(I,k) = \{J \mid J \subset I, |J| = k\}$ and $0 \le k \le r \le n$; for all other triples (n,r,k) we set $e^n_{r,k}(u,v) = 0$. Note that $e^n_{r,k}$ is homogeneous of degree r in (u,v), of degree k in u, and of degree $r-k$ in v, but that in general $e^n_{r,k}(u,v) \neq e^n_k(u)e^n_{r-k}(v)$.

Clearly, $e^n_{r,r}(u,v) = e^n_r(u)$, $e^n_{r,0}(u,v) = e^n_r(v)$, $e^n_{0,0}(u,v) = 1$, and, for example,

$$\begin{aligned} e^3_{2,1}(u,v) &= (u_1v_2 + u_2v_1) + (u_1v_3 + u_3v_1) + (u_2v_3 + u_3v_2) \\ e^3_{3,2}(u,v) &= u_1u_2v_3 + u_1u_3v_2 + u_2u_3v_1 \ , \end{aligned}$$

where the brackets indicate a common I. Symmetry means invariance under arbitrary permutations $\pi \in S_n$ of indices of the two sets of variables $u_{(n)}$ and $v_{(n)}$.

Lemma 3.2. *The bivariate elementary symmetric polynomials of order n have the generating function*

$$E_n(t,s) = \prod_{i=1}^n (1 + (u_i + v_i s)t) = \sum_{r=0}^n \left(\sum_{k=0}^r e^n_{r,k}(u,v)\, s^{r-k} \right) t^r \tag{3.9}$$

and satisfy the recursion

$$e^n_{r,k}(u,v) = e^{n-1}_{r,k}(u,v) + v_n\, e^{n-1}_{r-1,k}(u,v) + u_n\, e^{n-1}_{r-1,k-1}(u,v) \ . \tag{3.10}$$

Proof. Setting $w_i = u_i + v_i s$ one computes

$$E_n(t,s) = \prod_{i=1}^n (1 + w_i t) \overset{(3.2)}{=} \sum_{r=0}^n e^n_r(w)\, t^r \overset{(3.1)}{=} \sum_{r=0}^n \left(\sum_{I \in \mathcal{C}(n,r)} w_I \right) t^r \ .$$

But similar to (3.2)

$$w_I = \prod_{i \in I} (u_i + v_i s) = \sum_{k=0}^r \left(\sum_{J \in \mathcal{C}(I,k)} u_J v_{I \setminus J} \right) s^{r-k}$$

A reordering of summations gives (3.9) and a calculation similar to the one proving (3.3) gives the recursion (3.10). □

Note that (3.9) specializes to (3.2) for $s = 0$ and that (3.10) specializes to (3.3) for $k = r$.

The *(bivariate) j-blossom* of a polynomial $P(x,y) \in \mathbb{R}[x,y]$ of degree n is now defined as the polynomial $p(u,v) \in \mathbb{R}[u_{(n)}, v_{(n)}]$ that results from P by substituting for its monomials according to

$$x^k y^l \mapsto \frac{1}{\binom{j}{k+l}\binom{k+l}{k}}\, e^j_{k+l,k}(u,v) \ . \tag{3.11}$$

Again for $j < n$ the j-blossom is the blossom of the j-truncation $P^{[j]}$ of P that contains only monomials $x^k y^l$ with $k + l \le j$.

The j-blossom of P is symmetric, multiaffine and has the diagonal property $P(x,y) = p(x^{(n)}, y^{(n)})$, because the division by $|\mathcal{C}(n,r) \times \mathcal{C}(I,k)| = \binom{n}{r}\binom{r}{k}$ normalizes $e^n_{r,k}(u,v)$.

Before we state the main result of this paper we remark that a simple copying of the step from (3.6) to (3.7) as in Thm.3.1 will not give a correct blossoming of the $\widetilde{B}_k^n(x,y;\bar{a})$. Namely, the resulting polynomials are not symmetric due to the asymmetry of (2.13) induced by the binomial coefficients. The solution is to switch from the exponential form of the parameters $\bar{a}_i$ to the ordinary form of the parameters $\widehat{a}_i = \frac{\bar{a}_i}{i!}$ as introduced in (2.1) and discussed in section 2.3.

Subsequently, calculations in the $\mathbb{R}$-algebra $l(\mathbb{R})$ will show significant advantages. We will use exclusively the ordinary convolution and can therefore write $\widehat{a}^k$ instead of $\widehat{a}^{k\otimes}$ for the k^{th} power of $\widehat{a}$ and $\widehat{a}_n^k$ for its n-th component. Then (2.15) reads

$$p_{n,k}(\bar{a}) = \frac{n!}{k!}\,\widehat{a}_n^k\ . \tag{3.12}$$

With $\widehat{a} = (0, \widehat{a}_1, \widehat{a}_2, \dots) \in l(\mathbb{R})$ let

$$X = \mathbf{1} + \dot{X} \text{ with } \dot{X} = sx\,\widehat{a} = (0, sx\,\widehat{a}_1, sx\,\widehat{a}_2, \dots),$$
$$Y = \mathbf{1} + \dot{Y} \text{ with } \dot{Y} = sy\,\widehat{a} = (0, sy\,\widehat{a}_1, sy\,\widehat{a}_2, \dots).$$

Also abbreviate $X_r^j = X_r^j(s,x;\bar{a})$ and $Y_r^j = Y_r^j(s,y;\bar{a})$ for $j, r \in \mathbb{N}_0$. Then (3.6) and (2.12-13) can be replaced by recursions without binomial coefficients as follows:

Theorem 3.3. (a) *For $j = 1,\dots,n$ and $r = 1,\dots,n$ for fixed j the expressions X_r^j can be computed recursively from $X^0 = \mathbf{1}$, i.e., $X_r^0 = \delta_{r,0}$, by*

$$X_r^j = X_r^{j-1} + sx\sum_{i=1}^{r} \widehat{a}_i\, X_{r-i}^{j-1}. \tag{3.13}$$

For $r = 1,\dots,n$ the polynomials $p_r(x;\bar{a})$ satisfy the equation

$$p_r(x;\bar{a}) = X_r^r(s,x;\bar{a}) \text{ with substitutions } s^l \leftarrow (r-l)!\ . \tag{3.14}$$

Moreover, the expressions X_r^j are essentially the j-truncations $p_r^{[j]}(x;\bar{a}) = \sum_{k=1}^{j} p_{r,k}(\bar{a})x^k$ according to

$$p_r^{[j]}(x;\bar{a}) = \frac{r!}{j!} x_r^j(s,x;\bar{a}) \text{ with substitutions } s^l \leftarrow (j-l)!\ . \tag{3.15}$$

(b) *For $j = 1,\dots,n$ and $r = 1,\dots,n$, $k = 0,\dots,r$ for fixed j one can compute recursively the expressions*

$$H_k^{r,j} = H_k^{r,j}(s,x,y;\bar{a}) = \sum_{l=0}^{j}\binom{j}{l} C_{k,l}^r := \sum_{l=0}^{j}\binom{j}{l}\left(\sum_{\nu=0}^{l}\binom{l}{\nu}\dot{X}_k^\nu\,\dot{Y}_{r-k}^{l-\nu}\right). \tag{3.16}$$

from $H_k^{r,0} = \delta_{0,r}$ by

$$H_k^{r,j} = H_k^{r,j-1} + sy\sum_{i=1}^{r-k}\widehat{a}_i\, H_k^{r-i,j-1} + sx\sum_{i=1}^{k}\widehat{a}_i\, H_{k-i}^{r-i,j-1}. \tag{3.17}$$

Then for $r = 1, \dots, n$ and $k = 0, \dots, r$ the non-normalized homogeneous $\bar{a}$-Bernstein polynomials $\widetilde{B}^r_k(x, y; \bar{a})$ satisfy the equation

$$\widetilde{B}^r_k(x, y; \bar{a}) = H^{r,r}_k(s, x, y; \bar{a}) \text{ with substitutions } s^l \leftarrow (r-l)! \ . \tag{3.18}$$

Moreover, the expressions $H^{r,j}_k$ are essentially the j-truncations of the $\widetilde{B}^r_k$ according to

$$\widetilde{B}^{r,[j]}_k(x, y; \bar{a}) = \frac{r!}{j!} H^{r,j}_k(s, x, y; \bar{a}) \text{ with substitutions } s^l \leftarrow (j-l)! \ . \tag{3.19}$$

Proof. (a) The first equality (3.13) follows directly from the definition of convolution

$$X^j_r = (X \otimes X^{j-1})_r = X^{j-1}_r + sx \sum_{i=1}^{r} \widehat{a}_i \, X^{j-1}_{r-i}.$$

On the other hand with (3.12)

$$X^j_r = \big((\mathbf{1} + sx\,\widehat{a})^j\big)_r = (\sum_{k=0}^{j} \binom{j}{k} \widehat{a}^k (sx)^k \,)_r = \sum_{k=0}^{j} \binom{j}{k} \widehat{a}^k_r s^k x^k$$
$$= \sum_{k=0}^{j} \binom{j}{k} \frac{k!}{r!} s^k p_{r,k}(\bar{a}) x^k = \sum_{k=0}^{j} s^k \frac{j!}{r!(j-k)!} p_{r,k}(\bar{a}) x^k \ .$$

This shows (3.15) and also (3.14) with $j = r$.

(b) (3.16) gives for $k \le r$ and $r \ge 0$

$$H^{r,0}_k = C^r_{k,0} = X^0_k \, Y^0_{r-k} = \delta_{0,r}$$

and for $j, r > 0$, $k = 0$

$$C^r_{0,l} = \sum_{\nu=0}^{l} \binom{l}{\nu} \dot{X}^\nu_0 \, \dot{Y}^l_r = \dot{Y}^l_r \text{ and}$$
$$H^{r,j}_0 = \sum_{l=0}^{j} \binom{j}{l} C^r_{0,l} = \sum_{l=0}^{j} \binom{j}{l} \dot{Y}^l_r = \sum_{l=0}^{j} \binom{j}{l} \dot{Y}^l_r \mathbf{1}^{j-l} = (\mathbf{1} + \dot{Y})^j_r = Y^j_r \ .$$

Similarly one gets for $j, r > 0$ and $k = r$

$$H^{r,j}_r = X^j_r.$$

Therefore (3.17) specializes for $k = r$ to (3.13) and for $k = 0$ to (3.13) where X is replaced by Y. Since $\widetilde{B}^{r,[j]}_0 = X^j_r$ and $\widetilde{B}^{r,[j]}_r = Y^j_r$ with the substitutions of (3.15), the assertions of part (b) of the theorem are true for the right and left side of the triangular recurrence scheme defined by (3.17).

For the rest of the proof we can concentrate on the inner part of the triangular scheme, namely the case $k \ne 0, r$ and $r > 0$. The j-truncation $\widetilde{B}^{r,[j]}_k$ of $\widetilde{B}^r_k$ is with (3.12) and the definitions of $\dot{X}$ and $\dot{Y}$

$$\begin{aligned}
\widetilde{B}_k^{r,[j]} &= \binom{r}{k}\,[p_k(x)p_{r-k}(y)]^{[j]} \\
&= \binom{r}{k}\sum_{l=0}^{j}\left(\sum_{\nu=0}^{l} p_{k,\nu}p_{r-k,l-\nu}x^\nu y^{l-\nu}\right) \\
&= \binom{r}{k}\sum_{l=0}^{j}\left(\sum_{\nu=0}^{l} \frac{k!}{\nu!}\frac{(r-k)!}{(l-\nu)!}\bar{a}_k^\nu \bar{a}_{r-k}^{l-\nu}x^\nu y^{l-\nu}\right) \\
&= \binom{r}{k}\sum_{l=0}^{j}\left(\sum_{\nu=0}^{l} \frac{k!}{\nu!}\frac{(r-k)!}{(l-\nu)!}\dot{X}_k^\nu \dot{Y}_{r-k}^{l-\nu}\frac{1}{s^l}\right)
\end{aligned}$$

But taking into account the factor $\frac{r!}{j!}$ and the substitution $s^l \leftarrow (j-l)!$ gives

$$\binom{r}{k}\sum_{l=0}^{j}\left(\sum_{\nu=0}^{l}\frac{k!(r-k)!}{r!}\binom{j}{l}\frac{l!}{\nu!(l-\nu)!}\dot{X}_k^\nu \dot{Y}_{r-k}^{l-\nu}\right) = \sum_{l=0}^{j}\binom{j}{l}\left(\sum_{\nu=0}^{l}\binom{l}{\nu}\dot{X}_k^\nu\,\dot{Y}_{r-k}^{l-\nu}\right) = H_k^{r,j}.$$

This proves (3.18) and (3.19). It remains to verify the recursion (3.17). One computes with (3.16)

$$\begin{aligned}
H_k^{r,j} &= \sum_{l=0}^{j}\binom{j}{l}C_{k,l}^r = \sum_{l=0}^{j-1}\binom{j-1}{l}C_{k,l}^r + \sum_{l=0}^{j-1}\binom{j-1}{l-1}C_{k,l}^r \\
&= H_k^{r,j} + \sum_{l=0}^{j-1}\binom{j-1}{l-1}C_{k,l}^r \\
&= H_k^{r,j} + \sum_{l=0}^{j-1}\binom{j-1}{l-1}\left(\sum_{\nu=0}^{l-1}\binom{l-1}{\nu}\dot{X}_k^\nu\,\dot{Y}_{r-k}^{l-\nu}\right) + \sum_{l=1}^{j-1}\binom{j-1}{l-1}\left(\sum_{\nu=0}^{l-1}\binom{l-1}{\nu-1}\dot{X}_k^\nu\,\dot{Y}_{r-k}^{l-\nu}\right)
\end{aligned}$$

But

$$\dot{Y}_{r-k}^{l-\nu} = (\dot{Y}\otimes\dot{Y}^{l-\nu-1})_{r-k} = sy\sum_{i=1}^{r-k}\widehat{a}_i\,\dot{Y}_{r-k-i}^{l-1-\nu}$$

gives again with (3.16)

$$\begin{aligned}
&\sum_{l=0}^{j-1}\binom{j-1}{l-1}\left(\sum_{\nu=0}^{l-1}\binom{l-1}{\nu}\dot{X}_k^\nu\,\dot{Y}_{r-k}^{l-\nu}\right) \\
&= sy\sum_{i=1}^{r-k}\widehat{a}_i\sum_{l=0}^{j-1}\binom{j-1}{l-1}\left(\sum_{\nu=0}^{l-1}\binom{l-1}{\nu}\dot{X}_k^\nu\,\dot{Y}_{r-k-i}^{l-1-\nu}\right) \\
&= sy\sum_{i=1}^{r-k}\widehat{a}_i\sum_{l=0}^{j-1}\binom{j-1}{l-1}C_{k,l-1}^{r-i} = sy\sum_{i=1}^{r-k}\widehat{a}_i\,H_k^{r-i,j-1}
\end{aligned}$$

Similarly with some index shifting ones sees that

$$\sum_{l=1}^{j-1} \binom{j-1}{l-1} \left(\sum_{\nu=0}^{l-1} \binom{l-1}{\nu-1} \dot{X}_k^{\nu}\, \dot{Y}_{r-k}^{l-\nu} \right) = sx \sum_{i=1}^{k} \widehat{a}_i\, H_{k-i}^{r-i,j-1}.$$

□

The parametrized forms of (3.13-19) are obtained easily with the substitutions $x = t-a$, $y = b-t$ for a general interval $[a,b] \subset \mathbb{R}$ or with the canonical substitution $x = t$, $y = 1-t$ for the interval $[0,1] \subset \mathbb{R}$. For the normalized form one has to divide by $p_r(x+y)$ according to (2.6-7).

Note that the variable s in the above formulas is a convenient device to count the number l of factors x, y or, equivalently, of factors $\widehat{a}_i$ in any term of a polynomial X_r^j or $H_k^{r,j}$. Hence one can set $s = 1$ or simply omit s if the computation does not use concrete values for the mentioned variables. Subsequently, the substitution in formulas (3.18-19) will be called *s-normalization.*

Example 3.4. For parameters $\bar{a} = (1, 0, \ldots, 0)$,i.e., in the ordinary case, the recursion (3.17) becomes

$$H_k^{r,j} = H_k^{r,j-1} + sy\, H_k^{r-1,j-1} + sx\, H_{k-1}^{r-1,j-1}. \tag{3.20}$$

The explicit solution of this recursion is with $p_{r,k}(\bar{a}) = \bar{a}_1^r \delta_{k,r} = \delta_{k,r}$

$$\begin{aligned} H_k^{r,j}(s,x,y) &= \sum_{l=0}^{j} \binom{j}{l} \left(\sum_{\nu=0}^{l} \binom{l}{\nu} \dot{X}_k^{\nu}\, \dot{Y}_{r-k}^{l-\nu} \right) \\ &= \sum_{l=0}^{j} \binom{j}{l} \left(\sum_{\nu=0}^{l} \binom{l}{\nu} s^k x^k \delta_{\nu,k}\, s^{r-k} y^{r-k} \delta_{l-\nu,r-k} \right) \\ &= s^r \binom{j}{r} \binom{r}{k} x^k y^{r-k}. \end{aligned}$$

Note that $H_k^{r,j}(s,x,y) = 0$ for $r > j$. With s-normalization this gives the ordinary bivariate Bernstein polynomials

$$\widetilde{B}_k^{r,[j]} = \frac{r!}{j!}(j-r)! \binom{j}{r} \binom{r}{k} x^k y^{r-k} = \binom{r}{k} x^k y^{r-k}$$

if $r \leq j$ and zero otherwise. Therefore in the case $j = r$ the recursion (3.20) becomes with s-normalization the usual recursion of ordinary bivariate Bernstein polynomials

$$B_k^r(x,y) = y\, B_k^{r-1}(x,y) + x\, B_{k-1}^{r-1}(x,y). \tag{3.21}$$

The proof of the main recurrence (3.17) above is essentially an extension for additional parameters $\bar{a}_2, \bar{a}_3, \ldots$ of the following simple calculation for the ordinary case $\bar{a} = (1, 0, \ldots, 0)$:

$$\begin{aligned}
&\binom{n}{r}\binom{r}{k}x^k y^{r-k}\\
&=\binom{n-1}{r}\binom{r}{k}x^k y^{r-k}+\binom{n-1}{r-1}\binom{r}{k}x^k y^{r-k}\\
&=\binom{n-1}{r}\binom{r}{k}x^k y^{r-k}+\binom{n-1}{r-1}y\binom{r-1}{k}x^k y^{r-1-k}+\binom{n-1}{r-1}x\binom{r-1}{k-1}x^{k-1}y^{r-1-k}.
\end{aligned}$$

Observe that setting all $u_i = x$ and all $v_i = y$ in the recurrence (3.10) for bivariate elementary symmetric polynomials

$$e^n_{r,k}(u,v) = e^{n-1}_{r,k}(u,v) + v_n\, e^{n-1}_{r-1,k}(u,v) + u_n\, e^{n-1}_{r-1,k-1}(u,v)\ .$$

gives exactly the preceding formula. In other words, (3.10) is the blossoming of the preceding formula, and this gives a clue why the blossoming of (3.17) is now easy. Let

$$\begin{aligned}
X^j(u) &= X(u_1)\otimes\cdots\otimes X(u_j) \text{ with } X(u_\nu) = \mathbf{1}+\dot X(u_\nu) = \mathbf{1}+su_\nu\widehat{a}\\
Y^j(v) &= Y(v_1)\otimes\cdots\otimes Y(v_j) \text{ with } Y(v_\nu) = \mathbf{1}+\dot Y(v_\nu) = \mathbf{1}+sv_\nu\widehat{a}.
\end{aligned}$$

Corollary 3.5. (a) *For $j = 1,\dots,n$ and $r = 1,\dots,n$ for fixed j the blossoms $x^j_r(u)$ of the X^j_r in* Thm.3.3 (a) *can be computed recursively from $x^0 = \mathbf{1}$, i.e., $x^0_r = \delta_{r,0}$, by*

$$x^j_r(u) = x^{j-1}_r(u) + su_j \sum_{i=1}^{r} \widehat{a}_i\, x^{j-1}_{r-i}(u). \tag{3.22}$$

Then for $r = 1,\dots,n$ the blossoms $p_r(u;\bar a)$ of the polynomials $p_r(x;\bar a)$ satisfy the equation

$$p_r(u;\bar a) = x^r_r(s,u;\bar a) \text{ with substitutions } s^l \leftarrow (r-l)!\ . \tag{3.23}$$

Moreover, the expressions $x^j_r(u)$ are essentially the j-truncations $p^{[j]}_r(u;\bar a)$ of the blossoms $p_r(u;\bar a)$ according to

$$p^{[j]}_r(u;\bar a) = \frac{r!}{j!}x^j_r(s,u;\bar a) \text{ with substitutions } s^l \leftarrow (j-l)!\ . \tag{3.24}$$

(b) *For $j = 1,\dots,n$ and $r = 1,\dots,n$, $k = 0,\dots,r$ for fixed j the blossoms $h^{r,j}_k$ of the $H^{r,j}_k$ in* Thm.3.3 (b) *are given by*

$$h^{r,j}_k = h^{r,j}_k(s,u,v;\bar a) = \sum_{l=0}^{j}\binom{j}{l}c^r_{k,l}(u,v) = \sum_{l=0}^{j}\binom{j}{l}\left(\sum_{\nu=0}^{l}\binom{l}{\nu}\dot X^\nu_k(u)\,\dot Y^{l-\nu}_{r-k}(v)\right). \tag{3.25}$$

These blossoms can be computed recursively from $h^{r,0}_k = \delta_{0,r}$ by

$$h^{r,j}_k = h^{r,j-1}_k + sv_j\sum_{i=1}^{r-k}\widehat{a}_i\, h^{r-i,j-1}_k + su_j\sum_{i=1}^{k}\widehat{a}_i\, h^{r-i,j-1}_{k-i}. \tag{3.26}$$

Then for $r = 1, \dots, n$ *and* $k = 0, \dots, r$ *the blossoms* $\widetilde{B}_k^r(u, v; \bar{a})$ *of the non-normalized bivariate* $\bar{a}$*-Bernstein polynomials* $\widetilde{B}_k^r(x, y; \bar{a})$ *satisfy the equation*

$$\widetilde{B}_k^r(u, v; \bar{a}) = h_k^{r,r}(s, u, v; \bar{a}) \text{ with substitutions } s^l \leftarrow (r-l)!\ . \tag{3.27}$$

Moreover, the expressions $h_k^{r,j}$ *are essentially the* j*-truncations of the* $\widetilde{B}_k^r(u, v; \bar{a})$ *according to*

$$\widetilde{B}_k^{r,[j]}(u, v; \bar{a}) = \frac{r!}{j!} h_k^{r,j}(s, u, v; \bar{a}) \text{ with substitutions } s^l \leftarrow (j-l)!\ . \tag{3.28}$$

Proof. The proof of (a) is similar to the proof of Thm.3.1. For the proof of (b) one observes first that the $h_k^{r,j}$ of (3.25) are indeed the blossoms of the $H_k^{r,j}$ as defined in (3.16): the polynomials $h_k^{r,j}$ are symmetric and multiaffine by the properties of convolutional multiplication and they become identical with the $H_k^{r,j}$ by setting all $u_j = x$ and all $v_j = y$. That the polynomials $h_k^{r,j}$ satisfy the recursion (3.26) can be seen by a calculation similar to the calculation in the proof of Thm.3.3 (b) that uses the recurrence (3.10) for bivariate elementary symmetric polynomials. □

We observed above that the down recurrence (1.3) for the non-normalized $\bar{a}$-Bernstein polynomials $\widetilde{B}_k^r$ cannot be used directly for blossoming. Instead we had to switch to the down recurrence (3.17-18) that could be blossomed easily.

Similarly, the up recurrence (1.1) or $\bar{a}$-de Casteljau algorithm for the $\bar{a}$-Bézier curve (1.2) cannot be blossomed directly. Again we have to switch to a suitable up recurrence:

Corollary 3.6. *Consider a sequence* $\mathbf{P} = (P_0, \dots, P_n)$ *of control points in* $\mathbb{R}^N$ *and a sequence of real parameters* $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$*,* $\bar{a}_1 \neq 0$ *with* $p_n(1) \neq 0$*.*
For $j = 1, \dots, n$*,* $r = 0, \dots, n$*,* $k = 0, \dots, n-r$ *compute the expressions* $K_k^{r,j} = K_k^{r,j}(s, x, y; \bar{a}; \mathbf{P})$ *recursively from the initial values* $K_k^{r,0} = \delta_{r,0} P_k$ *for* $j = 0$ *by*

$$K_k^{r,j} = K_k^{r,j-1} + s \sum_{i=1}^{r} \widehat{a}_i \left(y\, K_k^{r-i,j-1} + x\, K_{k+i}^{r-i,j-1} \right). \tag{3.29}$$

Then the (non-normalized bivariate) $\bar{a}$-Bézier curve $\widetilde{\mathbf{X}}(x, y; \bar{a}; \mathbf{P})$ *is given by*
(3.30)

$$\widetilde{\mathbf{X}}(x, y; \bar{a}; \mathbf{P}) = \widetilde{\mathbf{X}}_0^n(x, y; \bar{a}; \mathbf{P}) = K_0^{n,n}(s, x, y; \bar{a}; \mathbf{P}) \text{ with substitutions } s^l \leftarrow (n-l)!\ .$$

Moreover, the (non-normalized bivariate) intermediary $\bar{a}$-Bézier curves $\widetilde{\mathbf{X}}_k^r(x, y; \bar{a}; \mathbf{P})$ *of order* r *are given by*

$$\widetilde{\mathbf{X}}_k^r(x, y; \bar{a}) = K_k^{r,n}(s, x, y; \bar{a}; \mathbf{P}) \text{ with substitutions } s^l \leftarrow (r-l)!\ . \tag{3.31}$$

To get the $\bar{a}$*-Bézier curve* (1.2) *on the interval* $[0, 1] \subset \mathbb{R}$ *one has to use the canonical substitution* $x = t$*,* $y = 1 - t$ *and to divide by* $p_n(1)$*. With the same substitution but* *division by $p_r(1)$ one gets the intermediary $\bar{a}$-Bézier curve of order r*

$$\mathbf{X}_k^r(t;\bar{a};\mathbf{P}) = \sum_{i=k}^{k+r} P_i B_{k+i}^r(t;\bar{a}) \text{ for } k = 0,\ldots,n-r. \tag{3.32}$$

On a general interval $[a,b] \subset \mathbb{R}$ with $p_r(b-a) \neq 0$ for $r = 1,\ldots,n$ one has to substitute $x = t-a$ and $y = b-t$ and to divide by $p_r(b-a)$ to get these curves.

The proof of this corollary is a straightforward calculation similarly to the one in [13] Thm.4.6 using (3.17-18). Blossoming the $\bar{a}$-de Casteljau algorithm using (3.26-27) is now easy.

Corollary 3.7. *Consider a sequence $\mathbf{P} = (P_0,\ldots,P_n)$ of control points in $\mathbb{R}^N$ and a sequence of real parameters $\bar{a} = (\bar{a}_1,\ldots,\bar{a}_n)$, $\bar{a}_1 \neq 0$ with $p_n(1) \neq 0$.*
For $j = 1,\ldots,n$, $r = 1,\ldots,n$, $k = 0,\ldots,n-r$ one computes the blossoms $k_k^{r,j} = k_k^{r,j}(s,u,v;\bar{a};\mathbf{P})$ of the expressions $K_k^{r,j} = K_k^{r,j}(s,x,y;\bar{a};\mathbf{P})$ recursively from the initial values $k_k^{r,0} = \delta_{r,0}P_k$ for $j = 0$ by

$$k_k^{r,j} = k_k^{r,j-1} + s\sum_{i=1}^{r} \widehat{a}_i \left(v_j\, k_k^{r-i,j-1} + u_j\, k_{k+i}^{r-i,j-1}\right). \tag{3.33}$$

Then the blossom $\widetilde{\mathbf{x}}(u,v;\bar{a};\mathbf{P})$ of the $\bar{a}$-Bézier curve $\widetilde{\mathbf{X}}(x,y;\bar{a};\mathbf{P})$ is given by

$$\widetilde{\mathbf{x}}(u,v;\bar{a};\mathbf{P}) = k_0^{n,n}(s,u,v;\bar{a};\mathbf{P}) \text{ with substitutions } s^l \leftarrow (n-l)!\ . \tag{3.34}$$

Moreover, the blossoms $\widetilde{\mathbf{x}}_k^r(u,v;\bar{a};\mathbf{P})$ of the intermediary $\bar{a}$-Bézier curve $\widetilde{B}_k^r(x,y;\bar{a};\mathbf{P})$ of order r are given by

$$\widetilde{\mathbf{x}}_k^r(u,v;\bar{a};\mathbf{P}) = k_k^{r,n}(s,u,v;\bar{a};\mathbf{P}) \text{ with substitutions } s^l \leftarrow (r-l)!\ . \tag{3.35}$$

For the blossom $\mathbf{x}(u;\bar{a};\mathbf{P})$ of the $\bar{a}$-Bézier curve (1.2) on the interval $[0,1] \subset \mathbb{R}$ one sets $v_j = 1-u_j$ and divides by $p_n(1)$. With the substitutions $u_j \leftarrow u_j - a$, $v_j = b - u_j$ and division by $p_n(b-a) \neq 0$ one gets the blossom $\mathbf{x}(u;\bar{a};\mathbf{P})$ on a general interval $[a,b] \subset \mathbb{R}$. Similarly one gets the blossoms of the intermediary Bézier curve.

Remark 3.8. The ordinary de Casteljau algorithm generates a new (intermediary) control point on level r from two adjacent points on level $r-1$. This pattern can be mimicked in the recursions (3.29) and (3.33) by distributing $K_k^{r,j-1}$ and $k_k^{r,j-1}$ suitably to the left y- resp. v-part and the right x- resp. u-part of the sums:

$$K_k^{r,j} = y\left(K_k^{r,j-1} + s\sum_{i=1}^{r} \widehat{a}_i\, K_k^{r-i,j-1}\right) + x\left(K_k^{r,j-1} + s\sum_{i=1}^{r} \widehat{a}_i\, K_{k+i}^{r-i,j-1}\right) \tag{3.36}$$

$$k_k^{r,j} = v_j\left(k_k^{r,j-1} + s\sum_{i=1}^{r} \widehat{a}_i\, k_k^{r-i,j-1}\right) + u_j\left(k_k^{r,j-1} + s\sum_{i=1}^{r} \widehat{a}_i\, k_{k+i}^{r-i,j-1}\right) \tag{3.37}$$

where it is assumed that $x+y=1$ and $u_j+v_j=1$, otherwise one has to divide explicitly by the sums. The expressions in the brackets can be seen as generalized left and right intermediary control points on level $r-1$.

4. Applications of $\bar{a}$-blossoming

4.1. **Dual functional property.** For a sequence $\mathbf{P}=(P_0,\ldots,P_n)$ of control points in $\mathbb{R}^N$ let $\mathbf{x}(u;\mathbf{P})$ be the blossom of the Bézier curve

$$\mathbf{X}(t;\mathbf{P})=\sum_{k=0}^{n} P_k B_k^n(t)$$

on the (canonical) interval $[0,1]\subset\mathbb{R}$. It is well-known that the control points P_k can be recovered via the *dual functional property* from the blossom by

$$\mathbf{x}(1^{(k)},0^{(n-k)})=P_k \text{ for } k=0,\ldots,n. \tag{4.1}$$

In [8] the h-Bernstein functions of order n with parameter $h\in\mathbb{R}$ on an arbitrary interval $[a,b]\subset\mathbb{R}$ are defined by

$$B_k^n(t;[a,b];h)=\binom{n}{k}\frac{\prod_{i=0}^{k-1}(t-a+ih)\prod_{i=0}^{n-k-1}(b-t+ih)}{\prod_{i=0}^{n-1}(b-a+ih)} \text{ for } k=0,\ldots,n \tag{4.2}$$

and the h-Bézier curves by

$$\mathbf{X}(t;[a,b];h;\mathbf{P})=\sum_{k=0}^{n} P_k B_k^n(t;[a,b];h).$$

Then the control points P_k can be recovered from the h-blossom $\mathbf{x}(u;h)=\mathbf{x}(u;[a,b];h;\mathbf{P})$ of the h-Bézier curve via

$$\mathbf{x}(\underbrace{b,\ldots,b+(k-1)h}_{k},\underbrace{a-kh,\ldots,a-(n-1)h}_{n-k};h)=P_k \text{ for } k=0,\ldots,n. \tag{4.3}$$

Clearly, (4.1) is a special case (4.3) by setting $a=0$, $b=1$, and $h=0$. According to Thm.2.1 of [12] the h-Bernstein functions, h-Bézier curves, and h-blossoms are special cases of $\bar{a}$-Bernstein polynomials, $\bar{a}$-Bézier curves, and $\bar{a}$-blossoms. Thus the question arises how to understand the special formula (4.3) in the $\bar{a}$-context. The next definitions provide a general framework for this task.

Definition 4.1. *(a) Let $c=(c_n)_{n\geq 0}\in l(\mathbb{R})$ be a sequence of real numbers with $c_0\neq 0$. Then a sequence of polynomials $(p_n)_{n\geq 0}$ with $p_n=p_n(t;c)\in\mathbb{R}[x]$ is* c-cumulative *if recursively $p_n(t;c)=(t-c_n)p_{n-1}(t;c)$ for $n\in\mathbb{N}$ and $p_0(t;c)=c_0$.*
(b) For c-cumulative $p_n(t;c)$ the Bernstein-type c-polynomials *of order n are defined by*

$$B_k^n(t;c):=\frac{1}{\rho_n}\binom{n}{k}p_k(t;c)p_{n-k}(1-t;c)$$

with suitable normalizing factor ρ_n. *For c-cumulative* $p_n(x;c)$ *and d-cumulative* $q_n(y;d)$ *the bivariate Bernstein-type* (c,d)-polynomials *of order* n *are defined by*

$$B_k^n(x,y;c) := \frac{1}{\rho_n}\binom{n}{k} p_k(x;c)p_{n-k}(y;c).$$

(c) For a polynomial $P \in \mathbb{R}[t]$ *the* c-blossom $p(u_{(n)};c)$ *of order* $n \geq \deg P$ *is the symmetric multiaffine polynomial that satisfies the* c-diagonal property

$$P(t) = p(t - c_1, \ldots, t - c_n; c).$$

For a bivariate $P \in \mathbb{R}[x,y]$ *the bivariate* (c,d)-blossom $p(u_{(n)}, v_{(n)}; c, d)$ *of order* $n \geq \deg P$ *is the bivariate symmetric multiaffine polynomial that satisfies the* (c,d)-diagonal property

$$P(x,y) = p(x - c_1, \ldots, x - c_n, y - d_1, \ldots, y - d_n; c, d).$$

To construct the c-blossom $p(u_{(n)};c)$ of order n of $P(t)$ one proceeds as follows:

(1) Compute the polynomials $\overline{e}_k^n(t)$ by substituting $u_i = t - c_i$ for $i = 1, \ldots, n$ in the elementary symmetric polynomials $e_k^n(u)$ for $k = 0, \ldots, n$.
(2) Since $\deg \overline{e}_k^n(t) = k$, it is possible to express the monomials t^k as linear combinations of the $\overline{e}_k^n(t)$. (It is usually difficult to find explicit formulas).
(3) To get $p(u_{(n)};c)$ express the monomials t^k in $P(t)$ by the linear combinations as determined in step 2, and then replace the $e_k^n(u)$ by the $\overline{e}_k^n(t)$.

Similarly, the (c,d)-blossom $p(u_{(n)}, v_{(n)}; c, d)$ of $P(x,y)$ can be constructed using the bivariate elementary symmetric polynomials $e_{r,k}^n(u,v)$.

Clearly with the canonical substitution $(x,y) = (t, 1-t)$, the ordinary Bernstein polynomials of any order are a special case of c-polynomials with $c = (1,0,0,0,\ldots)$.

The h-Bernstein basis functions can be rewritten as

$$B_k^n(t;[a,b];h) = \binom{n}{k}\frac{p_k(t-a)p_{n-k}(b-t)}{p_n(b-a)} = \binom{n}{k}\frac{p_k(x)p_{n-k}(y)}{p_n(x+y)} \quad \text{with}$$
$$p_n(x) = \prod_{i=0}^{n-1}(x+ih) = \prod_{i=1}^{n}(x+(i-1)h),\ x = t-a,\ y = b-t.$$

Therefore these basis functions are of the form $B_k^n(x,y;c,c)$ with $c_0 = 1$, $c_i = (i-1)h$ for $i \geq 1$, and normalizing factor $\rho_n = p_n(x+y)$. Alternatively, they are of the form $B_k^n(t-a, b-t; c, d)$ with $c_0 = d_0 = 1$ and $c_i = a - (i-1)h$, $d_i = b + (i-1)h$ for $i \geq 1$.

Theorem 4.2. *Let* $b_s^n(u,v;c,d)$ *be the blossom of* $B_s^n(t-a, b-t; c, d)$. *If the* c_i *and* d_i *are not zeros of the* $p_{n-s}(b-t;d)$ *and* $p_s(t-a;c)$, *then for* $k = 1, \ldots, n$:

$$b_s^n(d_1, \ldots, d_k, c_{k+1}, \ldots, c_n) = \delta_{s,k}. \tag{4.4}$$

Proof. From the definitions one sees that

$$b^n_s(u,v;c,d) = \frac{1}{\rho_n}\frac{\binom{n}{s}}{\binom{n}{n}\binom{n}{s}}e^n_{n,s}(u,v) = \frac{1}{\rho_n}\sum_{I\in\mathcal{C}(n,s)} u_I v_{[n]\setminus I} \tag{4.5}$$

with $u_i = t - c_i$, $v_i = d_i - t$, and $\rho_n = \prod_{i=1}^n (d_i - c_i)$. If now $s > k$ and $I \in \mathcal{C}(n,s)$, then $I \cap \{k+1,\dots,n\} \neq \emptyset$ and $t = c_i$ or $u_i = 0$ for some $i \in I$. Similarly, if $s < k$ and $J = [n] \setminus I \in \mathcal{C}(n, n-s)$, then $J \cap \{1,\dots,k\} \neq \emptyset$ and $t = d_i$ or $v_i = 0$ for some $i \in J$. Finally, for $k = s$ the product $u_I v_{[n]\setminus I}$ does not vanish iff $I = \{1,\dots,k\}$, but then $u_I v_{[n]\setminus I} = \rho_n$, which proves (4.4). □

Now (4.3) follows as a special case of (4.4) for (c,d)-Bézier curves.

Although the (c,d)-theory can be pursued in general and independently of the $\bar{a}$-theory, we investigate it here only in the context of the $\bar{a}$-theory. The next theorem shows that the only c-cumulative $\bar{a}$-Bernstein polynomials are the h-Bernstein functions.

Theorem 4.3. *For $\bar{a} = (1, \bar{a}_2, \bar{a}_3, \dots)$ the sequence of associated polynomials $p_n(x;\bar{a})$ is c-cumulative iff $p_n(x;\bar{a}) = \prod_{i=1}^n (x + (i-1)h)$, $\bar{a}_n = (-1)^{n-1}(n-1)!\ h^{n-1}$, and $c_n = (n-1)h$ for $n \geq 1$.*

Proof. By induction on n. Always with $\bar{a}_1 = 1$ one computes from the definition (2.2-3) of associated polynomials $p_0(x;\bar{a}) = 1$, $p_1(x;\bar{a}) = x$, and $p_2(x;\bar{a}) = x(x + \bar{a}_2)$; hence $c_0 = 1$ by definition (not a zero), $c_1 = 1$ and $c_2 = -\bar{a}_2 =: h$. With the notation

$$(x;h)_n = \prod_{i=1}^n (x + (i-1)h) \text{ and } (x)_n = (x;1)_n$$

for the falling factorials the well-known recursion ([7], p.58)

$$(x)_{n+1} = x\sum_{k=0}^n (-1)^{n-k}(n)_{n-k}(x)_k \tag{4.6}$$

can be rewritten as

$$(x)_n = x\sum_{k=1}^n (-1)^{k-1}(n-1)_{k-1}(x)_{n-k}$$

and further as

$$\begin{aligned}(x;h)_n &= h^n \left(\frac{x}{h}\right)_n \\ &= h^n\ x\sum_{k=1}^n (-1)^{k-1}(n-1)_{k-1}\left(\frac{n}{h}\right)_{n-k} \\ &= x\sum_{k=1}^n (-1)^{k-1}\binom{n-1}{k-1}(k-1)!\ h^{k-1}(n;h)_{n-k}.\end{aligned}$$

Now assume that the assertions of the theorem for $\bar{a}_k$, c_k, and $p_k(x;\bar{a})$ are true for $k = 0, \ldots, n-1$. Then by the recursion (2.12)

$$\begin{aligned} p_n(x;\bar{a}) &= x \sum_{k=1}^{n} \binom{n-1}{n-k} \bar{a}_k \; p_{n-k}(x;\bar{a}) \\ &= x \sum_{k=1}^{n} \binom{n-1}{k-1} \; (-1)^{k-1}(k-1)! h^{k-1}(x;h)_{n-k} = (x;h)_n \\ &= (x-(n-1)h) \; (x;h)_{n-1} \; . \end{aligned}$$

Hence, $p_n(x;\bar{a})$ is c-cumulative iff $p_n(x;\bar{a}) = (x-(n-1)h) \; (x;h)_{n-1} = p_n(x;c)$ with $c_n = (n-1)h$. But since $p_n(x;\bar{a}) = x^n + \cdots + \bar{a}_n x$ a comparison of coefficients with $p_n(x;c)$ shows that $\bar{a}_n = (-1)^{n-1}(n-1)! \; h^{n-1}$. □

4.2. **$\bar{a}$-dual functional property and $\bar{a}$-subdivision.** For an arbitrary sequence of control points $\mathbf{P} = (P_0, \ldots, P_n)$ any $\bar{a}$-Bézier curve is also an ordinary Bézier curve for a suitably transformed sequence of $\bar{a}$-control points $\mathbf{P}(\bar{a}) = (P_0(\bar{a}), \ldots, P_n(\bar{a}))$:

$$\begin{aligned} X(t;\bar{a};\mathbf{P}) &= \sum_{k=0}^{n} P_k B_k^n(t;\bar{a}) \\ &= \sum_{k=0}^{n} P_k(\bar{a}) B_k^n(t). \end{aligned}$$

In [12] it has been shown that

$$\mathbf{P}(\bar{a}) = \mathbf{P} \; M(\bar{a}) \tag{4.7}$$

with the $\bar{a}$-transformation matrix $M(\bar{a}) = \frac{1}{\rho_n}(\widetilde{m}_{kl}) \in \mathbb{R}^{(n+1)\times(n+1)}$,

$$\widetilde{m}_{kl} = \frac{\binom{n}{k}}{\binom{n}{l}} \sum_{j=0}^{k} \pi_{k,j} \; \pi_{n-k,n-k-l+j}, \text{ and} \tag{4.8}$$

$$\pi_{k,j} = \sum_{\nu=0}^{j} \binom{k-\nu}{k-j} p_{k,\nu}. \tag{4.9}$$

Note that we use the notation $\mathbf{P}$ and $\mathbf{P}(\bar{a})$ both for sequences of points in $\mathbb{R}^N$ and for matrices with the respective sequences of column vectors. At the same time $M(\bar{a})$ is the matrix of basis transformation from ordinary Bernstein polynomials to $\bar{a}$-Bernstein polynomials according to

$$\mathbf{B}^n(t;\bar{a}) = M(\bar{a}) \; \mathbf{B}^n(t),$$

where $\mathbf{B}^n(t;\bar{a}) = (B_0^n(t;\bar{a}), \ldots, B_n^n(t;\bar{a}))^T$ and $\mathbf{B}^n(t) = (B_0^n(t), \ldots, B_n^n(t;))^T$.

Now the proper $\bar{a}$-generalization of the ordinary dual functional property (4.1) is not the one that leads to the original control polygon $\mathbf{P}$, but the *$\bar{a}$-dual functional property* described by the following theorem that leads to the $\bar{a}$-control polygon $P(\bar{a})$.

Theorem 4.4. *For a given sequence of control points* $\mathbf{P} = (P_0, \dots, P_n)$ *and a feasible parameter sequence* $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n)$*, the* $\bar{a}$*-control points* $P(\bar{a}) = (P_0(\bar{a}), \dots, P_n(\bar{a}))$ *are given by*

$$\mathbf{x}(1^{(k)}, 0^{(n-k)}; \bar{a}; \mathbf{P}) = P_k(\bar{a}) \text{ for } k = 0, \dots, n, \tag{4.10}$$

where $\mathbf{x}(u, v; \bar{a}; \mathbf{P})$ *is the blossom of the* $\bar{a}$*-Bézier curve* $X(t; \bar{a}; \mathbf{P})$ *with* $v_j = 1 - u_j$*.*

Proof. This assertion follows from the blossom of

$$X(t; \bar{a}; \mathbf{P}) = \sum_{k=0}^{n} P_k(\bar{a}) B_k^n(t)$$

and (4.1). □

Hence $\mathbf{P}(\bar{a})$ can be computed from $\mathbf{P}$ with $M(\bar{a})$ by $P(\bar{a}) = P\, M(\bar{a})$ or, alternatively, using blossoming.

Corollary 4.5. *Assume that* $X(x, y; \bar{a}; \mathbf{P})$ *has been computed by the recursion* (3.29-30) *of Cor.*3.7 and

$$X(x, y; \bar{a}; \mathbf{P}) = \sum_{k=0}^{n} \alpha_k x^k y^{n-k} \tag{4.11}$$

with $y = 1 - x$. Then

$$P_k(\bar{a}) = \alpha_k / \binom{n}{k} \text{ for } k = 0, \dots, n. \tag{4.12}$$

Proof. Since the bivariate blossom of $\binom{n}{k} x^k y^{n-k}$ is

$$e_{n,k}^n(u, v) = \sum_{I \in \mathcal{C}(n,n)} \sum_{J \in \mathcal{C}(I,k)} u_J v_{I \setminus J} = \sum_{J \in \mathcal{C}(n,k)} u_J v_{I \setminus J}$$

this assertion follows from (4.9). □

If an ordinary Bézier curve is subdivided at some point $t_0 \in (0, 1)$, it is well-known [2, 3] that the control points $Q_0, \dots, Q_n$ of the left part of the curve and the control points $R_0, \dots, R_n$ of the right part of the curve can be read off from the left resp. right side of the triangular scheme of the ordinary de Casteljau algorithm. This simple approach does not work for the $\bar{a}$-de Casteljau algorithm, but it is possible to do something similar using the generalized left and right intermediary control points that generate $K_0^{n,n}(x, y; \bar{a}; \mathbf{P})$ as discussed in Rem.3.9. We will not give details here, because it is more efficient to apply ordinary subdivision directly to the points of $\mathbf{P}(\bar{a})$ or use the blossom $\mathbf{x}(u; \bar{a}; \mathbf{P})$ of the $\bar{a}$-Bézier curve from Cor.3.7 to compute

$$Q_k = \mathbf{x}(a^{(n-k)}, t_0^{(k)}; \bar{a}; \mathbf{P})$$
$$R_k = \mathbf{x}(t_0^{(n-k)}, b^{(k)}; \bar{a}; \mathbf{P}).$$

With respect to h-Bernstein functions the $\bar{a}$-control points $\mathbf{P}(\bar{a})$ and their (ordinary) subdivisions certainly have better geometric approximation properties than the original control points $\mathbf{P}$ and their h-subdivisions (compare Figs.6-9 of [8]).

5. $\bar{a}$-DEGREE ELEVATION

For a given sequence of control points $\mathbf{P} = (P_0, \ldots, P_n)$ the *degree elevated sequence of control points* $\mathbf{Q} = (Q_0, \ldots, Q_{n+1})$ is defined by $Q_0 = P_0$, $Q_{n+1} = P_n$, and

$$Q_k = \frac{n+1-k}{n+1} P_k + \frac{k}{n+1} P_{k-1} \quad \text{for } k = 1, \ldots, n$$

so that the Bézier curve of degree n for $\mathbf{P}$ is equal to the Bézier curve of degree $n+1$ for $\mathbf{Q}$. With the *degree elevation matrix* $D(n) \in \mathbb{R}^{(n+1)\times(n+2)}$ defined by

$$D(n)_{k,j} = \begin{cases} \frac{n+1-k}{n+1}, & j = k \\ \frac{k+1}{n+1}, & j = k+1 \\ 0, & \text{otherwise} \end{cases}$$

and the notation in section 4.2 one has $\mathbf{B}^n(t) = D(n)\mathbf{B}^{n+1}(t)$ (compare [3]). But this formula gives for the Bézier curve

$$\mathbf{X}(t; \mathbf{P}) = \mathbf{P}\mathbf{B}^n(t) = \mathbf{P}D(n)\mathbf{B}^{n+1}(t) = \mathbf{Q}\mathbf{B}^{n+1}(t)$$

or

$$\mathbf{Q} = \mathbf{P}D(n). \tag{5.1}$$

Therefore we define in accordance with $\mathbf{P}(\bar{a}) = \mathbf{P}M(\bar{a})$

$$\mathbf{Q}(\bar{a}) := \mathbf{P}(\bar{a})D(n) = \mathbf{P}M(\bar{a})D(n) \tag{5.2}$$

$$\mathbf{Q}(\bar{b}) := \mathbf{Q}M(\bar{b}) = \mathbf{P}D(n)M(\bar{b}). \tag{5.3}$$

In other words, $\mathbf{Q}(\bar{a})$ is the degree elevated $\bar{a}$-transformation of $\mathbf{P}$ and $\mathbf{Q}(\bar{b})$ is the $\bar{b}$-transformation of the degree elevated $\mathbf{P}$. Equations (5.2) and (5.3) lead to the following $(\mathbf{Q}(\bar{a}), \mathbf{Q}(\bar{b}))$-**Problem** (of degree n): For a given parameter sequence $\bar{a} = (1, \bar{a}_2, \ldots, \bar{a}_n)$ does there exist a parameter sequence $\bar{b} = (1, \bar{b}_2, \ldots, \bar{b}_{n+1})$ such that $\mathbf{Q}(\bar{a}) = \mathbf{Q}(\bar{b})$?

Or with a diagram:

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{D(n)} & \mathbf{Q} \\ {\scriptstyle M(\bar{a})}\downarrow & & \searrow{\scriptstyle M(\bar{b})} \\ \mathbf{P}(\bar{a}) & \xrightarrow{D(n)} & \mathbf{Q}(\bar{a}) \overset{?}{=} \mathbf{Q}(\bar{b}) \end{array} \tag{5.4}$$

Using the non-normalized matrices $\widetilde{D}(n) = (n+1)D(n)$, $\widetilde{M}(\bar{a}) = \rho_n(\bar{a})M(\bar{a})$, $\widetilde{M}(\bar{b}) = \rho_{n+1}(\bar{b})M(\bar{b})$, and (5.2-3) the equation $\mathbf{Q}(\bar{a}) = \mathbf{Q}(\bar{b})$ is equivalent to the matrix equation

$$\rho_{n+1}(\bar{b})\, \widetilde{M}(\bar{a})\, \widetilde{D}(n) = \rho_n(\bar{a})\, \widetilde{D}(n)\, \widetilde{M}(\bar{b})\ . \tag{5.5}$$

Since the $\bar{a}$-transformation matrix $M(\bar{a}) = (m_{k,l}(\bar{a}))$ is *point-symmetric* ([12]) with $m_{k,l}(\bar{a}) = m_{n-k,n-l}(\bar{a})$ for $k, l = 0, \dots, n$, equation (5.5) a system of $\frac{1}{2}n(n+1)$ polynomial equations of degree $\leq n$ in the variables $\bar{a}_2, \dots, \bar{a}_n, \bar{b}_2, \dots, \bar{b}_{n+1}$ with rational coefficients.

Calculations (with MAPLE) provide the following solutions for the $(\mathbf{Q}(\bar{a}), \mathbf{Q}(\bar{b}))$-problem for degrees $n = 1, \dots, 5$:

n	$\bar{b}_2, \dots, \bar{b}_n$
1	$\bar{b}_2$ arbitrary
2	$\bar{b}_2$ arbitrary $\bar{b}_3 = 2\bar{a}_2\bar{b}_2 + \bar{a}_2 - \bar{b}_2$
3	$\bar{b}_2$ arbitrary $\bar{b}_3 = 3\bar{a}_2\bar{b}_2 - \bar{b}_2^2 + \bar{a}_2 - \bar{b}_2$ $\bar{b}_4 = -3\bar{a}_2\bar{b}_2 + \bar{b}_2^2 - \bar{a}_2 + \bar{b}_2 + 3\bar{a}_3\bar{b}_2 + \bar{a}_3$
4	$\bar{b}_2 = \bar{a}_2$ $\bar{b}_3 = 2\bar{a}_2^2$ $\bar{b}_4 = -2\bar{a}_2^2 - 2\bar{a}_2^3 + 4\bar{a}_2\bar{a}_3 + \bar{a}_3$ $\bar{b}_5 = 2\bar{a}_2^2 + 2\bar{a}_2^3 - 4\bar{a}_2\bar{a}_3 - \bar{a}_3 + 4\bar{a}_2\bar{a}_4 + \bar{a}_4$
5	$\bar{b}_2 = \bar{a}_2$ $\bar{b}_3 = 2\bar{a}_2^2$ $\bar{b}_4 = -2\bar{a}_2^2 - 4\bar{a}_2^3 + 5\bar{a}_2\bar{a}_3 + \bar{a}_3$ $\bar{b}_5 = 2\bar{a}_2^2 + 4\bar{a}_2^4 + 5\bar{a}_2\bar{a}_4 - \bar{a}_3 + 6\bar{a}_2^3 - 5\bar{a}_2^2\bar{a}_3 - 6\bar{a}_2\bar{a}_3 + \bar{a}_4$ $\bar{b}_6 = -2\bar{a}_2^2 - 4\bar{a}_2^4 - 5\bar{a}_2\bar{a}_4 + \bar{a}_3 - 6\bar{a}_2^3 + 5\bar{a}_2^2\bar{a}_3 + 6\bar{a}_2\bar{a}_3 - \bar{a}_4 + 5\bar{a}_2\bar{a}_5 + \bar{a}_5$

A closer analysis of these solutions shows that they follow the pattern

$$\bar{b}_{k+1} = (1 + n\bar{a}_2)\,\bar{a}_k - (1 + (n-k)\bar{a}_2)\,\bar{b}_k \quad \text{for } k = 1, \dots, n \tag{5.6}$$

when one always sets $b_2 = a_2$. Unfortunately, this pattern does not work for $n = 6$ and we conjecture that the $(\mathbf{Q}(\bar{a}), \mathbf{Q}(\bar{b}))$-problem of degree $n \geq 6$ has no solution. This conjecture is also the reason why we presented the explicit solutions for $n \leq 5$. To solve of the system (5.5) it seems therefore necessary to impose conditions on the choice of $\bar{a}$ as given by the following result. Note that (5.7) is a specialization of (5.6) for $\bar{a}_2 = 0$.

Theorem 5.1. *Let $N = 2n$ or $N = 2n + 1$ for some $n \geq 2$. Then the $(\mathbf{Q}(\bar{a}), \mathbf{Q}(\bar{b}))$-problem of degree N for the parameter sequence $\bar{a} = (1, \bar{a}_2, \dots, \bar{a}_N)$ is solved by*

$$\bar{b}_{k+1} = \bar{a}_k - \bar{b}_k \quad \textit{for } k = 1, \dots, N \textit{ and } \bar{b}_1 = 1, \tag{5.7}$$

if $\bar{a}_2 = \cdots = \bar{a}_n = 0$. In particular, $\bar{b}_2 = \cdots = \bar{b}_{n+1} = 0$.

Proof. For $\bar{a}_1 = 1$ and $\bar{a}_2 = \cdots = \bar{a}_n = 0$ one sees from (2.8) that $\bar{a}_\lambda$ makes a contribution to $p_{N,k}(\bar{a})$ only if

$$\begin{aligned} &\lambda_1 = 1,\ k = N \text{ and } \lambda = 1^N \text{ or} \\ &n+1 \le \lambda_1 \le N,\ 1 \le k \le n \text{ and } \lambda = \lambda_1 1^{k-\lambda_1}. \end{aligned}$$

With formula (2.8) for the c_λ one sees

$$p_{N,k}(\bar{a}) = \begin{cases} \binom{N}{k-1}\bar{a}_{N+1-k}, & \text{for } 1 \le k < N \\ \bar{a}_1 = 1, & \text{for } k = N \end{cases} \tag{5.8}$$

and therefore

$$\rho_N(\bar{a}) = 1 + \sum_{k=1}^{n-1} \binom{N}{k-1} \bar{a}_{N+1-k}. \tag{5.9}$$

From (5.7) it is not hard to see that

$$\bar{b}_{i+1} = \sum_{j=n+1}^{i} (-1)^{i-j} \bar{a}_j \quad \text{for } n+1 \le i < N \tag{5.10}$$

To compute $\rho_N(\bar{b})$ one combines (5.9) and (5.10) and with some straightforward transformations and Pascal's identity for binomial coefficients one gets

$$\rho_{N+1}(\bar{b}) = \rho_N(\bar{a}). \tag{5.11}$$

Under the assumption of the theorem the proof of (5.5) is then reduced to the proof of

$$A_{i,j} = B_{i,j} \quad \text{for } i = 0, \dots, N \text{ and } j = 0, \dots, N+1, \tag{5.12}$$

where $A = \widetilde{M}(\bar{a})\,\widetilde{D}(n)$ and $B = \widetilde{D}(n)\,\widetilde{M}(\bar{b})$. Since the matrices $\widetilde{M}(\bar{a})$, $\widetilde{M}(\bar{b})$ and $\widetilde{D}(n)$ are point-symmetric and a product of point-symmetric matrices is again point-symmetric and since $\widetilde{m}_{i,0}(\bar{a}) = \rho(\bar{a})\,\delta_{i,0}$ and $\widetilde{m}_{i,0}(\bar{b}) = \rho(\bar{b})\,\delta_{i,0}$, the cases $j = 0$ and $j = N+1$ are done and we focus on $1 \le j \le N$.

From (4.8) and (5.8) it follows that

$$\pi_{i,j}(\bar{a}) = \begin{cases} \sum_{\nu=1}^{j} \binom{i-\nu}{j-\nu}\binom{i}{\nu-1}\bar{a}_{i+1-\nu}, & \text{for } n+1 \le i \le N \\ \delta_{i,j}, & \text{for } 0 \le i \le n. \end{cases} \tag{5.13}$$

By point-symmetry we can restrict our proof of (5.12) to the case $i \ge n+1$, i.e., the lower half of (5.12). But then (4.7) with (5.13) gives

$$\widetilde{m}_{i,j}(\bar{a}) = \frac{\binom{N}{i}}{\binom{N}{j}} \pi_{i,j}(\bar{a}). \tag{5.14}$$

With the formulas obtained so far one computes

$$\begin{aligned}A_{i,j} &= \sum_{k=0}^{N} \widetilde{m}_{i,k}(\bar{a})\widetilde{d}_{k,j}\\ &= \widetilde{m}_{i,j}(\bar{a})\widetilde{d}_{j,j} + \widetilde{m}_{i,j-1}(\bar{a})\widetilde{d}_{j-1,j}\\ &= (N+1-j)\widetilde{m}_{i,j}(\bar{a}) + j\widetilde{m}_{i,j-1}(\bar{a})\\ &= (N+1-j)\frac{\binom{N}{i}}{\binom{N}{j}}\pi_{i,j}(\bar{a}) + j\frac{\binom{N}{i}}{\binom{N}{j-1}}\pi_{i,j-1}(\bar{a})\end{aligned}$$

and similarly

$$\begin{aligned}B_{i,j} &= \sum_{k=0}^{N+1} \widetilde{d}_{i,k}\widetilde{m}_{k,j}(\bar{b})\\ &= \widetilde{d}_{i,i}\widetilde{m}_{i,j}(\bar{b}) + \widetilde{d}_{i,i+1}\widetilde{m}_{i+1,j}(\bar{b})\\ &= (N+1-j)\widetilde{m}_{i,j}(\bar{b}) + (i+1)\widetilde{m}_{i+1,j}(\bar{b})\\ &= (N+1-j)\frac{\binom{N+1}{i}}{\binom{N+1}{j}}\pi_{i,j}(\bar{b}) + (i+1)\frac{\binom{N+1}{i+1}}{\binom{N+1}{j}}\pi_{i+1,j}(\bar{b}).\end{aligned}$$

A miraculous cooperation of the entries of $\widetilde{D}(n)$ and the fractions of binomial coefficients allows one to reduce the equations of (5.12) to the equivalent equations

$$\pi_{i,j}(\bar{a}) + \pi_{i,j-1}(\bar{a}) = \pi_{i,j}(\bar{b}) + \pi_{i+1,j}(\bar{b}). \tag{5.15}$$

With (4.8) one computes

$$\begin{aligned}\pi_{i,j}(\bar{a}) + \pi_{i,j-1}(\bar{a}) &= \sum_{\nu=1}^{j}\binom{i-\nu}{j-\nu}p_{i,\nu}(\bar{a}) + \sum_{\nu=1}^{j-1}\binom{i-\nu}{j-1-\nu}p_{i,\nu}(\bar{a})\\ \pi_{i,j}(\bar{b}) + \pi_{i+1,j}(\bar{b}) &= \sum_{\nu=1}^{j}\binom{i-\nu}{j-\nu}p_{i,\nu}(\bar{b}) + \sum_{\nu=1}^{j}\binom{i+1-\nu}{j-\nu}p_{i+1,\nu}(\bar{b}).\end{aligned}$$

By (5.8) one has a contribution of $\bar{a}_1 = 1$ and $\bar{b}_1 = 1$ in the above equations exactly when $j = i$ or $j = i+1$. In other words: a summand 1 appears in equations (5.15) on both sides exactly when $j = i$ or $j = i+1$. Therefore for the proof of the equations (5.15) we investigate subsequently only whether the coefficients of the $\bar{a}_l$ for $l \geq n+1 > 1$ coincide on both sides.

For better comparison of these coefficients we compute for $\pi'_{i,j}(\bar{a})$, i.e., $\pi_{i,j}(\bar{a})$ without a possible summand $\bar{a}_1 = 1$, and with $l = i - n + 1 - \nu$

$$\pi'_{i,j}(\bar{a}) = \sum_{\nu=1}^{j} \binom{i-\nu}{j-\nu}\binom{i}{\nu-1}\bar{a}_{i+1-\nu} = \sum_{\nu=1}^{i-n} \binom{i-\nu}{j-\nu}\binom{i}{\nu-1}\bar{a}_{i+1-\nu}$$
$$= \sum_{\nu=1}^{i-n} \binom{n-1+l}{j-i+n-1+l}\binom{i}{i-n-1}\bar{a}_{n+l} = \sum_{l=n+1}^{i} \binom{l-1}{i-j}\binom{i}{l}\bar{a}_l.$$

This yields

$$\pi'_{i,j}(\bar{a}) + \pi'_{i,j-1}(\bar{a}) = \sum_{l=n+1}^{i} \binom{l-1}{i-j}\binom{i}{l}\bar{a}_l + \sum_{l=n+1}^{i} \binom{l-1}{i-j+1}\binom{i}{l}\bar{a}_l$$
$$= \sum_{l=n+1}^{i} \binom{l}{i-j+1}\binom{i}{l}\bar{a}_l$$

and similarly with (5.10)

$$\pi'_{i,j}(\bar{b}) + \pi'_{i+1,j}(\bar{b})$$
$$= \sum_{l=n+1}^{i-1}\sum_{\nu=1}^{i-l} \binom{i-\nu}{j-\nu}\binom{i}{\nu-1}(-1)^{i-\nu-l}\bar{a}_l + \sum_{l=n+1}^{i}\sum_{\nu=1}^{i+1-l} \binom{i+1-\nu}{j-\nu}\binom{i+1}{\nu-1}(-1)^{i+1-\nu-l}\bar{a}_l$$
$$= \binom{i}{i-j+1}\bar{a}_i + \sum_{l=n+1}^{i-1}\left[\binom{l}{i+1-j}\binom{i+1}{l+1} + \right.$$
$$\left.\sum_{\nu=1}^{i-l}\left(\binom{i-\nu}{j-\nu}\binom{i}{\nu-1} - \binom{i+1-\nu}{j-\nu}\binom{i+1}{\nu-1}\right)(-1)^{i-\nu-l}\right]\bar{a}_l$$

For $\bar{a}_i$ we get the desired equality of coefficients. For $i > l \geq n+1$ we get with the abbreviation $k = i - j$ the equations

$$\binom{l}{k+1}\binom{i}{l} =$$
$$\binom{l}{k+1}\binom{i+1}{l+1} + \sum_{\nu=1}^{i-l}(-1)^{i-\nu-l}\left[\binom{i-\nu}{k}\binom{i}{\nu-1} - \binom{i+1-\nu}{k+1}\binom{i+1}{\nu-1}\right].$$

Finally, elementary calculations with the cross product formula for binomial coefficients

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

and cancellation of the appearing common factor $\binom{i}{k+1}$ gives

$$\sum_{\nu=0}^{i-l}(-1)^{i-l-\nu}\binom{i-k}{\nu} = \binom{i-k-1}{i-l}. \tag{5.16}$$

But this is a special case of the formula

$$\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{x}{\nu} = \binom{x-1}{n}$$

which can be derived from comparing coefficients of the power series expansions of

$$\frac{1}{1-t}(1-t)^x = (1-t)^{x-1}.$$

□

In view of the conjectured non-solvability of the $(\mathbf{Q}(\bar a), \mathbf{Q}(\bar b))$-problem in general and Theorem 5.1 in particular it is reasonable to ask for a relaxation that imposes no restriction on the parameter sequence $\bar a = (1, \bar a_2, \ldots, \bar a_n)$. This relaxation could be done, e.g., by modifying the degree elevation of $\mathbf{P}(\bar a)$ so that

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{D(n)} & \mathbf{Q} \\ {\scriptstyle M(\bar a)}\downarrow & & \searrow{\scriptstyle M(\bar b)} \\ \mathbf{P}(\bar a) & \xrightarrow{D'(n;\bar a)} & \mathbf{Q}(\bar a) = \mathbf{Q}(\bar b) \end{array} \tag{5.17}$$

for a geometrically or algebraically nice choice of $D'(n;\bar a)$ or by modifying the degree elevation of $\mathbf{P}$ so that

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{D''(n;\bar a)} & \mathbf{Q} \\ {\scriptstyle M(\bar a)}\downarrow & & \searrow{\scriptstyle M(\bar b)} \\ \mathbf{P}(\bar a) & \xrightarrow{D(n)} & \mathbf{Q}(\bar a) = \mathbf{Q}(\bar b) \end{array} . \tag{5.18}$$

The interested reader is invited to pursue this topic.

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