MATHEMATICS AND MUSIC

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Science and art are two clearly separated and at the same time complementary areas of human culture: Science aims at "truth", which can be characterized very broadly as an accordance of theory and reality. Theory consists of a collections of facts and a few principles, from which the facts can be deduced. A good theory "explains" a huge diversity of facts by simple principles and effective rules of deduction. Contrary to the direction taken by science, which reduces diversity to abstract principles, art explores — much like nature itself — the diversity inherent in a given structural framework to its largest possible extent. The main criterion for the selection of items from all possible creations is "beauty" (in a broad sense), which depends very much on the taste of individuals and groups and the fashion of time.

Clear-cut as this distinction is, in the practice of science there can be observed a strong influence of the element of beauty — where of course truth remains the main criterion. In physics beauty of a theory is seen as a reliable indicator for a good theory. This is even more true in mathematics, where a theory is good, if it tells a truth of practical importance or a beautiful truth. However, "artistic freedom" in mathematics is a relatively recent phenomenon. Over thousands of years mathematics has been concerned with the facts about heavenly bodies, geometric shapes and numbers. Persons concerned with mathematics did not view themselves as inventors. The crucial of mental creativity in mathematics as exposed first by the emergence of imaginary and complex numbers — probably Euler was the first great "artist" in mathematics (compare [C]) — and non-Euclidean geometry was received as a shock. Since then it has become common wisdom that one is largely free to choose the structural framework, whose inherent diversity one would like to explore. And Gödel's results have shown that most axiomatic systems of moderate scope are in fact rich enough to guarantee an open end for the scientific-artistic mathematical exploration.

Whereas mathematics comes as close to art as one can expect from a genuine science, music comes as close to mathematics as a genuine art possibly can. Clearly, there are big differences between mathematics and music. It is hard to attach sense to the concept of truth in music. And admittedly the concept of beauty has a quite different touch and feel in mathematics than in music: a beautiful proof never has driven anybody to tears and a beautiful piece of music has never explained any worldly matters. But on the other hand mathematics and music appear to be closely related. The present article is an attempt to substantiate this claim by focussing on two special topics. First, on

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the parallels between the development of mathematics and music in the 20th century. (For the earlier periods of Antiquity, Renaissance, Baroque, Classic, and Romantic there is a surprising synchronicity for mathematics and music on one side and for arts and literature on the other side, if one compares the timetables and characterizations of these periods in different standard accounts of the history of the respective subjects [H]; see also [Gr] for further interesting historical informations.) And second, the common "abstractness" of mathematics and music.

Mathematics and music in the 20th century

Functional harmonics and tonality had reached its utmost limit at the beginning of the 20th century in the music of Reger, Mahler, and the young Schönberg. Therefore new methods of composing were created and explored: impressionism blurred the concepts of functionality in harmonics and the distinction between melody and accompaniment (Debussy, Ravel), new types of scales, often inspired by folk music, and polytonality were used (Bartok, Stravinsky, Ives), the predominance of pitch was diminished by emphasizing percussion instruments (Varèse). But most influential was the new method of 12-tone composition invented by Arnold Schönberg, which used sequences of all twelve half tones of the octave (and the old idea of counterpoint) to create musical pieces without tonal centre. The development of this *atonal* music seemed to be logical and inevitable, or as Schönberg has put it: "Somebody had to be Schönberg." Beginning with this invention it is not hard to subdivide 20th century composers into 12-tone composers (including Schönberg's disciples Alban Berg and Anton Weber) and non-12-tone composers, the most notable of several exceptions being Schönberg himself, who remarked about his later tonal works: "There is still much good music to be written in C-major."

Similarly, mathematics based on concrete computations and manipulations with special functions, power series, determinants, and more general invariants had reached its limit at the turn to the 20th century, and the mathematician who brought about the change to more abstract conceptual mathematics — comparable to Schönberg in music — was David Hilbert. Hilbert himself has contributed to both the old computational and the new conceptual mathematics: for example his work on invariant theory includes not only the celebrated Syzygy Theorem, which attracted the wrath of the old invariant theorists, but also a less known explicit algorithm for the computation of the fundamental set of invariants for all classical groups. The distinction between 'computational' and 'conceptual' was in many cases a dividing line between the generations: the disciples of the "old" invariant theorists Weitzenböck and Gordan were the "new" algebra heroes van der Waerden and Emmy Noether.

The predominance of atonal music as the only "serious" music — as far as the majority of the professionals and not the audience is concerned — from the 30s to the

70s is paralleled by the predominance of Bourbakism as the only "serious" (=rigorous and economical) way of doing and presenting mathematics (see [SC]). In the late 50s and during the 60s the summit of formalization and abstraction was reached in music by the *serial* method of composition (Karl-Heinz Stockhausen, Pierre Boulez), which subjected not only pitches to the "principle of uniform distribution", but all parameters of music including rhythm, dynamics (volume levels), attack (how notes are struck and released), and colour. This is paralleled in mathematics by the dominance of category theory, sheafs, and cohomology (Alexander Grothendieck).

Of course the above statements give only a very gross characterization and there are many noteworthy exceptions, which partially or completely avoided the main stream of formalism: e.g., musicians like Richard Strauss, Prokovief, Stravinsky, Olivier Messiaen and more "applied" mathematicians like Norbert Wiener and Andrei N. Komolgorov. Moreover, it is possible to find similar lines of development for the 20th century in arts, literature, physics, psychology, and philosophy; for example:

- The "category theory of psychology" is the *behaviourism* of B.F. Skinner: The human mind is viewed as a 'black box' (=objects), which can be scientifically studied only through the means of 'stimulus' and 'response' (=morphisms). The later *cognitive psychology* is actually not very different from behaviourism, but at least admits that there is something in the box. Interestingly, the social engineering phantasies of Skinner in "Beyond Freedom and Dignity" (1971), where he propagates the ideal of a problem free society through the behavioural drill of its members, corresponds to the at that time widespread opinion that humans are completely determined by their social circumstances.
- The dominance of serialism in music and of category theory in mathematics in the 60s has to be seen on the background of the dominance of *structuralism* in all intellectual disciplines during this time: in anthropology (Lèvi-Strauss), linguistics (Chomsky), neo-Freudian psychology (Lacan), Marxism (Althusser), semiology (Barthes), and philosophy (Foucault, Derrida). Structuralism argued that the diversity of human culture is the expression of underlying abstract and in general unobservable structures. This "unobservability" corresponds in serialism to the fact that the underlying serial patterns are usually too complex to be heard and in category theory to the fact that one does not see the internal structure of the objects except through the revelations given by the morphisms. The concept of the "underlying structure" has also led in structuralism to the attitude that in order to understand something one has to "decode" it (the corresponding tantalizing style of writing is all too well known in mathematics).

One can not pass over the period from the 40s to the 60s without acknowledging the counter movement to serialism in music, namely the introduction of *chance* into the composition and performance of music by John Cage. (One is tempted to correlate this with the great upswing of probability theory and stochastics in this century beginning with Komolgorov's axioms.) Cage also introduced eastern philosophy into the thinking about music and its composition — in fact 'chance' was seen by him as a means to minimize and finally eliminate the display of the pretentious ego of the composer. This paved the way to the emergence of *minimal music* in the late 60s (Steve Reich, Terry Riley, Phillip Glass with precursors La Monte Young and Morton Feldman, the latter being a disciple of Cage). The central idea of minimal music is *self-similarity*, the repetition of a few patterns with constant change in details. This reminds one of *fractals*, foremost of the popular Mandelbrot set, where now self similarity in time has been replaced by self similarity in scale. Another common aspect of minimalism (and their more recent offsprings such as New Age Music and Techno Music) with fractal images is that both of them rely on the use of computers.

This leads directly to the question of "popular styles" in music and mathematics. Due to the difference in accessibility it seems however difficult if not pointless to compare directly the popular musical styles, which the 20th century was so prolific for, with "popular" mathematics, the latter including the infamous work of hobby mathematicians on Fermat's Last Theorem and on the trisection of angles as well as beautiful puzzles like Rubrik's cube and games with strong underlying mathematical (mostly combinatorial) ideas.

The analogy between mathematics and music

Before we plunge into a closer inquiry of the connection between mathematics and music let us briefly recapitulate the opinions of some well known thinkers in the past. First there is the party, which emphasises the irrationality and therefore the dangers of music:

Plato looked on music as a department of ethics. And like Confucius he was anxious to regulate the use of particular modes (i.e., arrangements of notes and scales) because of their supposed effects on men. Plato was a stern musical disciplinarian; he saw a correspondence between the character of a man and the music that represented him. Straightforward simplicity was best. In the Laws, Plato declared that rhythmic and melodic complexities were to be avoided because they led to depression and disorder. Augustine, who was attracted by music and valued its utility to religion, was fearful of its sensuous element and anxious that the melody never take precedence over the words. Immanuel Kant ranked music as lowest in his hierarchy of the arts. What he distrusted most about music was its wordlessness; he considered it useful for enjoyment but negligible in the service of culture. G. W. F. Hegel preferred vocal music to instrumental, deprecating wordless music as subjective and indefinite. Schopenhauer said that music is

> ... by no means like the other arts a copy of the [Platonic] Ideas, but a copy of the Will itself. This is why the effect of music is so much more powerful and penetrating than that of the other arts, for they speak only of shadows, but it speaks of the thing itself.

Nietzsche posed an Apollonian-Dionysian dichotomy, the former representing form and rationality and the latter chaos and ecstasy. For him, music was the Dionysian art par excellence.

On the other hand Confucius valued music as an eminent means to restore harmony and order in the universe. This acknowledges the aspect of "emotionality" of music as well as its appearance as "orderly sound", which reflects and influences the order of the universe. The latter point has been emphasized in the occident by Pythagoras and his school. Augustine and later Thomas Aquinas held the basis of music to be mathematical; music reflects celestial movement and order, a point taken up again by Copernicus, Kepler, and others. For the philosopher-mathematician Leibniz music reflected a universal rhythm and mirrored a reality that was fundamentally mathematical, to be experienced in the mind as a subconscious apprehension of numerical relationships. Probably the most consequent and elaborate expression of this line of reasoning about mathematics and music can be found in the common self-understanding of Indian classical music. The *Sangita-Ratnakara of* Sarngadev (Indian musician and physician, 1175–1247) is the most popular and widely used textbook of *Ghandarva-Ved* or Indian classical music. Besides all the practical and technical material, e.g., scales and rhythms, necessary for the education of musicians, it contains a description of how step by step the whole cosmos originates from the sound *nada*: first the primordial syllables of the *ved* appear, then its words and verses, and finally the whole manifest world. Here 'ved' means literally "knowledge" and stands for the unmanifest state of unity of all the laws of nature, where creation has not yet started.

Let us state the preliminary agreement of the second party about mathematics and music this way: both disciplines reflect a "deeper level" of understanding of the laws of nature and of man, where music shows an unreasonable effectiveness to move man and mathematics shows an unreasonable effectiveness to move nature. When Aristotle says: "It is not easy to determine the nature of music or why anyone should have a knowledge of it" this applies obviously equally well to mathematics and it seems to be an acknowledgment of our ignorance. But from a different point of view this statement can be seen as a key to understanding: exactly the "things", which are constitutive for our world, are the ones, which are difficult to understand; the more fundamental and unavoidable something is, the more difficult it is to explain: for example TV's and cars, ties and money are among the less fundamental things — many cultures lived happily without these things —, therefore nobody bothers about explaining them. On the other hand there are items like time, consciousness, language, mathematics, and music, which have its place in every culture whatsoever; they must be considered as "fundamental" and hence difficult to explain.

A closer look on the analogy between mathematics and music reveals that the most important feature, which both disciplines seem to share and which distinguishes them from other branches of human culture, is the special relationship between *syntax* (the formal rules of a language) and *semantics* (meaning); in one word: their common *abstractness*. In fact, in every other branch of arts and science one can point to something in the material world and say: "This is a typical subject of our discipline." This abstractness of mathematics and music has lead to a unique feature in the attempt to understand them: the claim that these disciplines are purely *formal* or purely syntactical, i.e., that everything substantial is encoded in the set of used signs and their formal relationships. In mathematics the formalistic standpoint is expressed in the foundationalist endeavour or even more radically in logicism saying that mathematics is "nothing but" logic and set theory. It is not necessary to invoke Gödel to see that this "nothing but" is preposterous. Nobody ever came up with the idea that the great masterpieces of literature and poetry are "nothing but" an elaboration of orthography and grammar, or that everything one expects from a musical composition is that it follows certain rules of harmony and counterpoint. Nevertheless the obvious difficulty to say *exactly* what the subject or meaning of mathematics and music *is* has motivated a great many people in the past to adopt the formalistic standpoint. And even worse: the prejudice of the universality of formalistic rigor has been put hard pressure onto other disciplines (for philosophy cf. [R]).

Let us name some important non-formal and non-formalizable ingredients of mathematics and music. First there is the selection of what is *interesting* — this involves not only the individual preferences of some person, but a whole historical and cultural background: there are fashionable subjects and styles in both mathematics and music. Second there is the problem of *creativity* or how something "new" is found or invented. Rather than attempting to cope with these problems we restrict here to an analysis of two everyday situations, in which non-formalisability becomes apparent, namely the translation of a score into music and the translation of a mathematical paper into "understanding".

If a musician intends to play the music contained in a score, he first studies every single instruction about key, pitches, rhythm, dynamics, attack, etc., and tries to fully master all technical difficulties. Correspondingly, if a mathematician wants to understand a mathematical paper, he first reads through all definitions, theorems, examples, the intermediary text, and verifies all steps of proofs. But so far the result in the first case can not be considered as real music, and in the second case not as real understanding. The music obtained so far will sound robot like or at best amateur like. And with regard to the mathematical paper what one has obtained so far is at best a *verification of correctness*. (Even the verification of correctness involves a certain socialization of the mathematician and a lot of tacit knowledge encoded in a certain historical situation, since most proofs are full of abbreviations and omissions of “trivial” or “folklore” details [Be].) Moreover, the goal of mathematics is certainly not only to know that something is correct, but to understand the reason why something is correct: the search for full understanding or full *evidence* of a mathematical fact can cause decades of research activities; compare for example the case of the prime number theorem [PT] or the ongoing work to replace the computer verification of the Four-Colour-Theorem by a conceptual proof.

In case of playing a musical score as well as in understanding a mathematical paper the meaning is communicated by reliable means of notation, which contains all “necessary” information, but the real success of this communication relies on a whole context of personal training, history, and culture. (In this generality the argument applies to every kind of communication; and isn't it astonishing how well everything works given the utterly deficient objective means of our communication?)

Above we have used considerately the formal and syntactical term ‘correctness’ as opposed to *truth*, which invokes a kind of agreement between expressions in a language and "reality". In mathematics ‘correctness’ is the first step to truth or the full understanding of a fact. These objectively vague terms are perfectly clear, when considered on a subjective level. Formal verification leaves one usually with a certain tiredness and even some uncertainty, whether there has been something overlooked. To the contrary, understanding or the encounter with truth is something, for which one can very aptly choose emphatic terms like *evidence* [PT] and *enlightenment* [PB].

Correctness is not a concept alien to music:

> The creation of concrete form in music involves seeing the microcosm within the macrocosm, using small relations as models for large ones, transforming music according to the rules it establishes, through the particular domain of a given work. To carry forward our analogy with mathematical explorations, a composition begins out of its premises and proceeds to “prove” itself. When correctly understood, this is the way a composition is sensed – as an inarguable fact, which could hardly be different. Great compositions create their own form of necessity, the binding coming not from logic but from the unfolding of ideas with all the variations and style and imagination we find in sophisticated mathematical proofs. Much of the profound music of our culture is “intellectual” only in that it takes the powers of intellect seriously in the texture of music itself. [EM, p.132]

Correctness in music can be seen as a preparatory step for *beauty*; and beauty in mathematics as another aspect of enlightenment [PB] or the appreciation of truth.

It would be completely misleading now to dismiss evidence, enlightenment, and beauty as something purely subjective or psychological. In fact the attempt to *harmonize subject and object*, to *rebuilt the objective world in terms of our subjective emotionality and intelligence* is the underlying theme of all arts and science:

> Man tries to make for himself in the fashion that suits him best a simplified and intelligible picture of the world; he then tries to some extent to substitute this cosmos of his for the world of experience, and thus to overcome it. This is what the painter, the poet, the speculative philosopher, and the natural scientist do, each in his own fashion.
> ALBERT EINSTEIN *Address to the Physical Society, 1918*

> To review Kant: beauty makes it appear as if the object we are gazing at were perfectly adapted to our understanding, as if there were some unexpected harmony between the mind of the observer and the object observed. Beauty seems to have its home out there in the world; it inspires one to think every viewer should consider the object beautiful. [EM, p.186]

So far we have argued that the analogy between mathematics and music is due to their common abstractness, which is to be understood as remoteness of their subject or their meaning — not their means of communication — from the level of sensory perception of the material world. One big question, which remains is, whether there is a reason for this common abstractness.

Nowadays a fashonable explanation could go like this: Evolutionary biology tells us that the possibility of using abstract symbols — primarily in the form of sounds and later spoken words — has been a decisive advantage for the survival of the human race. Therefore the ability to utter sounds in an orderly manner and symbolic thinking is located in closely related regions of the brain. (I do not know, whether this is an established fact of neurology. However, it is a fact established by several long term studies that school kids with an intensified musical education perform much better in intellectual disciplines like mathematics than their less musically trained comrades and in addition they show a much better social behaviour — which supports in an unexpected way Plato's claim that Beauty, Truth, and Good are in fact identical.) By some exaggeration in the workings of nature man got the ability to perform mathematics and music in a way much beyond the level necessary for survival.

This nice tale has however at least two major difficulties, which we do not undertake to resolve here. First, the role of selection in evolution. If it is possible to "select" highly specialised organisms and functions and at the same time highly non-specialised and universal organisations as "well adapted", then nothing at all is explained (compare Gödel on consistency). Second, there is a yet unbridged difference between neural and mental activity. This relates directly to our discussion of formalism above and the mind-matter problem in general.

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References

[Be] H. Berger, Tacit Knowledge in Mathematical Theory, *in:* J. Echeverria et. al. (eds.), "The Space of Mathematics", de Gruyter, Berlin, 1992.

[C] P. Cartier, Mathemagics (A Tribute to L. Euler and R.Feynman), *Seminaire Lotharingien de Combinatoire* **44** (2000), 71 pp.

[EM] E. Rothstein, "Emblems of Mind. The inner life of music and mathematics", Avon Books, New York, 1996. — [Gi] is an instructive review of [EM], which exemplifies the distinction between correctness and truth: the reviewer rightly criticizes mathematical faults in [EM] and praises the professional analysis of certain musical pieces — and it demonstrates how to miss the main point, namely the thoughts in [EM] about the relationship between the two fields.

[Gi] L. Gillman, Review of "Emblems of Mind: The inner life of music and mathematics", *Math. Intelligencer* **20**(3) (1998), 60 - 64.

[Gr] H. Grant, What is Modern about "Modern" Mathematics?, *Math. Intelligencer* **17**(3) (1995), 62 - 66.

[H] J. Henle, Classical Mathematics. Baroque Mathematics. Romantic Mathematics? Mathematics Jazz! Also Atonal, New Age, Minimalist, and Punk Mathematics. *Amer. Math. Monthly* **103** (1996), 18 - 29. — The less convincing "guesses" of Henle about the parallels between mathematics and music in the 20th century have prompted the first part of the present paper.

[PB] G.-C. Rota, The Phenomenology of Mathematical Beauty, *in:* [R], 121 - 133.

[PT] G.-C. Rota, The Phenomenology of Mathematical Truth, *in:* [R], 108 - 120.

[R] G.-C. Rota, "Indiscrete Thoughts", Birkhäuser, Boston, 1997.

[SC] M. Senechal, The Continuing Silence of Bourbaki – An Interview with Pierre Cartier, June 18, 1997, *Math. Intelligencer* **20**(1) (1998), 22 - 28.

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