

Rate-form equilibrium for an isotropic Cauchy-elastic formulation: Part I: modeling

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Abstract

We derive the rate-form spatial equilibrium system for a nonlinear Cauchy elastic formulation in isotropic finite-strain elasticity. For a given explicit Cauchy stress-strain constitutive equation, we determine those properties that pertain to the appearing fourth-order stiffness tensor. Notably, we show that this stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$ acting on the Zaremba-Jaumann stress rate is uniformly positive definite. We suggest a mathematical treatment of the ensuing spatial PDE-system which may ultimately lead to a local existence result, to be presented in part II of this work. As a preparatory step, we show existence and uniqueness of a subproblem based on Korn's first inequality and the positive definiteness of this stiffness tensor. The procedure is not confined to Cauchy elasticity, however in the Cauchy elastic case, most theoretical statements can be made explicit.

Our development suggests that looking at the rate-form equations of given Cauchy-elastic models may provide additional insight to the modeling of nonlinear isotropic elasticity. This especially concerns constitutive requirements emanating from the rate-formulation, here being reflected by the positive definiteness of $\mathbb{H}^{ZJ}(\sigma)$.

Keywords: nonlinear elasticity, hyperelasticity, rate-formulation, Eulerian setting, hypo-elasticity, Cauchy elasticity, material stability, updated Lagrangean, total Lagrangean

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1 Introduction

Problems of nonlinear elastic deformations have attracted a great deal of attention from the applied mathematics community, above all due to their diverse and deep challenges for analysis. In this work, we focus on homogeneous isotropic compressible finite strain elasticity. The modelling framework of these problems can be considered to be complete, see e.g. the classical books by Ogden [61], Ciarlet [9] and Marsden-Hughes [44]. Indeed, starting with a hyperelastic formulation, the static equilibrium equations appear as Euler-Lagrange equations of the energy functional

$$\int_{\Omega} W(D\varphi) - \tilde{f} \cdot \varphi \, dx \quad \longrightarrow \quad \min . \varphi. \quad (1.1)$$

Here, $\varphi : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the (diffeomorphic) deformation, $F = D\varphi(x) \in GL^+(3)$ is the deformation gradient, and \tilde{f} represents a dead load body force in the referential setting. In particular, the elastic energy $W : GL^+(3) \rightarrow \mathbb{R}$ completely describes the constitutive response. Finding suitable forms of W which encode physically realistic behaviour of actual materials is *Truesdell's Hauptproblem* [75], which, to the best of our knowledge, is not completely solved up to date. The nonlinear Euler-Lagrange equations corresponding to (1.1) are then given by

$$\begin{aligned} \text{Div}_x S_1(F(x)) &= \tilde{f}(x) && \text{in } \Omega, \\ S_1(F(x)).n &= \tilde{S}_1(x).n = \tilde{p}(x) && \text{on } \Gamma_N, \\ \varphi(x) &= \tilde{\varphi}(x) && \text{on } \Gamma_D, \quad \text{with } \Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N. \end{aligned} \quad (1.2)$$

In (1.2), $\Gamma_D \subset \partial\Omega$ is the part of the boundary where Dirichlet conditions are prescribed, and Γ_N is the part of the boundary where Neumann conditions apply (see Figure 1). Here, $S_1(F) = D_F W(F)$ is the non-symmetric first Piola-Kirchhoff stress tensor and \tilde{S}_1 or \tilde{p} allows to specify the tractions.

Based on the Piola transformation, it is possible to obtain the equilibrium equations in the current configuration $\Omega_\xi = \varphi(\Omega)$ by introducing the Cauchy stress tensor

$$\sigma(F) = \frac{1}{J} S_1(F) \cdot F^T = \frac{1}{J} D_F W(F) \cdot F^T, \quad J = \det F, \quad F = D\varphi. \quad (1.3)$$

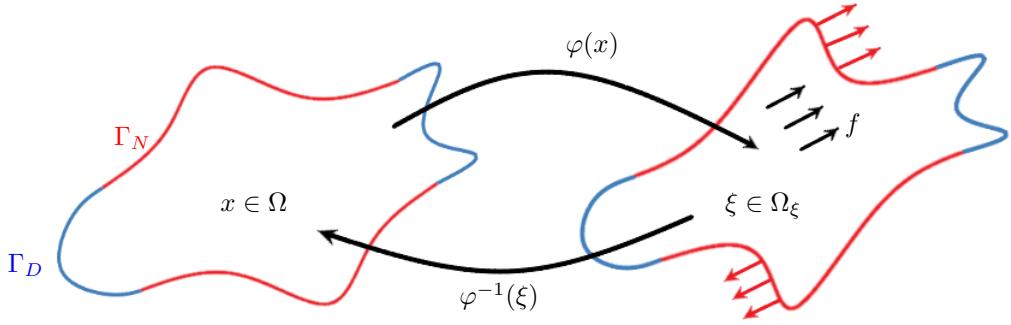


Figure 1: Illustration of the domain Ω with different types of prescribed boundary conditions and its deformation under the diffeomorphism φ . The part Γ_D describes the Dirichlet boundary, while Γ_N defines the Neumann boundary.

The equilibrium equations in the Eulerian setting read for $\sigma(\xi) := \sigma(F(\varphi^{-1}(\xi)))$ by abuse of notation

$$\text{Div}_\xi \sigma(\xi) = f(\xi) \quad \text{in } \Omega_\xi, \quad (1.4)$$

which can be obtained from (1.2) by setting $f(\xi) = (\det F(x))^{-1} \cdot \tilde{f}(x)$. We note that this comes to the effect of losing the variational structure of (1.2), a discrepancy which would not be seen by linearized elasticity.

Objectivity (frame-indifference) and isotropy together amount to

$$W(Q_1 F Q_2) = W(F), \quad \forall Q_1, Q_2 \in O(3), \quad (1.5)$$

and imply that the Cauchy stress σ is symmetric and admits the representation

$$\sigma = \sigma(B) = \frac{2}{J} D_B \widetilde{W}(B) \cdot B, \quad B := F F^T, \quad W(F) := \widetilde{W}(B), \quad (1.6)$$

where σ is an isotropic tensor function of the left Cauchy-Green tensor B , i.e.

$$\sigma(Q^T B Q) = Q^T \sigma(B) Q, \quad \forall Q \in O(3). \quad (1.7)$$

For future reference, we note that any such σ can be written as (Richter's representation [25, 52, 65, 66, 67, 68], Rivlin-Ericksen representation [3])

$$\sigma(B) = \beta_0 \mathbb{1} + \beta_1 B + \beta_{-1} B^{-1}, \quad (1.8)$$

where $\beta_0, \beta_1, \beta_{-1}$ are scalar-valued functions of the principal invariants I_1, I_2, I_3 of B , with

$$\begin{aligned} I_1(B) &= \text{tr}(B) = \|F\|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2(B) &= \text{tr}(\text{Cof } B) = \|\text{Cof } F\|^2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \\ I_3(B) &= \det B = (\det F)^2 = \lambda_1^2 \cdot \lambda_2^2 \cdot \lambda_3^2. \end{aligned} \quad (1.9)$$

Here, $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigenvalues of B . Based on these considerations, the objective of the present paper is to model isotropic, compressible nonlinear elasticity in a rate format in an Eulerian configuration, see (1.30) below. In particular, this specific formulation gives us access to the positive definiteness of a constitutive fourth-order tangent stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$. In order to put our modeling into its natural context, we pause to discuss related approaches to nonlinear elasticity first.

1.1 Mathematical approaches towards nonlinear elasticity

1.1.1 The direct method of the calculus of variations

It is by now standard to use the framework of the calculus of variations to show the existence of energy minimizers to (1.1) via the direct method. Ball's seminal introduction of polyconvexity [4] (together with

growth conditions and coercivity estimates) is the fundamental notion here. Polyconvexity means that the elastic energy $W : GL^+(3) \rightarrow \mathbb{R}$ can be written as

$$W(F) = \mathcal{P}(F, \text{Cof } F, \det F), \quad (1.10)$$

where $\mathcal{P} : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R} \rightarrow \mathbb{R}$ is convex and $\text{Cof } F$ is defined by $F^T \text{Cof } F = \det F \cdot \mathbb{1}$. Polyconvexity is sufficient for weak lower semicontinuity in spaces of weakly differentiable functions, and implies rank-one convexity of the energy, i.e.,

$$D_F^2 W(F) \cdot (\xi \otimes \eta, \xi \otimes \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{R}^3, \quad (1.11)$$

if $W \in C^2(GL^+(3); \mathbb{R})$. Alternatively, one may express this as rank-one monotonicity of the first Piola-Kirchhoff stress S_1 , in the sense that

$$\langle S_1(F + \xi \otimes \eta) - S_1(F), \xi \otimes \eta \rangle \geq 0 \quad \forall \xi, \eta \in \mathbb{R}^3 \setminus \{0\}. \quad (1.12)$$

Dunn [15] has reformulated this requirement in terms of the Cauchy stress. Condition (1.11) and (1.12) are also referred to as *Legendre-Hadamard ellipticity* (LH-ellipticity for brevity). A notable condition between polyconvexity and rank-one convexity is *Morrey's quasiconvexity*, meaning that

$$\int_{\Omega} W(F + D\vartheta) dx \geq \int_{\Omega} W(F) dx \quad \forall \vartheta \in C_0^{\infty}(\Omega; \mathbb{R}^3) \quad \forall F \in \mathbb{R}^{3 \times 3}. \quad (1.13)$$

This means that the homogeneous configuration is energy-optimal with respect to perturbations that leave affine-linear boundary data $x \mapsto \varphi(x) = F.x + b$ invariant. Even though quasiconvexity is often equivalent to lower semicontinuity of numerous energies on spaces of weakly differentiable functions, it is often too general to support existence results under physically relevant conditions. Being based on the minors of matrix, the main benefit of polyconvexity is that it gives us direct access to the physical requirement that extreme stretches ($\det F \rightarrow 0^+$) should be accompanied by extreme elastic energy ($W(F) \rightarrow +\infty$), so that energy minimizers satisfy $\det D\varphi(x) > 0$ almost everywhere.

The general drawback of the method is that, at present, no general result is available that would show that the equilibrium equations (1.2) are satisfied even weakly. This is due to the fact that one cannot, in general, find minimizers satisfying $\det D\varphi(x) \geq c^+ > 0$ in advance, not to speak of the regularity of the solution. From the perspective of modeling, another issue is that certain stable equilibria found in nature are not global minimizers. For instance, this is the case for the everted configuration of a tube, cf. Nedjar et al. [49]; see Figures 2 and 3.

1.1.2 Implicit function theorem

A different approach towards existence of solutions to (1.2) is based on the implicit function theorem in appropriately chosen Sobolev spaces (cf. Stoppelli [72], Marsden-Hughes [44, p. 371], Ciarlet [9, 10] and Valent [78]). This methodology establishes the existence of smooth $W^{2,p}$ -solutions, $p > 3$, to (1.2) in the neighbourhood of a given smooth equilibrium configuration $\varphi_0 \in C^2(\Omega; \mathbb{R}^3)$ with $\det D\varphi_0(x) \geq c^+ > 0$ under two fundamental conditions:

- uniform Legendre-Hadamard ellipticity at the given configuration φ_0 :

$$D_F^2 W(D\varphi_0(x)) \cdot (\xi \otimes \eta, \xi \otimes \eta) \geq c^+ |\xi|^2 |\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^3 \setminus \{0\}, \quad (1.14)$$

- and the linearization at φ_0 must be well-posed, i.e.,

$$\text{Div } D_F^2 W(D\varphi_0(x)) \cdot Du = 0, \quad u \in H_0^1(\Omega, \mathbb{R}^3) \quad (1.15)$$

has only the trivial solution.

Condition (1.15) excludes any (interior) instability, and is usually satisfied only in the neighbourhood of the identity. On the other hand, (1.14) effectively excludes shear-band type instabilities as an occurrence of microstructure. In general, this method applies only to either pure Dirichlet problems or pure traction problems.

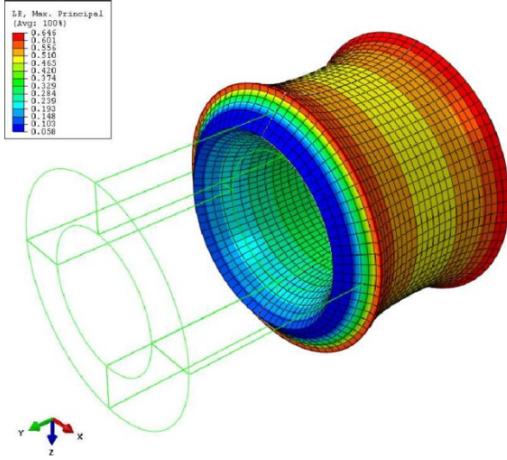


Figure 2: The fully everted elastic tube with maximum occurring principal stretches from [49], finite element solution based on $W(F) \sim e^{\|\log V\|^2}$, $V = \sqrt{FF^T}$, the exponentiated Hencky energy.

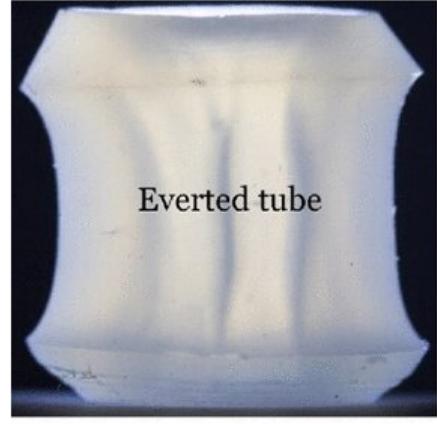


Figure 3: Photo of an everted tube, cf. [49]. The everted tube is not a global energy minimizer, since for traction-free boundary conditions the natural state has less energy, cf. Truesdell [77, p. 103], but presumably it is a stable local energy minimizer.

1.1.3 Topological continuation methods

Thirdly, there is the topological continuation method (cf. Healey [29, 30, 31]) which uses degree-theoretical arguments to show the existence of global $C^{2,\alpha}$ -Hölder-smooth equilibrium solutions to the pure Dirichlet problem (1.2) driven by body forces parametrized by $t \geq 0$. In this approach, one considers that the elastic energy W , the load f and the domain Ω are sufficiently smooth, that W is strongly elliptic in F , i.e., $D_F^2 W(F).(\xi \otimes \eta) > 0$ for all unit vectors $\xi, \eta \in \mathbb{R}^3 \setminus \{0\}$, that the reference configuration is stress-free and has a positive definite elasticity tensor $\mathbf{A} := D_F^2 W(\mathbf{1})$. Then the pure Dirichlet problem in the reference configuration

$$\operatorname{Div}_x D_F W(D\varphi(x)) = f(x, \varphi, D\varphi, t), \quad \varphi(x) = x \quad \text{on} \quad \partial\Omega \quad (1.16)$$

admits a *solution in the large* (in the language of Healey [29, 30, 31]) in spaces of smooth functions, meaning that either the solution exists for all load parameters $t \in \mathbb{R}$, or the solution explodes at some definite load parameter. In this sense, Healey calls it a *global inverse function theorem*.

Note that in all described existence theorems some form of LH-ellipticity is crucially involved, allowing to treat large classes of elastic energies. Altogether, not even for a compressible Neo-Hooke model do we know the existence of weak equilibrium solutions in situations of interest with mixed boundary conditions. Here, to the contrary, the existence theory that we prepare will be largely independent of LH-ellipticity, but will work only for very specific, yet natural examples of constitutive requirements. These, however, are compatible with the assumptions of nonlinear elasticity for extreme deformations, which usually turn out problematic in a variety of other theories.

1.2 Isotropic Cauchy-elasticity

Truesdell has extended and generalized the hyperelastic framework by giving up the requirement that a strain energy W exists. While the physics of a theory without strain energy is certainly questionable, nevertheless, insight into the nonlinear elasticity problem can be gained and useful results for purely mechanical problems may be shown [79]. Here, one directly postulates a Cauchy stress-stretch constitutive law $B \mapsto \sigma(B)$ and with

$$S_1(F) = \sigma(B) \cdot \operatorname{Cof} F, \quad B = FF^T, \quad (1.17)$$

one can either study the problem in the Lagrangian setting (1.2) or in the Eulerian setting (1.4). Constitutive assumptions on σ are now called upon for a decent modeling framework. Richter [25, 52, 65, 66, 67, 68] and Reiner [63] have early suggested that

$$\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3), \quad B \mapsto \sigma(B), \quad (1.18)$$

should be bijective for idealized perfect elasticity, in a straight-forward extension of ideas from linear elasticity in which $\sigma_{\text{lin}} : \text{Sym}(3) \rightarrow \text{Sym}(3)$, $\varepsilon \mapsto \sigma_{\text{lin}}(\varepsilon) = \mathbb{C}^{\text{iso}} \cdot \varepsilon$ is bijective for $\varepsilon = \text{sym } \mathbf{D}u$. This is, e.g., the case for a slightly compressible Neo-Hooke type solid with elastic energy

$$\begin{aligned} W_{\text{NH}}(F) &= \frac{\mu}{2} \left(\frac{\|F\|^2}{(\det F)^{\frac{2}{3}}} - 3 \right) + \frac{\kappa}{2} e^{(\log \det F)^2}, \\ \sigma_{\text{NH}}(B) &= \mu (\det B)^{-\frac{5}{6}} \text{dev}_3 B + \frac{\kappa}{2} (\det B)^{-\frac{1}{2}} (\log \det B) e^{\frac{1}{4} (\log \det B)^2} \cdot \mathbb{1}, \end{aligned} \quad (1.19)$$

where $\text{dev}_n X := X - \frac{1}{n} \text{tr}(X) \cdot \mathbb{1}$ (cf. [53]). However, most well-known elastic energies do not satisfy bijectivity of (1.18) for all positive definite stretches $B \in \text{Sym}^{++}(3)$, e.g. the slightly simple compressible Neo-Hooke model

$$\widetilde{W}_{\text{NH}}(F) = \frac{\mu}{2} \left(\frac{\|F\|^2}{(\det F)^{\frac{2}{3}}} - 3 \right) + \frac{\kappa}{2} (\det F - 1)^2, \quad \widetilde{\sigma}_{\text{NH}}(B) = \mu (\det B)^{-\frac{5}{6}} \text{dev}_3 B + \kappa (\sqrt{\det B} - 1) \cdot \mathbb{1} \quad (1.20)$$

does not yield bijectivity, due to the merely quadratic nature of the volumetric term [38].

Another frequently-used physical requirement are the (weak) empirical inequalities (cf. Mihai and Goriely [46, 47, 48]). If $\sigma(B)$ is given as in (1.8), so

$$\sigma(B) = \beta_0 \cdot \mathbb{1} + \beta_1 \cdot B + \beta_{-1} \cdot B^{-1}, \quad (1.21)$$

it is required that $\beta_1 > 0$, $\beta_{-1} \leq 0$ (cf. [73]). This seems to be in accordance with most available experimental data. Notably, a positive Poynting effect (cf. Zurlo et al. [84]) is predicted, i.e. a tube in torsion will lengthen. The Baker-Ericksen inequalities [3] also make a statement for the Cauchy stress (but only at one given configuration). However, they consider the principal Cauchy stresses and demand that the higher principal Cauchy stress is occurring for the higher principal stretch, i.e.

$$(\sigma_i - \sigma_j) \cdot (\lambda_i - \lambda_j) > 0 \iff (\widehat{\sigma}_i - \widehat{\sigma}_j) \cdot (\log \lambda_i - \log \lambda_j) > 0. \quad (1.22)$$

In this paper, we will also deal with a version of “stress increases with strain”, which we interpret firstly as a condition on the Cauchy-stress tensor σ (the “true” stress) and secondly as a subsequent monotonicity requirement in the Frobenius scalar product (**Hilbert-monotonicity** of $\widehat{\sigma}(\log B) := \sigma(B)$ in terms of $\log B$), cf. [53], where $\log B = 2 \log V$ is the logarithmic strain (Hencky strain, true-strain)

$$\langle \widehat{\sigma}(\log B_1) - \widehat{\sigma}(\log B_2), \log B_1 - \log B_2 \rangle > 0, \quad \forall B_1, B_2 \in \text{Sym}^{++}(3), \quad B_1 \neq B_2, \quad (1.23)$$

which is not equivalent to Hilbert-monotonicity of σ in terms of B , i.e.

$$\langle \sigma(B_1) - \sigma(B_2), B_1 - B_2 \rangle > 0, \quad \forall B_1, B_2 \in \text{Sym}^{++}(3), \quad B_1 \neq B_2. \quad (1.24)$$

For more information about the logarithm of a matrix we refer to Richter [67], Lankeit et al. [39] and Neff et al. [58]. However, (1.23) already implies the Baker-Ericksen inequalities and bijectivity of $B \mapsto \sigma(B)$ if for $\widehat{\sigma} : \text{Sym}(3) \rightarrow \text{Sym}(3)$, $\widehat{\sigma}(\log B) := \sigma(B)$ we have

$$\|\widehat{\sigma}(\log B)\| \rightarrow +\infty \quad \text{for} \quad \|\log B\| \rightarrow +\infty. \quad (1.25)$$

This will be shown in a forthcoming work [45]. A strengthening of the condition (1.23) is the TSTS-M⁺⁺ (true-stress true-strain monotonicity condition)

$$\begin{aligned} \log B \mapsto \widehat{\sigma}(\log B) \text{ is strongly Hilbert-monotone} \\ \iff \text{sym } \mathbf{D}_{\log B} \widehat{\sigma}(\log B) \in \text{Sym}_4^{++}(6), \end{aligned} \quad (1.26)$$

where $\text{Sym}_4^{++}(6)$ denotes the set of positive definite, minor and major symmetric fourth-order tensors. Recently, in [12] it is shown that TSTS-M⁺⁺ is equivalent to the corotational stability postulate (CSP)

$$\left\langle \frac{D^\circ}{Dt}[\sigma], D \right\rangle > 0 \quad \forall D \in \text{Sym}(3) \setminus \{0\}, \quad (1.27)$$

see also [54, 56], where $\frac{D^\circ}{Dt}[\sigma] = \frac{D}{Dt}[\sigma] + \sigma \Omega - \Omega \sigma$ with $\Omega \in \mathfrak{so}(3)$ denotes any corotational derivative of the Cauchy stress σ . The corotational stability postulate implies that in all uniform deformations without rotation of principal axes the incremental Cauchy stress moduli are positive [56].

1.3 Hypoelasticity: rate-formulation in the current configuration

Finally, Truesdell extended and formalized the concept of Cauchy elasticity towards a rate-formulation. This means the constitutive law is written in terms of an ODE that must be integrated along a specific loading path for each material point. Such an ODE must satisfy the principle of frame-indifference (objectivity) and necessitates the introduction of objective time rates for stresses. One such stress rate (among many others) is the Zaremba-Jaumann rate (cf. Jaumann [32, 33], Zaremba [81] and many other authors [18, 19, 20, 21, 35, 69, 70]). For its derivation, we now interpret the diffeomorphism φ as a time-dependent quantity, so that $\varphi = \varphi(x, t) : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3$. Then the corotational Zaremba-Jaumann rate is given by

$$\frac{D^{\text{ZJ}}}{Dt}[\sigma] := \frac{D}{Dt}[\sigma] + \sigma W - W \sigma, \quad W = \text{skew } L, \quad L = \dot{F} F^{-1}, \quad (1.28)$$

where $\frac{D}{Dt}[\sigma]$ denotes the material derivative of σ . $\frac{D^{\text{ZJ}}}{Dt}$ is arguably the simplest objective rate since the spin W can be determined directly from $D_\xi v = L = D + W$, $W = \text{skew } D_\xi v$, where v is the spatial velocity in the current configuration Ω_ξ . The Zaremba-Jaumann rate of the Cauchy stress $\frac{D^{\text{ZJ}}}{Dt}[\sigma]$ provides a measure of the change of stress in a material point with respect to a (corotated) frame that rotates with the spin W , such that rigid body rotations are properly taken into account. Then a hypoelastic model in the Eulerian configuration Ω_ξ can be written as

$$\underbrace{\frac{D^{\text{ZJ}}}{Dt}[\sigma] = \mathbb{H}^{\text{ZJ}}(\sigma).D, \quad D = \text{sym } D_\xi v,}_{\text{constitutive law}} \quad \underbrace{\text{Div}_\xi \sigma(\xi, t) = f(\xi, t),}_{\text{spatial equilibrium}} \quad \text{in } \Omega_\xi, \quad (1.29)$$

together with suitable initial and boundary conditions. Note that $\mathbb{H}^{\text{ZJ}}(\sigma)$, defined by (1.29)₁, is a constitutive fourth-order tangent stiffness tensor, mapping symmetric arguments to symmetric arguments (minor symmetry) and depending on the used stress rate, here the Zaremba-Jaumann rate.

We will show subsequently that for a smooth diffeomorphism $\varphi : \Omega \rightarrow \Omega_\xi$ the system (1.29) is equivalent to

$$\begin{aligned} \partial_t \sigma + D_\xi \sigma \cdot v + \sigma W - W \sigma &= \mathbb{H}^{\text{ZJ}}(\sigma). \text{sym } D_\xi v, & W &= \text{skew } D_\xi v, \\ \text{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma). \text{sym } D_\xi v] &= \text{Div}_\xi [\sigma \cdot (D_\xi v)^T - \text{div}_\xi v \cdot \sigma + \sigma (\text{skew } D_\xi v) - (\text{skew } D_\xi v) \sigma] \\ &\quad + \text{div}_\xi v \cdot f(\xi, t) + D_\xi f(\xi, t) \cdot v + \partial_t f(\xi, t), \end{aligned} \quad (1.30)$$

where only the stress rate and the spatial velocity appear.

It is well-known that any hyperelastic or Cauchy elastic model in which $\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3)$, $B \mapsto \sigma(B)$ is bijective can be written in the format (1.29) (cf. Truesdell [76] and Noll [60], see also [53]), however with an expression for $\mathbb{H}^{\text{ZJ}}(\sigma)$ that is not easily manageable (cf. [53]). Hypoelasticity of grade zero, which means $\mathbb{H}^{\text{ZJ}}(\sigma) = \mathbb{C}^{\text{iso}} = \text{const.}$ (cf. Truesdell [74]) is usually not compatible with Cauchy elasticity (the Cauchy stress would depend on the loading path and not only on the local configuration), and not compatible with the existence of a strain energy function. This is why hypoelasticity (also referred to as hypoelasticity of grade zero) has often been abandoned in favour of the total Lagrangean approach in (1.1), (1.2). Yet, as we argue below, (1.30) arises from natural modeling requirements and comes with a satisfactory existence theory.

For more details on hypoelasticity, we refer the reader to [6, 26, 37, 62, 64, 69, 70] for a non-exhaustive list. Moreover, different aspects of the FEM-implementation of hypoelastic models are addressed e.g. in [5, 16, 24, 36, 62, 83], and the recent work [53] provides more in-depth explanations of hypoelasticity.

1.4 Approach in this paper

Based on our above discussion, the main purpose of the present paper is to understand whether writing the equations of isotropic, compressible nonlinear elasticity in the rate format (1.30) and the Eulerian configuration can support any new existence theorem, provided that suitable constitutive assumptions on the induced tangent stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$ are made. Therefore, we first need to reformulate (1.29) into the equivalent system (1.30) that allows us to utilize the symmetry and positive definiteness of $\mathbb{H}^{ZJ}(\sigma)$, see Section 2.1.

In order to be sufficiently self-contained, we will confine ourselves to one exceptionally favorable Cauchy-elastic material for which most necessary calculations and observations can be made explicit. Namely, we consider the isotropic and compressible Cauchy elastic constitutive law

$$\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3), \quad \sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbb{1} = \mu \sinh(\log B) + \frac{\lambda}{2} \text{tr}(\log B) \mathbb{1}, \quad (1.31)$$

in which $\frac{1}{2}(B - B^{-1})$ is the Mooney-strain (cf. Curnier and Rakotomanana [11]). Thus, our development should only be taken as a first step for more general considerations, including the most important case of hyperelasticity.

Remark 1.1. The material response (1.31), while definitely not hyperelastic in the compressible case [80], presents a number of salient mechanical features, listed below in Section 2.2. Notably, (1.31) is valid for large rotations (since frame-indifferent) and finite strains (since extreme stretches go with extreme Cauchy stresses). Moreover, the corresponding induced tangent stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$ is smooth as a function of σ and not only minor symmetric, meaning that it maps the space of symmetric matrices onto itself, but also major symmetric, whereby $\mathbb{H}^{ZJ}(\sigma)$ is even self-adjoint; see the appendix, Section A.3.6 for more detail. Lastly, $\mathbb{H}^{ZJ}(\sigma)$ is uniformly positive definite, even though (1.31) is not Legendre-Hadamard elliptic throughout. In the sense that the rank-one monotonicity (1.12) is not satisfied throughout. Note that (1.12) remains applicable to merely Cauchy elastic response [15]. Moreover, we note that the law (1.31) may already be used for the stress analysis in a purely mechanical context.

While the present paper is centered around the underlying mathematical modeling, the follow-up paper [7] will address the corresponding existence theory. In particular, the aim of [7] is to establish that the new system (1.30) – to be derived in Section 2.1 – together with the material response (1.31) and suitable initial and boundary conditions admits a weak solution in terms of Cauchy stresses σ and the velocity v . For the complete initial boundary value problem (1.30) in the velocity $v(\xi, t)$ and the Cauchy stress $\sigma(\xi, t)$, we shall use a parabolic regularisation and pass to the limit in conjunction with a Schauder fixed point argument. This determines the velocity field $v(\xi, t)$. In an additional integration step, the deformation solution $\varphi(x, t)$ can be obtained for small $0 < t < T^*$, where T^* is determined by a Schauder-type fixed point argument provided that the velocity solution v is sufficiently smooth. Yet, in Theorem 2.8, below we foreshadow such results by establishing the well-posedness of an associated subproblem.

For technical simplicity, the present and the follow-up paper [7] will primarily be concerned with a fixed domain $\Omega_\xi = \text{const.}$, meaning that no shape or volumetric changes at the boundary are considered. We thereby give the foundation for more general scenarios, where the recently developed theory of time-varying domains will play a pivotal role. With the latter previously being confined to problems from fluid mechanics, see e.g. [8] for applications to fluid-structure-interaction, this will eventually lead to a unified existence theory for nonlinear solid and time-varying materials. The case of time-varying domains, however, comes with a multitude of technical challenges on top of the conceptual key novelties, which are already at the heart of the scenario as considered here.

2 The non-linear hypoelasticity model

As described in Section 1.4, the first major task consists in the derivation of a system that is equivalent to (1.29) and allows for the utilization of the symmetry and positive definiteness of the induced fourth-order tangent stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$. Therefore, let us consider $W(F)$ to be the strain energy function of an elastic material in which $F(x) := D\varphi(x) \in \text{GL}^+(3)$ is the gradient of a deformation $\varphi : \Omega \rightarrow \mathbb{R}^3$ with an open domain $\Omega \subset \mathbb{R}^3$. The deformation $\varphi(x)$ maps from a stress-free reference configuration Ω to a configuration $\varphi(\Omega) =: \Omega_\xi$ in the Euclidean 3-space; $W(F)$ is measured per unit volume of the reference configuration.

In the quasi-static setting, which is the setting we are working in, we understand $\varphi : \Omega \times \mathbb{R}_+^0 \rightarrow \varphi(\Omega, t) \subset \mathbb{R}^3$ as a deformation that allows for interior movements depending on the time t due to loads. We generally assume that for any fixed time $t_0 \geq 0$ the function $\varphi(\cdot, t_0) : \Omega \rightarrow \varphi(\Omega, t_0)$, $\varphi(x, t_0) = \xi$ is a diffeomorphism from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$. Furthermore, we restrict our analysis to the special case $\varphi(\Omega, t) \equiv \Omega_\xi \equiv \text{const.}$ This corresponds to the situation where only interior movements of the material are possible due to body forces $f(\xi, t)$, similar to a moving fluid confined in a container. This setting excludes structural and geometrical instabilities like e.g. buckling, barrelling and necking, while material instabilities are not a priori neglected.

2.1 Derivation of the hypo-elastic system - a rate-formulation in the spatial setting

Now, to start with the derivation of the alternative system of equations it is mandatory to obtain an expression for $\text{Div}_\xi \frac{D}{Dt}[\sigma]$, since our goal is to replace $\frac{D}{Dt}[\sigma]$ by

$$\mathbb{H}^{\text{ZJ}}(\sigma).D = \frac{D}{Dt}[\sigma] + \sigma W - W \sigma \quad \iff \quad \frac{D}{Dt}[\sigma] = \mathbb{H}^{\text{ZJ}}(\sigma).D - \sigma W + W \sigma, \quad (2.32)$$

yielding an equation with the structure

$$\text{Div}_\xi[\mathbb{H}^{\text{ZJ}}(\sigma(\xi, t)).D] = -g(\xi, t) \quad (2.33)$$

with some vector-valued function $g(\xi, t)$. For future reference, we note that it is incorrect to simply write $\frac{D}{Dt}[\text{Div}_\xi \sigma] = \text{Div}_\xi \frac{D}{Dt}[\sigma]$. This will be shown by the subsequent calculations, and is due to the fact that $\text{Div}_\xi = \text{Div}_{\varphi(x, t)}$ is itself time-dependent and $\frac{D}{Dt}$ in the spatial configuration is meant to be the material or substantial derivative. Thus, we first recall how the corresponding equilibrium equation

$$\text{Div}_\xi \sigma(\xi, t) = f(\xi, t), \quad (2.34)$$

where $f(\xi, t)$ is a vector-valued body force, can be transformed from the current configuration Ω_ξ to the reference configuration Ω and vice versa. This will allow us to differentiate the equation with respect to time in the reference configuration. A key ingredient in this respect is the **Piola transformation**, which states that for a diffeomorphism $\varphi : \Omega \rightarrow \Omega_\xi$ and a continuously differentiable operator $S : \Omega_\xi \rightarrow \mathbb{R}^{3 \times 3}$ the following identity holds:

$$\text{Div}_x(S(\varphi(x, t), t) \cdot \text{Cof } F(x, t)) = \det F(x, t) \cdot \text{Div}_\xi S(\varphi(x, t), t) = \det F(x, t) \cdot \text{Div}_\xi S(\xi, t). \quad (2.35)$$

2.1.1 Transformation of the equilibrium equation

Consider the equilibrium equation in elastostatics

$$\text{Div}_\xi \sigma(\xi, t) = f(\xi, t) \quad (2.36)$$

in the current configuration Ω_ξ for a given body force $f(\xi, t)$. The non-symmetric first Piola-Kirchhoff stress tensor $S_1(x) := S_1(F(x))$ connects points $x \in \Omega$ with points $\varphi(x, t) = \xi \in \Omega_\xi$ via

$$S_1(x, t) = \sigma(\xi, t) \cdot \text{Cof } F(x, t). \quad (2.37)$$

Applying the Piola transformation (2.35) to $\sigma(\xi, t)$ yields

$$\begin{aligned} \text{Div}_x S_1(x, t) &= \text{Div}_x(\sigma(\varphi(x, t), t) \cdot \text{Cof } F(x, t)) \stackrel{\text{Piola}}{=} \det F(x, t) \cdot \text{Div}_\xi \sigma(\xi, t) \\ \iff \text{Div}_\xi \sigma(\xi, t) &= (\det F(x, t))^{-1} \cdot \text{Div}_x S_1(x, t). \end{aligned} \quad (2.38)$$

Simultaneously, we can transform the body force $f = f(\xi, t)$, suppressing the t -dependence for this calculation, via (cf. Ciarlet [9, p.73])

$$\begin{aligned} \int_{\Omega_\xi} \langle \text{Div}_\xi \sigma(\xi), \vartheta(\xi) \rangle d\xi &= \int_{\Omega_\xi} \langle f(\xi), \vartheta(\xi) \rangle d\xi \quad \forall \vartheta \in C_0^\infty(\Omega_\xi, \mathbb{R}^3) \\ &= \int_{\Omega} \langle f(\varphi(x)), \vartheta(\varphi(x)) \rangle \cdot \underbrace{\det D\varphi(x) dx}_{\cong d\xi} = \int_{\Omega} \underbrace{\langle \det D\varphi(x) f(\varphi(x)), \vartheta(\varphi(x)) \rangle}_{=: \tilde{f}(x)} dx \\ &= \int_{\Omega} \langle \tilde{f}(x), \tilde{\vartheta}(x) \rangle dx \quad \forall \tilde{\vartheta} \in C_0^\infty(\Omega, \mathbb{R}^3), \end{aligned} \quad (2.39)$$

whereby we have, in particular, the relation

$$\tilde{f}(x, t) = \det D\varphi(x, t) \cdot f(\varphi(x, t)) = \det F(x, t) \cdot f(\xi, t), \quad \text{so} \quad f(\xi, t) = (\det F(x, t))^{-1} \cdot \tilde{f}(x, t). \quad (2.40)$$

Thus, using (2.38) and (2.40), we can express (2.36) equivalently in the reference configuration Ω by

$$\operatorname{Div}_x S_1(F(x, t)) = \tilde{f}(x, t) = \det F(x, t) \cdot f(\xi, t). \quad (2.41)$$

2.1.2 Derivation of the rate formulation of force equilibrium for the Zaremba-Jaumann rate

Now that we have an equivalent system to (2.34) in the reference configuration Ω , we can interchange the operations as initially intended, i.e. $\frac{d}{dt} \operatorname{Div}_x S_1 = \operatorname{Div}_x \frac{d}{dt} S_1$, since in the material setting the domain Ω is independent of the time t . Even though clear to the expert reader, the following derivation of the rate formulation is crucial for the sequel and thus shall be carried out in detail: Since the first Piola-Kirchhoff stress tensor $S_1(x, t)$ fulfills the relation

$$S_1(x, t) = \sigma(\varphi(x, t), t) \cdot \operatorname{Cof} F(x, t) = J(x, t) \cdot \sigma(\xi, t) \cdot F^{-T}(x, t), \quad (2.42)$$

where $J = \det F$, we have

$$\frac{d}{dt} S_1 = \frac{d}{dt} [\det F \cdot \sigma \cdot F^{-T}] = \frac{d}{dt} [\det F] \cdot \sigma \cdot F^{-T} + \det F \cdot \frac{D}{Dt} [\sigma] \cdot F^{-T} + \det F \cdot \sigma \cdot \frac{d}{dt} [F^{-T}]. \quad (2.43)$$

Additionally, we obtain

$$\frac{d}{dt} [\det F] = \langle \operatorname{Cof} F, \dot{F} \rangle = \det F \langle F^{-T}, \dot{F} \rangle = \det F \langle \mathbb{1}, \dot{F} F^{-1} \rangle = \det F \cdot \operatorname{tr}(L) \quad (2.44)$$

as well as

$$\frac{d}{dt} [F^{-T}] = \left(\frac{d}{dt} F^{-1} \right)^T = (-F^{-1} \dot{F} F^{-1})^T = -L^T F^{-T}. \quad (2.45)$$

Hence, (2.43) yields the relation

$$\begin{aligned} \frac{d}{dt} S_1 &= \det F \cdot \operatorname{tr}(L) \cdot \sigma \cdot F^{-T} + \det F \cdot \frac{D}{Dt} [\sigma] \cdot F^{-T} + \det F \cdot \sigma \cdot (-L^T \cdot F^{-T}) \\ &= \operatorname{tr}(D) \cdot \sigma \cdot \operatorname{Cof} F + \frac{D}{Dt} [\sigma] \cdot \operatorname{Cof} F - \sigma \cdot L^T \cdot \operatorname{Cof} F \\ &= \left(\operatorname{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right) \cdot \operatorname{Cof} F. \end{aligned} \quad (2.46)$$

Applying the Piola transformation (2.35) once again to transform back to the current configuration Ω_ξ , we can use (2.41) and (2.46) to obtain

$$\begin{aligned} \operatorname{Div}_x \frac{d}{dt} S_1(F(x, t)) &\stackrel{(2.46)}{=} \operatorname{Div}_x \left(\left[\operatorname{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right] \cdot \operatorname{Cof} F \right) \\ &\stackrel{\text{Piola}}{=} (\det F) \cdot \operatorname{Div}_\xi \left(\operatorname{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right) = \frac{d}{dt} \tilde{f}(x, t) \\ \stackrel{\det F > 0}{\iff} \quad \operatorname{Div}_\xi \frac{D}{Dt} [\sigma] &= (\det F)^{-1} \cdot \frac{d}{dt} \tilde{f}(x, t) - \operatorname{Div}_\xi (\operatorname{tr}(D) \cdot \sigma - \sigma \cdot L^T), \end{aligned} \quad (2.47)$$

where now by (2.40)

$$\begin{aligned} \frac{d}{dt} \tilde{f}(x, t) &= \langle \operatorname{Cof} F, \dot{F} \rangle f(\varphi(x, t), t) + \det F \cdot \frac{d}{dt} [f(\varphi(x, t), t)] \\ &= \det F \cdot \left(\underbrace{\langle F^{-T}, \dot{F} \rangle}_{= \operatorname{tr}(D)} \cdot f(\varphi(x, t), t) + \frac{d}{dt} [f(\varphi(x, t), t)] \right). \end{aligned} \quad (2.48)$$

Thus (2.47)₃ becomes

$$\operatorname{Div}_\xi \frac{D}{Dt}[\sigma] = \operatorname{tr}(D) \cdot f(\varphi(x, t), t) + \frac{d}{dt}[f(\varphi(x, t), t)] - \operatorname{Div}_\xi (\operatorname{tr}(D) \cdot \sigma - \sigma \cdot L^T). \quad (2.49)$$

In a final step we use the defining relation for the Zaremba-Jaumann derivative $\frac{D^{\text{ZJ}}}{Dt}[\sigma]$

$$\mathbb{H}^{\text{ZJ}}(\sigma) \cdot D = \frac{D^{\text{ZJ}}}{Dt}[\sigma] := \frac{D}{Dt}[\sigma] + \sigma W - W \sigma, \quad W = \operatorname{skew} L, \quad L = \dot{F} F^{-1} \quad (2.50)$$

for an invertible constitutive law $B \mapsto \sigma(B)$ in (2.49) and where $\mathbb{H}^{\text{ZJ}}(\sigma)$ is the induced fourth-order tangent stiffness tensor, to obtain

$$\begin{aligned} \operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma) \cdot D] &= \operatorname{tr}(D) \cdot f(\varphi(x, t), t) + \frac{d}{dt}[f(\varphi(x, t), t)] - \underbrace{\operatorname{Div}_\xi (\operatorname{tr}(D) \cdot \sigma - \sigma \cdot L^T)}_{\text{"contribution from Piola"}} + \underbrace{\operatorname{Div}_\xi [\sigma W - W \sigma]}_{\text{"contribution from the rate}} \\ &= \underbrace{\operatorname{Div}_\xi (\sigma \cdot L^T - \operatorname{tr}(D) \cdot \sigma + \sigma W - W \sigma) + \operatorname{tr}(D) \cdot f(\varphi(x, t), t)}_{=: -g(\xi, t)} + \frac{d}{dt}[f(\varphi(x, t), t)]. \end{aligned} \quad (2.51)$$

So, we have determined the alternative rate form equilibrium system in the current configuration

$$\begin{aligned} \operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma) \cdot \operatorname{sym} D_\xi v] &= \operatorname{Div}_\xi [\sigma \cdot (D_\xi v)^T - \operatorname{div}_\xi v \cdot \sigma + \sigma (\operatorname{skew} D_\xi v) - (\operatorname{skew} D_\xi v) \sigma] \\ &\quad + \operatorname{div}_\xi v \cdot f(\xi, t) + D_\xi f(\xi, t) \cdot v + \partial_t f(\xi, t). \end{aligned} \quad (2.52)$$

The system is completed with the constitutive equation for the symmetric Cauchy stress σ

$$\frac{D^{\text{ZJ}}}{Dt}[\sigma] = \mathbb{H}^{\text{ZJ}}(\sigma) \cdot \operatorname{sym} D_\xi v \quad \iff \quad \partial_t \sigma + D_\xi \sigma \cdot v + \sigma W - W \sigma = \mathbb{H}^{\text{ZJ}}(\sigma) \cdot \operatorname{sym} D_\xi v, \quad (2.53)$$

in which $W = \operatorname{skew} D_\xi v$ and $\sigma \in \operatorname{Sym}(3)$.

Thus, the complete rate-form system in the Eulerian setting for $\sigma(\xi, t) \in \operatorname{Sym}(3)$ and $v(\xi, t) \in \mathbb{R}^3$ is given by

$$\begin{aligned} \partial_t \sigma + D_\xi \sigma \cdot v + \sigma W - W \sigma &= \mathbb{H}^{\text{ZJ}}(\sigma) \cdot \operatorname{sym} D_\xi v, \quad W = \operatorname{skew} D_\xi v, \\ \operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma) \cdot \operatorname{sym} D_\xi v] &= \operatorname{Div}_\xi [\sigma \cdot (D_\xi v)^T - \operatorname{div}_\xi v \cdot \sigma + \sigma (\operatorname{skew} D_\xi v) - (\operatorname{skew} D_\xi v) \sigma] \\ &\quad + \operatorname{div}_\xi v \cdot f(\xi, t) + D_\xi f(\xi, t) \cdot v + \partial_t f(\xi, t) \end{aligned} \quad (2.54)$$

as already announced in (1.30). In particular, we note that the deformation $\varphi(x, t)$ and therefore $F = D\varphi$ and $B = F F^T$ do not appear explicitly anymore. As a consequence, the deformation $\varphi(x, t)$ has to be found in a subsequent step, the latter being potentially conditional upon sufficient regularity of the velocity $v(\xi, t)$.

Remark 2.1. This system, paired with suitable initial and boundary conditions, will be the origin for the existence theory to be developed in the follow-up paper [7]. At this point, it is important to take careful note of the following perceived ambiguity: While it is necessary for the derivation of the system (2.54) to understand $\xi = \varphi(x, t)$ as a **time-dependent** variable, leading for example to the relation for the substantial derivative

$$\frac{D}{Dt}[f(\xi, t)] = \frac{d}{dt}[f(\varphi(x, t), t)] = \partial_t f(\xi, t) + D_\xi f(\xi, t) \cdot \varphi_t = \partial_t f(\xi, t) + D_\xi f(\xi, t) \cdot v, \quad (2.55)$$

it is evident that from now on, working only on the current configuration, ξ is understood as **time-independent, solely spatial variable**. As correctly pointed out by an anonymous referee, this approach can be encountered frequently in the context of fluid mechanics. In particular, every time-derivative of $\sigma = \sigma(\xi, t)$ or $v = v(\xi, t)$ is to be understood as partial derivative in time direction. We will clarify this causality by writing $\partial_t \sigma(\xi, t)$ instead of $\frac{d}{dt}[\sigma(\xi, t)]$, even though, strictly speaking, in the context of the initial boundary value problem (2.79) both are the same.

Remark 2.2 (Comparison to the linear theory). Considering zero prestress $\sigma \equiv 0$ we obtain

$$\operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(0) \cdot \operatorname{sym} D_\xi v(\xi, t)] = \operatorname{div}_\xi v \cdot f(\xi, t) + D_\xi f(\xi, t) \cdot v + \partial_t f(\xi, t), \quad (2.56)$$

which is formally *nearly* the rate formulation

$$\operatorname{Div}_x \mathbb{C}^{\text{iso}} \cdot \operatorname{sym} D_x u(x, t) = f(x, t) \quad \xrightarrow{\partial_t} \quad \operatorname{Div}_x \mathbb{C}^{\text{iso}} \cdot \operatorname{sym} D_x u_t(x, t) = \partial_t f(x, t) \quad (2.57)$$

of linear elasticity by identifying reference and deformed configuration together with $\mathbb{H}^{\text{ZJ}}(0) = \mathbb{C}^{\text{iso}}$. Note that for spatially homogeneous body force f and small $\operatorname{tr}(D) = \operatorname{div}_\xi v$ (nearly incompressible), the right hand side of (2.56) reduces to $\partial_t f(\xi, t)$.

Remark 2.3 (Comparable results). Setting $\frac{d}{dt}[\hat{f}] := (\det F)^{-1} \cdot \frac{d}{dt}[\tilde{f}(x, t)]$, equation (2.47) is equivalent to

$$\operatorname{Div}_\xi \left(\frac{D}{Dt}[\sigma] + \operatorname{tr}(L) \sigma - \sigma L^T \right) - \frac{d}{dt}[\hat{f}] = 0. \quad (2.58)$$

This equation is similar to that of many other authors.

For example Ji et al. [34, Eq. 44] write (in the weak form and in our notation)

$$\begin{aligned} 0 &= \int_{\Omega_\xi} \left\langle \frac{D^{\text{ZJ}}}{Dt}[\sigma], \operatorname{sym} D_\xi \vartheta \right\rangle - 2 \langle \sigma, D \operatorname{sym} D_\xi \vartheta \rangle + \langle \sigma, L^T D_\xi \vartheta \rangle + \operatorname{tr}(L) \langle \sigma, \operatorname{sym} D_\xi \vartheta \rangle d\xi \quad \forall \vartheta \in C_0^\infty(\Omega_\xi, \mathbb{R}^3). \\ &= \int_{\Omega_\xi} \left\langle \frac{D^{\text{ZJ}}}{Dt}[\sigma] - 2 D \sigma + \operatorname{tr}(D) \sigma, \operatorname{sym} D_\xi \vartheta \right\rangle + \langle L \sigma, D_\xi \vartheta \rangle d\xi \\ &= \int_{\Omega_\xi} \left\langle \frac{D^{\text{ZJ}}}{Dt}[\sigma] - 2 \cdot \frac{1}{2} (D \sigma + \sigma D) + \operatorname{tr}(D) \sigma + L \sigma, D_\xi \vartheta \right\rangle d\xi \\ &= \int_{\Omega_\xi} \underbrace{\left\langle \frac{D^{\text{ZJ}}}{Dt}[\sigma] - 2 \cdot \frac{1}{2} (D \sigma + \sigma D) + L \sigma + \operatorname{tr}(D) \sigma, D_\xi \vartheta \right\rangle d\xi}_{=: \mathcal{A}} \quad \forall \vartheta \in C_0^\infty(\Omega_\xi, \mathbb{R}^3). \end{aligned} \quad (2.59)$$

Rewriting \mathcal{A} then leads to ($\operatorname{tr}(L) = \operatorname{tr}(D)$)

$$\begin{aligned} \mathcal{A} &= \frac{D}{Dt}[\sigma] + \sigma W - W \sigma - 2 \cdot \frac{1}{2} \cdot (D \sigma + \sigma D) + D \sigma + W \sigma + \operatorname{tr}(D) \sigma \\ &= \frac{D}{Dt}[\sigma] + \sigma W - \sigma D + \operatorname{tr}(D) \sigma = \frac{D}{Dt}[\sigma] + \operatorname{tr}(D) \sigma - \sigma L^T, \end{aligned} \quad (2.60)$$

which is the argument of the divergence proposed in (2.58).

Another example is given by Aubram in [2, Prop 6.3], who considers (again in the weak form and in our notation)

$$0 = \int_{\Omega_\xi} \left\langle \frac{D^{\text{ZJ}}}{Dt}[\sigma] - 2 D \sigma + \sigma \operatorname{tr}(D), \operatorname{sym} D_\xi \vartheta \right\rangle + \langle \sigma, L^T \cdot (D_\xi \vartheta) \right\rangle d\xi, \quad \forall \vartheta \in C_0^\infty(\Omega_\xi, \mathbb{R}^3). \quad (2.61)$$

Using $\langle \sigma, L^T \cdot D_\xi \vartheta \rangle = \langle L \sigma, D_\xi \vartheta \rangle$, it is easily verified that (2.61) coincides with the previous two equations (2.58) and (2.59).

Lastly, Korobeynikov and Larichkin [37, p. 6] write (with Div denoting their divergence operator)

$$\operatorname{Div} \left(\frac{D}{Dt}[\sigma] + \sigma W - W \sigma + \sigma \operatorname{tr}(D) + \sigma L^T - (D \sigma + \sigma D) \right) + \varrho \frac{d}{dt}[f] = 0, \quad (2.62)$$

which may be simplified by observing

$$\sigma L^T + \sigma W - W \sigma - D \sigma - \sigma D = \sigma (D - W) + \sigma W - W \sigma - D \sigma - \sigma D = -W \sigma - D \sigma = -L \sigma, \quad (2.63)$$

yielding the identity

$$\underline{\text{Div}} \left(\frac{D}{Dt} [\sigma] + \sigma \text{tr}(D) - L \sigma \right) + \varrho \frac{d}{dt} [f] = 0. \quad (2.64)$$

Since in their notation $\underline{\text{Div}}(X) := \text{Div}_\xi(X^T)$, we recover (2.58) by observing that

$$\left(\frac{D}{Dt} [\sigma] + \sigma \text{tr}(D) - L \sigma \right)^T = \frac{D}{Dt} [\sigma] + \sigma \text{tr}(D) - \sigma L^T. \quad (2.65)$$

We also note here that the other authors assume that the force on the right hand side of the equation is given as a (spatially constant) body force per unit volume, i.e. we can write it in the form $\tilde{f}(t) := \varrho f(t)$, where $\varrho = \frac{\varrho_0}{\det F}$ and ϱ_0 is the density for $t = 0$. Then, starting with the spatial equilibrium equation

$$\text{Div}_\xi \sigma - \tilde{f}(t) = 0, \quad (2.66)$$

one may proceed with the Piola transformation, which yields

$$\text{Div}_\xi \sigma(\xi, t) = (\det F(x, t))^{-1} \cdot \text{Div}_x S_1(x, t). \quad (2.67)$$

Hence (2.66) becomes

$$\frac{1}{\det F} \cdot \text{Div}_x S_1 - \tilde{f}(t) = 0 \quad \iff \quad \text{Div}_x S_1 - \varrho_0 \cdot f(t) = 0. \quad (2.68)$$

Now differentiation with respect to time yields

$$\text{Div}_x \frac{d}{dt} S_1 - \varrho_0 \cdot \dot{f}(t) = 0, \quad \text{with} \quad \text{Div}_x \frac{d}{dt} S_1 = (\det F) \cdot \text{Div}_\xi \left(\text{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right). \quad (2.69)$$

Thus ($\varrho = \frac{\varrho_0}{\det F}$),

$$\begin{aligned} & (\det F) \cdot \text{Div}_\xi \left(\text{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right) - \varrho_0 \cdot \dot{f} = 0 \\ \iff & \text{Div}_\xi \left(\text{tr}(D) \cdot \sigma + \frac{D}{Dt} [\sigma] - \sigma \cdot L^T \right) - \varrho \cdot \dot{f} = 0. \end{aligned} \quad (2.70)$$

The approach in this paper is a more general one, in which $\frac{d}{dt} [\hat{f}]$ admits the explicit expression

$$\begin{aligned} \frac{d}{dt} [\hat{f}] &= \det F \cdot \left(\text{tr}(D) \cdot f(\varphi(x, t), t) + \frac{d}{dt} [f(\varphi(x, t), t)] \right) \\ &= \det F \cdot \left(\text{div}_\xi v \cdot f(\xi, t) + D_\xi f(\xi, t) \cdot v + \partial_t f(\xi, t) \right), \end{aligned} \quad (2.71)$$

where the force f may also change in spatial direction.

Remark 2.4. If we consider $\mathbb{H}^{\text{ZJ}}(\sigma) \cdot D = 2\mu D + \lambda \text{tr}(D) \mathbb{1}$ in (2.54) then we deal with a classical zero-grade hypo-elastic rate-formulation. To the best of our knowledge, no existence result for (2.54) is known but it is clear that $\mathbb{H}^{\text{ZJ}}(\sigma) \cdot D = 2\mu D + \lambda \text{tr}(D) \mathbb{1}$ may not be integrable towards a Cauchy-elastic formulation [71]. If we take $\mathbb{H}^{\text{ZJ}}(\sigma)$ as induced tangent stiffness tensor subordinate to the Cauchy-elastic law $\sigma = \frac{\mu}{2} (B - B^{-1}) + \frac{\lambda}{2} \log \det B \mathbb{1}$ then showing existence to (2.54) may lead to an existence result for the problem

$$\text{Div}_\xi \sigma(\xi, t) = f(\xi, t), \quad \sigma = \frac{\mu}{2} (B - B^{-1}) + \frac{\lambda}{2} \log \det B \mathbb{1} \quad (2.72)$$

provided that the initial conditions are compatible and the solutions (σ, v) to (2.54) are smooth enough to reconstruct the deformation $\varphi(x, t)$ along the particle moving in the velocity field. We therefore find it worthwhile to first investigate the rate-form equilibrium system under generic conditions on $\mathbb{H}^{\text{ZJ}}(\sigma)$, postponing the regularity issue to future works.

2.2 Properties of the constitutive law $\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2}(\log \det B) \cdot \mathbb{1}$

The constitutive choice of $\sigma(B)$ as made in (1.31) is not arbitrary. In fact, the constitutive law

$$\widehat{\sigma}(\log B) := \sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2}(\log \det B) \cdot \mathbb{1} = \mu \sinh(\log B) + \frac{\lambda}{2} \operatorname{tr}(\log B) \mathbb{1}$$

shares the following salient properties:

- The constitutive law $B \mapsto \sigma(B)$ is objective and isotropic.
- $\log B \mapsto \widehat{\sigma}(\log B) = \sigma(B)$ is strictly monotone in $\log B$ (it satisfies TSTS-M⁺), meaning that

$$\langle \widehat{\sigma}(\log B_1) - \widehat{\sigma}(\log B_2), \log B_1 - \log B_2 \rangle > 0 \quad \forall B_1, B_2 \in \operatorname{Sym}^{++}(3), \quad B_1 \neq B_2. \quad (2.73)$$

However, we point out that $B \mapsto \sigma(B)$ is not monotone in B .

- In $\mathbb{H}^{\text{ZJ}}(\sigma) \cdot D := \frac{D^{\text{ZJ}}}{Dt}[\sigma]$ the fourth-order tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$ is uniformly positive definite, meaning that there is a constant $c^+ > 0$ such that $\langle \mathbb{H}^{\text{ZJ}}(\sigma) \cdot D, D \rangle \geq c^+ \|D\|^2$ for all $\sigma \in \operatorname{Sym}(3)$ for all $D \in \operatorname{Sym}(3)$, and has major and minor symmetry (cf. [53]), i.e. $\mathbb{H}^{\text{ZJ}}(\sigma) \in \operatorname{Sym}_4^{++}(6)$.
- $\sigma(B)$ fulfills the “tension-compression symmetry” $\sigma(B) = -\sigma(B^{-1})$.
- Extreme stresses for extreme strains: $\begin{cases} \|\sigma(B)\| \rightarrow +\infty \text{ for } \det B \rightarrow 0 \text{ and } \|\sigma(B)\| \rightarrow +\infty \text{ for } \|B\| \rightarrow \infty, \\ \|\widehat{\sigma}(\log B)\| \rightarrow +\infty \text{ as } \|\log B\| \rightarrow +\infty, \end{cases}$

Therefore, $\sigma(B)$ is suitable for large rotations and large strains.

- $B \mapsto \sigma(B)$, $V \mapsto \sigma(V)$, $\log B \mapsto \widehat{\sigma}(\log B)$, $\log V \mapsto \widehat{\sigma}(\log V)$ are all bijective.
- There exists a smooth inverse mapping for $B \mapsto \sigma(B)$: $\mathcal{F}^{-1} : \operatorname{Sym}(3) \rightarrow \operatorname{Sym}^{++}(3)$, $B = \mathcal{F}^{-1}(\sigma(B))$.
- $\sigma(B)$ induces a rank-one convex formulation in a large neighbourhood of the stress-free reference configuration.
- Correct linearization: $2D_B\sigma(B)|_{B=\mathbb{1}} = \mathbb{C}^{\text{iso}} \in \operatorname{Sym}_4^{++}(6)$, $\mathbb{C}^{\text{iso}} \cdot \varepsilon = 2\mu\varepsilon + \lambda \operatorname{tr}(\varepsilon) \cdot \mathbb{1}$, $\varepsilon = \operatorname{sym} Du$.
- The weak-empirical inequalities are satisfied ($\beta_1 = \frac{\mu}{2} > 0$ and $\beta_{-1} = -\frac{\mu}{2} < 0$); no condition on β_0 .
- The tension-extension inequality, pressure-compression inequality and Baker-Ericksen inequalities are satisfied (cf. the Appendix A.3) and we have monotonicity in uniaxial loading and monotonicity of shear stress in simple shear (cf. [53] and [55, 56]).
- $B \mapsto \sigma(B)$ is additionally operator-monotone in B . By *operator-monotonicity* we mean that $B_1 \prec B_2 \implies \sigma(B_1) \prec \sigma(B_2)$, where $B_1 \prec B_2$ is the Löwner partial ordering [43]:

$$B_1 \prec B_2 \iff B_2 - B_1 \in \operatorname{Sym}^+(3). \quad (2.74)$$

The operator-monotonicity of B and B^{-1} with respect to B is clear (cf. Löwner [43]), it remains to show operator monotonicity of $B \mapsto \log \det B \cdot \mathbb{1} = \operatorname{tr}(\log B) \cdot \mathbb{1}$. Since if $B_1 \prec B_2$ we have $\log B_1 \prec \log B_2$ (due to the operator-monotonicity of $B \mapsto \log B$, it is clear that $\log B_2 - \log B_1 \in \operatorname{Sym}^+(3)$), then $\operatorname{tr}(\log B_2 - \log B_1) > 0$, so that $\operatorname{tr}(\log B_1) \cdot \mathbb{1} \prec \operatorname{tr}(\log B_2) \cdot \mathbb{1}$.

- There is a Cauchy pseudo-stress potential $\widetilde{\Psi} = \widetilde{\Psi}(B)$ for the Mooney strain $\frac{1}{2}(B - B^{-1})$:

$$\underbrace{B - B^{-1}}_{\text{monotone in } B} = D_B \underbrace{\left(\frac{1}{2} \|B\|^2 - \log \det B + 3 \right)}_{\text{convex in } B} = D_B \widetilde{\Psi}(B) \quad (2.75)$$

but $\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbb{1}$ is **not Hilbert-monotone** in terms of B , i.e.

$$\langle \sigma(B_1) - \sigma(B_2), B_1 - B_2 \rangle \not> 0 \quad \forall B_1, B_2 \in \operatorname{Sym}^{++}(3), \quad B_1 \neq B_2. \quad (2.76)$$

Crucially, comparing this with (2.73), we particularly find σ to be non-(Hilbert) monotone in B , but in $\log B$. Moreover, there does not exist a function $\Psi : \text{Sym}^{++}(3) \rightarrow \mathbb{R}$ such that

$$\sigma(B) = D_B \Psi(B), \quad (2.77)$$

while $\mathbb{H}^{\text{ZJ}}(\sigma)$ is major symmetric. Moreover, σ is **not hyperelastic**, i.e. there does not exist an isotropic energy $W(F)$ so that

$$\sigma(B) = \frac{2}{J} D_B W(B) \cdot B. \quad (2.78)$$

2.3 The complete spatial PDE-system and some first considerations

For a final time $0 < T \leq \infty$ we define $\Omega_T := \Omega_\xi \times (0, T)$ and $\Sigma_T := \partial\Omega_\xi \times (0, T)$. Then, based on the system (2.54), we study the following hypo-elasticity initial boundary value problem in terms of symmetric Cauchy stresses σ and velocities v :

Find the solution $(\sigma, v) = (\sigma(\xi, t), v(\xi, t))$ with values in $\text{Sym}(3) \times \mathbb{R}^3 \cong \mathbb{R}^6 \times \mathbb{R}^3 = \mathbb{R}^9$ of

$$\begin{aligned} \partial_t \sigma + D_\xi \sigma \cdot v + \sigma W - W \sigma &= \mathbb{H}^{\text{ZJ}}(\sigma) \cdot \text{sym} D_\xi v, & W = \text{skew } D_\xi v \in \mathfrak{so}(3), & \text{in } \Omega_T, \\ \text{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma) \cdot \text{sym} D_\xi v] &= \begin{cases} \text{Div}_\xi [\sigma(D_\xi v)^T - (\text{div}_\xi v)\sigma + \sigma(\text{skew } D_\xi v) - (\text{skew } D_\xi v)\sigma] \\ + (\text{div}_\xi v)f(\xi, t) + (D_\xi f(\xi, t))v + \partial_t f(\xi, t), \end{cases} & \text{in } \Omega_T, & (2.79) \\ v(\xi, 0) &= v_0(\xi), & \sigma(\xi, 0) = \sigma_0(\xi) \in \text{Sym}(3), & \text{in } \Omega_\xi, \\ v(\xi, t) &= 0, & & \text{on } \Sigma_T. \end{aligned}$$

Counting equations, (2.79)₁ yields six and (2.79)₂ yields three equations for the independent variables (σ, v) . Note that (2.79) constitutes a system resembling Differential Algebraic Equations (DAEs), a subject area with a rich literature. However, the first equation also contains spatial derivatives, so that DAE-techniques are difficult to be applied directly. We now discuss this system in more detail and give a well-posedness result for a subproblem in Theorem 2.8, leaving the full system (2.79) to the follow-up paper [7].

2.3.1 The induced tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$

Searching for a solution $(\sigma(\xi, t), v(\xi, t))$ of the system (2.79), one cannot expect that the initially assumed constitutive law

$$\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2}(\log \det B) \cdot \mathbb{1}, \quad \text{where } B = F F^T \text{ is the Finger tensor,} \quad (2.80)$$

is a-priori fulfilled for every time $0 < t < T$, notably if some form of approximation is used. However, the information which constitutive law was used to derive the system (2.79) is expressed by the initial condition $\sigma(\xi, 0) = \sigma_0(\xi)$ and more importantly, in an implicit way, by the induced tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$. As shown in the Appendix A.2 (see also [53] for an alternative method), the constitutive law (2.80) leads to the induced tangent stiffness tensor

$$\mathbb{H}^{\text{ZJ}}(B) \cdot D := \frac{\mu}{2} \{B D + D B + B^{-1} D + D B^{-1}\} + \lambda \text{tr}(D) \cdot \mathbb{1}. \quad (2.81)$$

Furthermore, since the constitutive law $\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3)$ (2.80) is invertible, the inverse function $\mathcal{F}^{-1}(\sigma) = B$ can be used to derive the explicit dependence $\sigma \mapsto \mathbb{H}^{\text{ZJ}}(\sigma)$, thus encoding which constitutive law was used initially to obtain the system (2.79).

Remark 2.5. When starting from hyperelasticity, $\mathbb{H}^{\text{ZJ}}(\sigma)$ always possesses *minor symmetry*, meaning that $\mathbb{H}^{\text{ZJ}}(\sigma) : \text{Sym}(3) \rightarrow \text{Lin}(\text{Sym}(3), \text{Sym}(3))$, but no *major symmetry*, i.e. $\mathbb{H}^{\text{ZJ}}(\sigma)$ is not necessarily symmetric as a matrix. However, the particular choice (2.80) for the constitutive law $B \mapsto \sigma(B)$ leads to an induced fourth-order tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$ that is **minor and major symmetric** as well as **positive definite** (cf. Section A.2).

It is important to note that the positive definiteness implies the existence of a constant $c_0 > 0$ with

$$\langle \mathbb{H}^{ZJ}(\sigma).D, D \rangle \geq c_0 \|D\|^2, \quad \forall D \in \text{Sym}(3), \quad (2.82)$$

not to be confused with LH-ellipticity of the constitutive law (as (2.80) is not LH-elliptic in the sense of (1.12)). Since (2.82) holds independently of the stress level σ , this condition resembles nevertheless the strong ellipticity of $\mathbb{H}^{ZJ}(\sigma)$, cf. [1].

Remark 2.6. Every hyperelastic formulation in the deformed configuration Ω_ξ can be written in the format (2.79), if the constitutive law $\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3)$, $B \mapsto \sigma(B)$ is invertible (cf. Truesdell [76] and Noll [60]). However, $\mathbb{H}^{ZJ}(\sigma)$ need not be symmetric or positive definite at given σ and the invertibility of $B \mapsto \sigma(B)$ is equivalent almost everywhere in σ to the invertibility of $\mathbb{H}^{ZJ}(\sigma)$ (cf. [53]).

For example, a slightly compressible Neo-Hooke type solid with elastic energy

$$\begin{aligned} W_{NH}(F) &= \frac{\mu}{2} \left(\frac{\|F\|^2}{(\det F)^{\frac{2}{3}}} - 3 \right) + \frac{\kappa}{2} e^{(\log \det F)^2}, \\ \sigma_{NH}(B) &= \mu (\det B)^{-\frac{5}{6}} \text{dev}_3 B + \frac{\kappa}{2} (\det B)^{-\frac{1}{2}} (\log \det B) e^{\frac{1}{4} (\log \det B)^2} \cdot \mathbb{1}, \end{aligned} \quad (2.83)$$

admits an invertible constitutive law $\sigma_{NH} : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3)$, $B \mapsto \sigma_{NH}(B)$ together with $\det \mathbb{H}^{ZJ}(\sigma_{NH}) \neq 0$, but it can be shown that the corresponding induced tangent stiffness tensor $\mathbb{H}^{ZJ}(\sigma)$ is not positive definite throughout and W_{NH} does therefore not satisfy the TSTS-M⁺⁺ condition, and therefore does not satisfy inequality (2.73), see [53] and [38]. However, (2.83) is polyconvex and LH-elliptic (cf. Hartmann and Neff [28]).

Remark 2.7. In the general theory of hypo-elasticity, one considers a rate equation of the form

$$\frac{D^\sharp}{Dt}[\sigma] = \mathbb{H}^*(\sigma).D, \quad (2.84)$$

where $\frac{D^\sharp}{Dt}[\sigma]$ is an appropriate objective rate of the Cauchy stress tensor σ and $\mathbb{H}^*(\sigma)$ is a constitutive fourth-order tangent stiffness tensor (for more information on this topic see [53] and [17]). As the notation already suggests, the choices of $\frac{D^\sharp}{Dt}[\sigma]$ and $\mathbb{H}^*(\sigma)$ are a-priori arbitrary and independent of each other. In this work, it is our choice to use the tangent stiffness tensor $\mathbb{H}^\sharp(\sigma)$ that is induced by $\frac{D^\sharp}{Dt}[\sigma]$, yielding several features discussed earlier.

However, since the fourth-order tangent stiffness tensor $\mathbb{H}^*(\sigma)$ can be prescribed arbitrarily in general, one commonly used choice is

$$\mathbb{H}^*(\sigma).D = \mathbb{C}^{\text{iso}}.D = 2\mu D + \lambda \text{tr}(D) \cdot \mathbb{1}, \quad \text{the so-called "zero grade hypo-elasticity".} \quad (2.85)$$

We again emphasize that for the choice $\mathbb{H}^* = \mathbb{C}^{\text{iso}}$ there is no explicit law for the Cauchy stress σ . However, $\mathbb{C}^{\text{iso}}.D$ from (2.85) would also be positive definite for $\mu > 0, 3\lambda + 2\mu > 0$, completely independent of the stress.

2.3.2 Initial, boundary and compatibility conditions

The equations (2.79)₃ and (2.79)₄ state initial and boundary conditions that have to be fulfilled by the solution $(\sigma(\xi, t), v(\xi, t))$ of the system (2.79), where σ_0 is a (smooth) initial Cauchy stress distribution and v_0 is the initial velocity, respectively. For an illustration of the initial configuration see Figure 4.

The functions σ_0 and v_0 need to fulfill a set of requirements necessary for the application of the Schauder fixed point theory in the follow-up [7] and the determination of the deformation $\varphi(x, t)$, which we will go over next.

- The first one is given by the **compatibility condition**

$$\sigma_0(\xi) = \frac{\mu}{2} (B_0 - B_0^{-1}) + \frac{\lambda}{2} \log \det(B_0) \cdot \mathbb{1}, \quad B_0(\xi) := D\varphi_0(\varphi_0^{-1}(\xi; t)) D\varphi_0^T(\varphi_0^{-1}(\xi; t)), \quad (2.86)$$

that prescribes the initial values for $\sigma(\xi, 0)$. Here we need to assume $\varphi_0(x) \in H^2(\Omega)$ implying $\sigma_0, B_0 \in W^{1,2}(\Omega_\xi)$, which is a necessary condition for the application of the Schauder fixed point theorem.

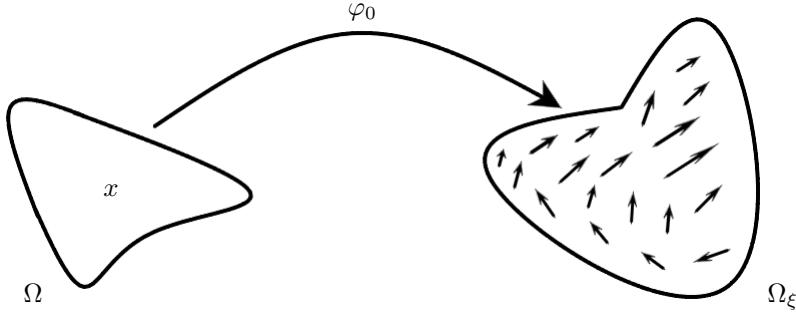


Figure 4: Picture of the initial diffeomorphism $\varphi_0 = \varphi(x, 0) : \Omega \rightarrow \Omega_\xi$. The arrows indicate the traction vectorfield $\sigma_0(\xi).e_1$, where $\sigma_0(\xi)$ is given by (2.86). Additionally, we have $v_0(\xi) = 0$ on $\partial\Omega_\xi$.

- The second one is another **compatibility condition** given by the assumption of an “initial equilibrium”

$$\operatorname{Div}_\xi \sigma_0(\xi) = f(\xi, 0) \iff \operatorname{Div}_x S_1(D\varphi_0(x)) = \tilde{f}(x, 0). \quad (2.87)$$

- Moreover, as stated in (2.79)₄ the boundary values of the diffeomorphism $\varphi(x, t)$ are assumed to be constant in time, which can be seen from

$$v(\xi, t) := \frac{d}{dt} \varphi(\varphi^{-1}(\xi; t), t) = \frac{d}{dt} \varphi(x, t) = 0 \quad \text{on } \Sigma_T \iff \varphi(x, t) \equiv \varphi(x) \quad \forall t \in [0, T]. \quad (2.88)$$

Hence on all of Σ_T we have $\varphi(x, t) \equiv \varphi(x, 0) =: \varphi_0(x)$ which is assumed to be a known quantity.

- Lastly, it will sometimes be convenient to assume $v_0(\xi) \equiv 0$ on Ω_ξ , meaning that the system is supposed to be in an equilibrium state at $t = 0$.

2.3.3 Well-posedness of a sub-problem

We end this section by proving the well-posedness of the system (2.79) in terms of the velocity v at a given history of σ, W and f . For fixed t and given $\sigma = \sigma(\xi, t) \in H^1(\Omega_\xi; \operatorname{Sym}(3))$, $W(\xi, t) \in H^1(\Omega_\xi; \mathfrak{so}(3))$, the left hand side of (2.79)₂ defines a linear operator $\mathbb{A}(\sigma) : H_0^1(\Omega_\xi; \mathbb{R}^3) \rightarrow H^{-1}(\Omega_\xi; \mathbb{R}^3)$ by

$$\mathbb{A}(\sigma).v := -\operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma).D]. \quad (2.89)$$

We wish to point out that, in the present setting, this result can be alternatively approached by the Lax-Milgram theorem. However, for the purposes of our follow-up [7] and to be ready to face more general nonlinear hyperelasticity, it is useful to directly embed the set-up into that of monotone operators.

Theorem 2.8. *Let $t \in (0, T)$ and $\sigma(\cdot, t) \in H^1(\Omega_\xi; \operatorname{Sym}(3))$, $W(\cdot, t) \in H^1(\Omega_\xi; \mathfrak{so}(3))$ be given and assume $\mathbb{H}^{\text{ZJ}}(\sigma)$ satisfies positive definiteness (2.82). Then the equation (2.89) defines a strictly monotone operator $v \mapsto \mathbb{A}(\sigma).v$ on $H_0^1(\Omega_\xi; \mathbb{R}^3)$. For $g \in L^2(\Omega_T)$ and a.e. $t \in (0, T)$, there exists a unique solution $v \in H_0^1(\Omega_\xi; \mathbb{R}^3)$ to the vector-valued equation*

$$-\operatorname{Div}_\xi [\mathbb{H}^{\text{ZJ}}(\sigma(\xi, t)).D] = g(\xi, t) \quad \text{in } \Omega_\xi, \quad (2.90)$$

where g is defined in (2.51) and assumed to be given.

Proof. In order to show that (2.89) defines a monotone operator $v \mapsto \mathbb{A}(\sigma).v$, we need to verify two properties.

The first observation is that $\mathbb{A}(\sigma)$ is **hemi-continuous**, i.e. for arbitrary $v_1, v_2 \in H_0^1(\Omega_\xi; \mathbb{R}^3)$ and any $w \in H_0^1(\Omega_\xi; \mathbb{R}^3)$, it holds

$$s \mapsto \langle\langle w, \mathbb{A}(\sigma).((1-s)v_1 + s v_2) \rangle\rangle \in C^0([0, 1]; \mathbb{R}). \quad (2.91)$$

Here and below, $\langle\langle w, w^* \rangle\rangle$ denotes the duality pairing for $w \in H_0^1(\Omega_\xi; \mathbb{R}^3)$, $w^* \in H^{-1}(\Omega_\xi; \mathbb{R}^3)$.

Secondly, $v \mapsto \mathbb{A}(\sigma).v$ is **monotone**. For $v_1, v_2 \in H_0^1(\Omega_\xi; \mathbb{R}^3)$, after integrating by parts and using the positive definiteness of $\mathbb{H}^{\text{ZJ}}(\sigma)$ from (2.82), we have

$$\begin{aligned} \langle\langle v_1 - v_2, \mathbb{A}(\sigma).v_1 - \mathbb{A}(\sigma).v_2 \rangle\rangle &= - \int_{\Omega_\xi} \langle (v_1 - v_2), \text{Div}_\xi [(\mathbb{H}^{\text{ZJ}}(\sigma)).(D_1 - D_2)] \rangle_{\mathbb{R}^3} d\xi \\ &= \int_{\Omega_\xi} \langle \mathbb{H}^{\text{ZJ}}(\sigma).(D_1 - D_2), Dv_1 - Dv_2 \rangle d\xi = \int_{\Omega_\xi} \langle \mathbb{H}^{\text{ZJ}}(\sigma).(D_1 - D_2), D_1 - D_2 \rangle d\xi \\ &\geq c_0 \int_{\Omega_\xi} \|D_1 - D_2\|^2 d\xi = c_0 \int_{\Omega_\xi} \|\text{sym}(D_\xi v_1 - D_\xi v_2)\|^2 d\xi \geq \frac{c_0}{2} \|D_\xi(v_1 - v_2)\|_{L^2(\Omega_\xi)}^2, \end{aligned} \quad (2.92)$$

where Korn's inequality is used in the last line (cf. Gmeineder et al. [13, 22, 23], Lewintan et al. [40, 41, 42] and Neff et al. [59]) and we write $D_i = \text{sym}(D_\xi v_i)$, $i = 1, 2$. By the Poincaré inequality, we hence find for $v_1 \neq v_2$

$$\langle\langle v_1 - v_2, \mathbb{A}(\sigma).v_1 - \mathbb{A}(\sigma).v_2 \rangle\rangle > 0, \quad (2.93)$$

which is the strict monotonicity of $v \mapsto \mathbb{A}(\sigma).v$.

From the hemi-continuity and the monotonicity of $\mathbb{A}(\sigma)$, the existence of a solution to (2.90) follows from the theory of monotone operators, see, e.g. [82]. The uniqueness of the solution is a direct consequence of the strict monotonicity of $\mathbb{A}(\sigma)$. \square

Remark 2.9. Theorem 2.8 and its proof reveal that the major symmetry of $\mathbb{H}^{\text{ZJ}}(\sigma)$ is not required and that instead the minor symmetry can be the right framework for solving the divergence equation (2.79).

Remark 2.10. Advanced elliptic regularity for PDE-systems in divergence form without major symmetry can be found in Haller-Dintelmann et al. [27]. If, in addition, we know that $\mathbb{H}^{\text{ZJ}}(\sigma)$ has major symmetry, as is the case for equation (2.79)₂ for given $\sigma(\xi, t), W(\xi, t), f(\xi, t)$ has variational structure. Indeed, the corresponding minimization problem is given by

$$\int_{\Omega_\xi} \langle \mathbb{H}^{\text{ZJ}}(\sigma(t)).\underbrace{\text{sym} D_\xi v}_{=:D}, \text{sym} D_\xi v \rangle - \langle g(\xi, t), v(\xi, t) \rangle d\xi \longrightarrow \min .v, \quad v \in H_0^1(\Omega_\xi; \mathbb{R}^3). \quad (2.94)$$

Again, existence and uniqueness follow from Korn's inequality. The advantage of using this framework is that we may use full elliptic regularity for the velocity field v if wanted.

2.4 Conditional determination of the diffeomorphism $\varphi(x, t)$ from the solution (σ, v) for $0 < t < T^*$

Assume that the spatial velocity $v(\xi, t)$ is regular enough to define $\varphi(x, t)$ as the (unique) solution of the characteristic system

$$\partial_t \varphi(x, t) = v(\varphi(x, t), t), \quad \varphi(x, 0) = \varphi_0(x). \quad (2.95)$$

This is for instance the case if $v(\xi, t)$ is Lipschitz continuous in ξ , which, for a domain Ω_ξ that is smooth enough (Lipschitz boundary suffices), is equivalent to $v(\cdot, t) \in W^{1,\infty}(\Omega_\xi)$ and continuous in t . Then the Picard-Lindelöf theorem can be used to prove unique solvability of (2.95) and we may use the reconstructed deformation $\varphi(x, t)$ to determine the quantities $F = D\varphi(x, t)$ and $B = FF^T$.

Reconstruction of the constitutive law is possible by using the initial condition

$$\sigma_0(\xi) = \frac{\mu}{2} (B_0 - B_0^{-1}) + \frac{\lambda}{2} \log \det(B_0) \cdot \mathbb{1}, \quad B_0(\xi) := D\varphi_0(\varphi_0^{-1}(\xi; t)) D\varphi_0^T(\varphi_0^{-1}(\xi; t)). \quad (2.96)$$

Furthermore, for a sufficiently smooth solution $\varphi(x, t)$ of (2.95) we then have with $\dot{J} = \text{tr}(L) J = \text{tr}(D) J$ and $J(x, t) = \det F(x, t)$:

$$\frac{d}{dt} [\log J] = \frac{\dot{J}}{J} = \text{tr}(D) \quad (2.97)$$

so that $\log J(t) = \log J(0) + \int_0^t \text{tr}(D(s)) \, ds$ and

$$J(t) = J(0) \cdot \exp \left(\int_0^t \text{tr}(D(s)) \, ds \right), \quad (2.98)$$

showing that automatically $\det D\varphi(x, t) = J(x, t) > 0$. Therefore, local smooth solutions of the hypoelastic problem satisfy automatically the local invertibility constraint.

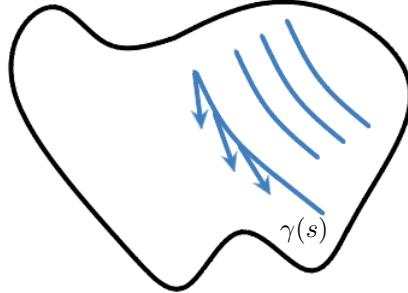


Figure 5: Reconstructing the diffeomorphism $\varphi(x, t)$ by solving the characteristic system $\partial_t \varphi(x, t) = v(\varphi(x, t), t)$ with $\varphi(x, 0) = \varphi_0(x)$.

3 Conclusion

Any hyperelastic isotropic nonlinear elasticity formulation can be written as a spatial rate-form equilibrium problem using objective stress rates and involving exclusively the Cauchy stress σ and the spatial velocity v . For this, the Piola transformation is needed. Here, for simplicity, we considered a Cauchy-elastic formulation (not necessarily hyperelastic) for which all appearing tensor quantities can be made explicit and for which their properties are directly determined. In addition, we have shown that there emerges a subproblem of elliptic type that admits a unique solution based on Korn's inequality.

In the follow-up paper [7] we will use the presented exceptional structure to provide a local existence result by applying a parabolic regularisation, using the properties of a differential Sylvester equation together with the positive definiteness of $\mathbb{H}^{\text{ZJ}}(\sigma)$ and passing to the limit together with a Schauder fixed-point argument.

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A Appendix

A.1 Notation

The deformation $\varphi(x, t)$, the material time derivative $\frac{D}{Dt}$ and the partial time derivative ∂_t

In accordance with [44] we agree on the following convention regarding an elastic deformation φ and time derivatives of material quantities:

Given two sets $\Omega, \Omega_\xi \subset \mathbb{R}^3$ we denote by $\varphi : \Omega \rightarrow \Omega_\xi, x \mapsto \varphi(x) = \xi$ the deformation from the *reference configuration* Ω to the *current configuration* Ω_ξ . A *motion* of Ω is a time-dependent family of deformations, written $\xi = \varphi(x, t)$. The *velocity* of the point $x \in \Omega$ is defined by $\bar{V}(x, t) = \partial_t \varphi(x, t)$ and describes a vector emanating from the point $\xi = \varphi(x, t)$ (see also Figure 6). Similarly, the velocity viewed as a function of $\xi \in \Omega_\xi$ is denoted by $v(\xi, t)$.

Considering an arbitrary material quantity $Q(x, t)$ on Ω , equivalently represented by $q(\xi, t)$ on Ω_ξ , we obtain by the chain rule for the time derivative of $Q(x, t)$

$$\frac{D}{Dt} q(\xi, t) := \frac{d}{dt} [Q(x, t)] = D_\xi q(\xi, t) \cdot v + \partial_t q(\xi, t). \quad (\text{A.99})$$

Since it is always possible to view any material quantity $Q(x, t) = q(\xi, t)$ from two different angles, namely by holding x or ξ fixed, we agree to write

- $\dot{q} := \frac{D}{Dt}[q]$ for the material (substantial) derivative of q with respect to t holding x fixed and

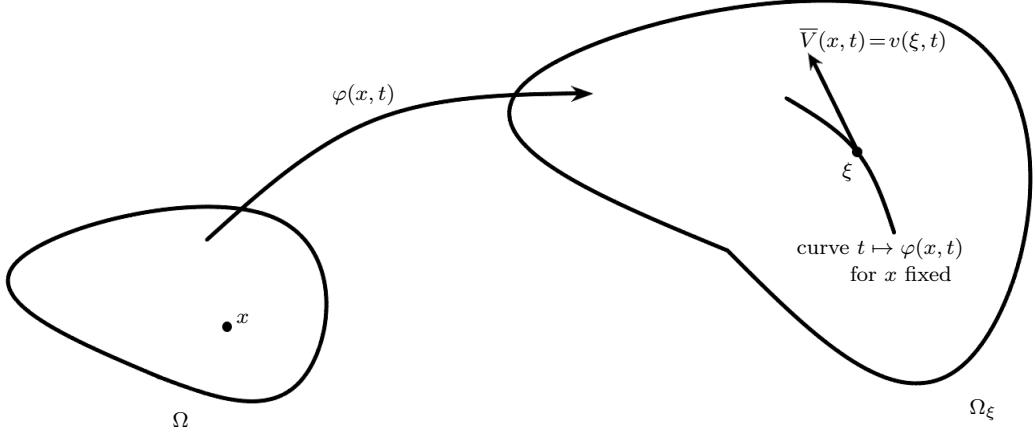


Figure 6: Illustration of the deformation $\varphi(x, t) : \Omega \rightarrow \Omega_\xi$ and the velocity $\bar{V}(x, t) = v(\xi, t)$.

- $\partial_t q$ for the derivative of q with respect to t holding ξ fixed.

For example, we obtain the velocity gradient $L := D_\xi v(\xi, t)$ by

$$\begin{aligned} L = D_\xi v(\xi, t) &= D_\xi \bar{V}(x, t) \stackrel{\text{def}}{=} D_\xi \frac{d}{dt} \varphi(x, t) = D_\xi \partial_t \varphi(\varphi^{-1}(\xi, t), t) = \partial_t D\varphi(\varphi^{-1}(\xi, t), t) D_\xi(\varphi^{-1}(\xi, t)) \\ &= \partial_t D\varphi(\varphi^{-1}(\xi, t), t) (D\varphi)^{-1}(\varphi^{-1}(\xi, t), t) = \dot{F}(x, t) F^{-1}(x, t) = L, \end{aligned} \quad (\text{A.100})$$

where we used that $\partial_t = \frac{d}{dt} = \frac{D}{Dt}$ are all the same, if x is fixed.

As another example, when determining a corotational rate $\frac{D^\circ}{Dt}$ we write

$$\frac{D^\circ}{Dt}[\sigma] = \frac{D}{Dt}[\sigma] + \sigma \Omega^\circ - \Omega^\circ \sigma = \dot{\sigma} + \sigma \Omega^\circ - \Omega^\circ \sigma. \quad (\text{A.101})$$

However, if we solely work on the current configuration, i.e. holding ξ fixed, we write $\partial_t v$ for the time-derivative of the velocity (or any quantity in general).

Inner product

For $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on \mathbb{R}^n with associated vector norm $\|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n}$. We denote by $\mathbb{R}^{n \times n}$ the set of real $n \times n$ second-order tensors, written with capital letters. The standard Euclidean scalar product on $\mathbb{R}^{n \times n}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{n \times n}} = \text{tr}(XY^T)$, where the superscript T is used to denote transposition. Thus the Frobenius tensor norm is $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$, where we usually omit the subscript $\mathbb{R}^{n \times n}$ in writing the Frobenius tensor norm. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$.

Frequently used spaces

- $\text{Sym}(n)$, $\text{Sym}^+(n)$ and $\text{Sym}^{++}(n)$ denote the symmetric, positive semi-definite symmetric and positive definite symmetric second order tensors, respectively. Note that $\text{Sym}^{++}(n)$ is considered herein only as an algebraic subset of $\text{Sym}(n)$, not endowed with a Riemannian geometry [18, 20, 21, 35].
- $\text{GL}(n) := \{X \in \mathbb{R}^{n \times n} \mid \det X \neq 0\}$ denotes the general linear group.
- $\text{GL}^+(n) := \{X \in \mathbb{R}^{n \times n} \mid \det X > 0\}$ is the group of invertible matrices with positive determinant.

- $\mathrm{SL}(n) := \{X \in \mathrm{GL}(n) \mid \det X = 1\}$.
- $\mathrm{O}(n) := \{X \in \mathrm{GL}(n) \mid X^T X = \mathbb{1}\}$.
- $\mathrm{SO}(n) := \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T X = \mathbb{1}, \det X = 1\}$.
- $\mathfrak{so}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid X^T = -X\}$ is the Lie-algebra of skew symmetric tensors.
- $\mathfrak{sl}(3) := \{X \in \mathbb{R}^{3 \times 3} \mid \mathrm{tr}(X) = 0\}$ is the Lie-algebra of traceless tensors.
- The set of positive real numbers is denoted by $\mathbb{R}_+ := (0, \infty)$, while $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$.

Frequently used tensors

- $F = \mathrm{D}\varphi(x, t)$ is the Fréchet derivative (Jacobian) of the deformation $\varphi(\cdot, t) : \Omega \rightarrow \Omega_\xi \subset \mathbb{R}^3$. $\varphi(x, t)$ is usually assumed to be a diffeomorphism at every time $t \geq 0$ so that the inverse mapping $\varphi^{-1}(\cdot, t) : \Omega_\xi \rightarrow \Omega$ exists.
- $C = F^T F$ is the right Cauchy-Green strain tensor.
- $B = F F^T$ is the left Cauchy-Green (or Finger) strain tensor.
- $U = \sqrt{F^T F} \in \mathrm{Sym}^{++}(3)$ is the right stretch tensor, i.e. the unique element of $\mathrm{Sym}^{++}(3)$ with $U^2 = C$.
- $V = \sqrt{F F^T} \in \mathrm{Sym}^{++}(3)$ is the left stretch tensor, i.e. the unique element of $\mathrm{Sym}^{++}(3)$ with $V^2 = B$.
- $\log V = \frac{1}{2} \log B$ is the spatial logarithmic strain tensor or Hencky strain.
- We write $V = Q \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) Q^T$, where $\lambda_i \in \mathbb{R}_+$ are the principal stretches.
- $L = \dot{F} F^{-1} = \mathrm{D}_\xi v(\xi)$ is the spatial velocity gradient.
- $v = \frac{\mathrm{D}}{\mathrm{D}t} \varphi(x, t)$ denotes the Eulerian velocity.
- $D = \mathrm{sym} L$ is the spatial rate of deformation, the Eulerian rate of deformation tensor (or stretching).
- $W = \mathrm{skew} L$ is the vorticity tensor or spin.
- We also have the polar decomposition $F = R U = V R \in \mathrm{GL}^+(3)$ with an orthogonal matrix $R \in \mathrm{O}(3)$ (cf. Neff et al. [57]), see also [39, 58].

Calculus with the material derivative

Consider the spatial Cauchy stress

$$\sigma(\xi, t) := \Sigma(B) = \Sigma(F(x, t) F^T(x, t)) = \Sigma(F(\varphi^{-1}(\xi, t), t) F^T(\varphi^{-1}(\xi, t), t)). \quad (\mathrm{A.102})$$

Then, on the one hand we have for the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} [\sigma] = \mathrm{D}_\xi \sigma(\xi, t) \cdot v(\xi, t) + \partial_t \sigma(\xi, t) \cdot \mathbb{1} \quad (\mathrm{A.103})$$

and on the other hand equivalently

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t} [\sigma] &= \frac{\mathrm{D}}{\mathrm{D}t} [\Sigma(F(x, t) F(x, t)^T)] \stackrel{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}t} [\Sigma(F(x, t) F^T(x, t))] \\ &\stackrel{\text{standard chain rule}}{=} \mathrm{D}_B \Sigma(F(x, t) F^T(x, t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t} [(F(x, t) F^T(x, t))] = \mathrm{D}_B \Sigma(F(x, t) F^T(x, t)) \cdot (\dot{F} F^T + F \dot{F}^T) \\ &= \mathrm{D}_B \Sigma(F(x, t) F^T(x, t)) \cdot [\dot{F} F^{-1} F F^T + F F^T F^{-T} \dot{F}^T] = \mathrm{D}_B \Sigma(F(x, t) F^T(x, t)) \cdot [L B + B L^T]. \end{aligned} \quad (\mathrm{A.104})$$

In (A.104)₁ we have used the fact that there is already a material representation which allows to set $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}$. Of course, (A.103) is equivalent to (A.104). From the context it should be clear which representation of σ

(referential or spatial) we are working with and by abuse of notation we do not distinguish between σ and Σ .

The same must be observed when calculating with corotational derivatives

$$\frac{D^\circ}{Dt}[\sigma] = \frac{D}{Dt}[\sigma] + \sigma \Omega^\circ - \Omega^\circ \sigma, \quad \Omega^\circ = \frac{D}{Dt}Q^\circ(x, t)(Q^\circ)^T(x, t) = \frac{d}{dt}Q^\circ(x, t)(Q^\circ)^T(x, t). \quad (\text{A.105})$$

Here, we have

$$\begin{aligned} \frac{D^\circ}{Dt}[\sigma] &\stackrel{(**)}{=} Q^\circ(x, t) \frac{D}{Dt}[(Q^\circ)^T(x, t) \sigma Q^\circ(x, t)] (Q^\circ)^T(x, t) \\ &= Q^\circ(x, t) \left\{ \frac{D}{Dt}(Q^\circ)^T(x, t) \sigma Q^\circ(x, t) + (Q^\circ)^T(x, t) \frac{D}{Dt}[\sigma] Q^\circ(x, t) + (Q^\circ)^T(x, t) \sigma \frac{D}{Dt}Q^\circ(x, t) \right\} (Q^\circ)^T(x, t) \\ &= Q^\circ(x, t) \left\{ \underbrace{\frac{d}{dt}(Q^\circ)^T(x, t) \sigma Q^\circ(x, t) + (Q^\circ)^T(x, t)}_{(***)} \underbrace{\frac{D}{Dt}[\sigma] Q^\circ(x, t) + (Q^\circ)^T(x, t) \sigma \frac{d}{dt}Q^\circ(x, t)}_{(***)} \right\} (Q^\circ)^T(x, t) \end{aligned} \quad (\text{A.106})$$

and we can decide for $(***)$ to continue the calculus with (A.103) or (A.104). In either case one has to decide viewing the functions as defined on the reference configuration Ω or in the spatial configuration Ω_ξ .

In (A.106) we used $Q = Q(x, t)$ and $\Omega = \Omega(x, t)$. This means that the ‘‘Lie-type’’ representation $(**)$ necessitates the definition of a reference configuration, so that we can switch between $\xi = \varphi(x, t)$ and x .

The interpretation $(**)$ is most clearly represented for the Green-Naghdi rate $\frac{D^{\text{GN}}}{Dt}$, in which the spin $\Omega^{\text{GN}} := \frac{d}{dt}R(x, t)R^T(x, t) = \dot{R}(x, t)R^T(x, t)$ is defined according to the polar decomposition $F = RU$ and in

$$\frac{D^{\text{GN}}}{Dt}[\sigma] = \frac{D}{Dt}[\sigma] + \sigma \Omega^{\text{GN}} - \Omega^{\text{GN}} \sigma = R \frac{D}{Dt}[R^T \sigma R] R^T \quad (\text{A.107})$$

the term $[R^T \sigma R]$ is called *corotational stress tensor* (cf. [44, p. 142]).

Tensor domains

Denoting the reference configuration by Ω with tangential space $T_x\Omega$ and the current/spatial configuration by Ω_ξ with tangential space $T_\xi\Omega_\xi$ as well as $\varphi(x) = \xi$, we have the following relations (see also Figure 7):

- $F: T_x\Omega \rightarrow T_\xi\Omega_\xi$
- $R: T_x\Omega \rightarrow T_\xi\Omega_\xi$
- $F^T: T_\xi\Omega_\xi \rightarrow T_x\Omega$
- $R^T: T_\xi\Omega_\xi \rightarrow T_x\Omega$
- $C = F^T F: T_x\Omega \rightarrow T_x\Omega$
- $B = F F^T: T_\xi\Omega_\xi \rightarrow T_\xi\Omega_\xi$
- $\sigma: T_\xi\Omega_\xi \rightarrow T_\xi\Omega_\xi$
- $\tau: T_\xi\Omega_\xi \rightarrow T_\xi\Omega_\xi$
- $S_2 = F^{-1} S_1: T_x\Omega \rightarrow T_x\Omega$
- $S_1 = D_F W(F): T_x\Omega \rightarrow T_\xi\Omega_\xi$
- $R^T \sigma R: T_x\Omega \rightarrow T_x\Omega$.

The strain energy function $W(F)$

We are only concerned with rotationally symmetric functions $W(F)$ (objective and isotropic), i.e.

$$W(F) = W(Q_1^T F Q_2), \quad \forall F \in \text{GL}^+(3), \quad Q_1, Q_2 \in \text{SO}(3).$$

List of additional definitions and useful identities

- For two metric spaces X, Y and a linear map $L: X \rightarrow Y$ with argument $v \in X$ we write $L.v := L(v)$. This applies to a second-order tensor A and a vector v as $A.v$ as well as to a fourth-order tensor \mathbb{C} and a second-order tensor H as $\mathbb{C}.H$.
- We denote the space of minor and major symmetric, positive definite fourth-order tensors \mathbb{C} by $\text{Sym}_4^{++}(6)$, i.e. $\mathbb{C} \in \text{Sym}_4^{++}(6)$ if and only if $\langle \mathbb{C}.D, D \rangle > 0$ for all $D \in \text{Sym}(3) \setminus \{0\}$.
- By $\text{div}(\cdot)$ we denote the common divergence of vectors, i.e. $\text{div}_\xi v = \sum_{i=1}^3 \partial_{\xi_i} v_i$.

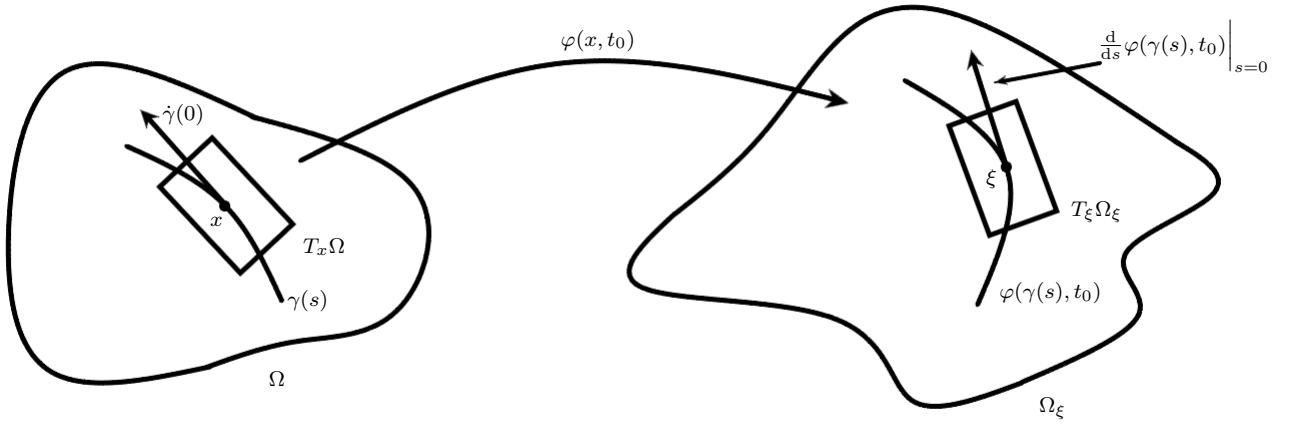


Figure 7: Illustration of the curve $s \mapsto \varphi(\gamma(s), t_0)$, $\gamma(0) = x$ for a fixed time $t = t_0$ with vector field $s \mapsto \frac{d}{ds} \varphi(\gamma(s), t) \in T_\xi \Omega_\xi$.

- By $\text{Div}(\cdot)$ we denote the divergence of matrices, i.e. for $A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^{n \times n}$, where $a_i \in \mathbb{R}^n$ are the rows of A , we have $\text{Div}_\xi A = (\text{div}_\xi a_1, \text{div}_\xi a_2, \dots, \text{div}_\xi a_n)$.
- We define $J = \det F$ and denote by $\text{Cof}(X) = (\det X)X^{-T}$ the cofactor of a matrix in $\text{GL}^+(3)$.
- We define $\text{sym } X = \frac{1}{2}(X + X^T)$ and $\text{skew } X = \frac{1}{2}(X - X^T)$ as well as $\text{dev } X = X - \frac{1}{3}\text{tr}(X) \cdot \mathbf{1}\mathbf{1}^T$.
- For all vectors $\xi, \eta \in \mathbb{R}^3$ we have the tensor or dyadic product $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$.
- $S_1 = D_F W(F) = \sigma \cdot \text{Cof } F$ is the non-symmetric first Piola-Kirchhoff stress tensor.
- $S_2 = F^{-1} S_1 = 2 D_C \widetilde{W}(C)$ is the symmetric second Piola-Kirchhoff stress tensor.
- $\sigma = \frac{1}{J} S_1 F^T = \frac{1}{J} F S_2 F^T = \frac{2}{J} D_B \widetilde{W}(B) B = \frac{1}{J} D_V \widetilde{W}(V) V = \frac{1}{J} D_{\log V} \widehat{W}(\log V)$ is the symmetric Cauchy stress tensor.
- $\sigma = \frac{1}{J} F S_2 F^T = \frac{2}{J} F D_C \widetilde{W}(C) F^T$ is the "Doyle-Ericksen formula" [14].
- For $\sigma : \text{Sym}(3) \rightarrow \text{Sym}(3)$ we denote by $D_B \sigma(B)$ with $\sigma(B + H) = \sigma(B) + D_B \sigma(B) \cdot H + o(H)$ the Fréchet-derivative. For $\sigma : \text{Sym}^+(3) \subset \text{Sym}(3) \rightarrow \text{Sym}(3)$ the same applies. Similarly, for $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ we have $W(X + H) = W(X) + \langle D_X W(X), H \rangle + o(H)$.
- $\tau = J \sigma = 2 D_B \widetilde{W}(B) B$ is the symmetric Kirchhoff stress tensor.
- $\tau = D_{\log V} \widehat{W}(\log V)$ is the "Richter-formula" [65, 66].
- $\sigma_i = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \lambda_i \frac{\partial g(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i} = \frac{1}{\lambda_j \lambda_k} \frac{\partial g(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i}$, $i \neq j \neq k \neq i$ are the principal Cauchy stresses (the eigenvalues of the Cauchy stress tensor σ), where $g : \mathbb{R}_+^3 \rightarrow \mathbb{R}$ is the unique function of the singular values of U (the principal stretches) such that $W(F) = \widetilde{W}(U) = g(\lambda_1, \lambda_2, \lambda_3)$.
- $\sigma_i = \frac{1}{\lambda_1 \lambda_2 \lambda_3} \frac{\partial \widehat{g}(\log \lambda_1, \log \lambda_2, \log \lambda_3)}{\partial \log \lambda_i}$, where $\widehat{g} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the unique function such that $\widehat{g}(\log \lambda_1, \log \lambda_2, \log \lambda_3) := g(\lambda_1, \lambda_2, \lambda_3)$.

$$\bullet \quad \tau_i = J \sigma_i = \lambda_i \frac{\partial g(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i} = \frac{\partial \widehat{g}(\log \lambda_1, \log \lambda_2, \log \lambda_3)}{\partial \log \lambda_i} \quad .$$

Conventions for fourth-order symmetric operators, minor and major symmetry

For a fourth-order linear mapping $\mathbb{C}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$, we agree on the following convention.

We say that \mathbb{C} has *minor symmetry* if

$$\mathbb{C}.S \in \text{Sym}(3) \quad \forall S \in \text{Sym}(3). \quad (\text{A.108})$$

This can also be written in index notation as $\mathbb{C}_{ijkm} = \mathbb{C}_{jikm} = \mathbb{C}_{ijmk}$.

We say that $\mathbb{C}: \text{Sym}(3) \rightarrow \text{Sym}(3)$ has *major symmetry* (or is *self-adjoint*) if

$$\langle \mathbb{C}.T, S \rangle = \langle \mathbb{C}.S, T \rangle \quad \forall T, S \in \text{Sym}(3). \quad (\text{A.109})$$

Major symmetry in index notation is understood as $C_{ijkm} = C_{kmij}$. For $\mathbb{C}: \text{Sym}(3) \rightarrow \text{Sym}(3)$, we define the adjoint operator $\mathbb{C}^T: \text{Sym}(3) \rightarrow \text{Sym}(3)$ via $\langle \mathbb{C}^T.T, S \rangle = \langle \mathbb{C}.S, T \rangle \forall S, T \in \text{Sym}(3)$. Then $\text{sym } \mathbb{C} := \frac{1}{2}(\mathbb{C}^T + \mathbb{C})$ denotes the major-symmetric part of \mathbb{C} . The set of major (and minor) symmetric fourth-order tensor mappings $\text{Sym}(3) \rightarrow \text{Sym}(3)$ which are *positive-definite* and *positive-semidefinite* are denoted by $\text{Sym}_4^{++}(6)$ and $\text{Sym}_4^+(6)$, respectively, i.e.

$$\mathbb{C} \in \text{Sym}_4^{++}(6) \iff \langle \mathbb{C}.H, H \rangle > 0 \quad \forall H \in \text{Sym}(3), \quad (\text{A.110})$$

$$\mathbb{C} \in \text{Sym}_4^+(6) \iff \langle \mathbb{C}.H, H \rangle \geq 0 \quad \forall H \in \text{Sym}(3). \quad (\text{A.111})$$

By identifying $\text{Sym}(3) \cong \mathbb{R}^6$, we can interpret $\mathbb{C}: \text{Sym}(3) \rightarrow \text{Sym}(3)$ as a linear mapping $\tilde{\mathbb{C}}: \mathbb{R}^6 \rightarrow \mathbb{R}^6$. More specifically, for $H \in \text{Sym}(3) \cong \mathbb{R}^6$ we can write, using the Mandel-notation,

$$h := \text{vec}(H) = (H_{11}, H_{22}, H_{33}, \sqrt{2}H_{12}, \sqrt{2}H_{23}, \sqrt{2}H_{31}) \in \mathbb{R}^6 \quad (\text{A.112})$$

so that with $\tilde{\mathbb{C}}.\text{vec}(H) := \text{vec}(\mathbb{C}.H)$ we have $\langle \mathbb{C}.H, H \rangle_{\text{Sym}(3)} = \langle \tilde{\mathbb{C}}.h, h \rangle_{\mathbb{R}^6}$. Then for any fourth-order tensor $\mathbb{C}: \text{Sym}(3) \rightarrow \text{Sym}(3)$, we can also define $\text{sym } \mathbb{C}$ by $(\text{sym } \mathbb{C}).H = \text{vec}^{-1}((\text{sym } \tilde{\mathbb{C}}).\text{vec}(H))$, implying

$$\langle \mathbb{C}.H, H \rangle_{\text{Sym}(3)} = \langle \tilde{\mathbb{C}}.h, h \rangle_{\mathbb{R}^6} = \langle (\text{sym } \tilde{\mathbb{C}}).h, h \rangle_{\mathbb{R}^6} = \langle \text{sym } \mathbb{C}.H, H \rangle_{\text{Sym}(3)} \quad \forall H \in \text{Sym}(3). \quad (\text{A.113})$$

Major symmetry in these terms can be expressed as $\tilde{\mathbb{C}} \in \text{Sym}(6)$. However, we omit the tilde-operation and sym and write in short $\text{sym } \mathbb{C} \in \text{Sym}_4(6)$ if no confusion can arise. In the same manner we speak about $\det \mathbb{C}$ meaning $\det \tilde{\mathbb{C}}$.

A.2 Derivation of the induced fourth-order tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$

For the derivation of the induced tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$ for the Cauchy stress-strain law

$$\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbb{1} \quad (\text{A.114})$$

we begin by splitting $\sigma(B)$ into three parts, namely

$$\sigma = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \text{tr}(\log B) \mathbb{1} = \frac{1}{2} \left\{ \underbrace{\mu(B - \mathbb{1})}_{=: \sigma_1} + \underbrace{\mu(\mathbb{1} - B^{-1})}_{=: \sigma_2} + \underbrace{\lambda \text{tr}(\log B) \mathbb{1}}_{=: \sigma_3} \right\}. \quad (\text{A.115})$$

Next, we calculate the Zaremba-Jaumann derivative, given by

$$\frac{D^{\text{ZJ}}}{Dt}[\sigma] = \frac{D}{Dt}[\sigma] + \sigma W - W \sigma \quad (\text{A.116})$$

for each of the three components. To do so, we use the easily verifiable identity $\frac{D}{Dt}[B] = LB + BL^T$, yielding for σ_1

$$\begin{aligned} \frac{D}{Dt}[\sigma_1] &= \mu \frac{D}{Dt}[B - \mathbb{1}] = \mu(LB + BL^T) = \mu(L(B - \mathbb{1}) + (B - \mathbb{1})L^T + L + L^T) \\ &= (L\sigma_1 + \sigma_1 L^T) + 2\mu D = (D + W)\sigma_1 + \sigma_1(D + W)^T + 2\mu D \\ &= D\sigma_1 + \sigma_1 D + 2\mu D + W\sigma_1 - \sigma_1 W \\ \iff \frac{D^{\text{ZJ}}}{Dt}[\sigma_1] &= \frac{D}{Dt}[\sigma_1] + \sigma_1 W - W\sigma_1 = 2\mu D + (\sigma_1 D + D\sigma_1). \end{aligned} \quad (\text{A.117})$$

Recalling the identities $\frac{D}{Dt}[B^{-1}] = -B^{-1}\dot{B}B^{-1}$ and $B^{-1} = \mathbb{1} - \frac{1}{\mu}\sigma_2$, we obtain for σ_2

$$\begin{aligned} \frac{D}{Dt}[\sigma_2] &= \frac{D}{Dt}[\mu(\mathbb{1} - B^{-1})] = -\mu \frac{D}{Dt}[B^{-1}] = \mu B^{-1}\dot{B}B^{-1} \\ &= \mu B^{-1}(LB + BL^T)B^{-1} = \mu(B^{-1}L + L^T B^{-1}) \\ &= \mu(B^{-1}W + B^{-1}D - WB^{-1} + DB^{-1}) \\ &= \mu\left(\left(\mathbb{1} - \frac{1}{\mu}\sigma_2\right)D + D\left(\mathbb{1} - \frac{1}{\mu}\sigma_2\right) + \left(\mathbb{1} - \frac{1}{\mu}\sigma_2\right)W - W\left(\mathbb{1} - \frac{1}{\mu}\sigma_2\right)\right) \\ &= 2\mu D - (\sigma_2 D + D\sigma_2) + W\sigma_2 - \sigma_2 W \\ \implies \frac{D^{\text{ZJ}}}{Dt}[\sigma_2] &= 2\mu D - (\sigma_2 D + D\sigma_2). \end{aligned} \quad (\text{A.118})$$

Also, since $\text{tr}(\log B) \cdot \mathbb{1} = \log(\det B) \cdot \mathbb{1}$ (cf. Neff et al. [50, 51]) and $\sigma_3 W - W\sigma_3 = 0$, since $\sigma_3 = g(t) \cdot \mathbb{1}$, $g(t) \in \mathbb{R}$, we have for σ_3

$$\begin{aligned} \frac{D}{Dt}[\log(\det B) \cdot \mathbb{1}] &= \frac{1}{\det B} \langle \text{Cof } B, \dot{B} \rangle \mathbb{1} = \frac{1}{\det B} \cdot \det B \cdot \langle B^{-1}, \dot{B} \rangle \cdot \mathbb{1} \\ &= \langle B^{-1}, LB + BL^T \rangle \cdot \mathbb{1} = (\langle \mathbb{1}, L \rangle + \langle \mathbb{1}, L^T \rangle) \cdot \mathbb{1} = 2\text{tr}(D) \cdot \mathbb{1} \\ \implies \frac{D^{\text{ZJ}}}{Dt}[\sigma_3] &= 2\lambda \text{tr}(D) \cdot \mathbb{1}. \end{aligned} \quad (\text{A.119})$$

Combining linearly these three expressions for the rates now yields for the Cauchy stress σ

$$\begin{aligned} \frac{D^{\text{ZJ}}}{Dt}[\sigma] &= \frac{1}{2} \frac{D^{\text{ZJ}}}{Dt}[\sigma_1 + \sigma_2 + \sigma_3] = 2\mu D + \frac{1}{2}\{(\sigma_1 - \sigma_2)D + D(\sigma_1 - \sigma_2)\} + \lambda \text{tr}(D) \cdot \mathbb{1} \\ &= 2\mu D + \frac{\mu}{2}\{(B + B^{-1})D + D(B + B^{-1}) - 4D\} + \lambda \text{tr}(D) \cdot \mathbb{1} \\ &= \frac{\mu}{2}\{BD + DB + B^{-1}D + DB^{-1}\} + \lambda \text{tr}(D) \cdot \mathbb{1} =: \mathbb{H}^{\text{ZJ}}(B).D = \mathbb{H}^{\text{ZJ}}(\sigma).D, \end{aligned} \quad (\text{A.120})$$

where the last equality holds since $\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3)$, $B \mapsto \sigma(B)$ is invertible.

A.3 Salient properties of $\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2}(\log \det B) \cdot \mathbb{1}$

The principal Cauchy stresses σ_i are the eigenvalues of σ . Since σ is coaxial to B (cf. Thiel et al [73]) we have

$$\sigma(Q^T B Q) = Q^T \sigma(B) Q = \text{diag}(\sigma_1, \sigma_2, \sigma_3) = \sigma(\text{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)). \quad (\text{A.121})$$

Therefore, we may simply insert $B = \text{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ into the constitutive law

$$\sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbb{1} \quad (\text{A.122})$$

and calculate the diagonal entries of $\sigma(B)$, yielding

$$\sigma_i = \frac{\mu}{2}(\lambda_i^2 - \lambda_i^{-2}) + \lambda \log(\lambda_1 \lambda_2 \lambda_3), \quad (\text{A.123})$$

where λ_i^2 , $i = 1, 2, 3$ are the eigenvalues of B .

A.3.1 Tension-extension inequality

The principal Cauchy stresses σ_i fulfill the tension-extension inequality $\frac{\partial \sigma_i}{\partial \lambda_i} > 0$. Indeed, for λ_k, λ_j fixed, $k, j \neq i$,

$$\frac{\partial \sigma_i}{\partial \lambda_i} = \mu \left(\lambda_i + \frac{1}{\lambda_i^3} \right) + \frac{\lambda}{\lambda_i} \stackrel{\lambda_i > 0}{>} 0, \quad \mu, \lambda > 0. \quad (\text{A.124})$$

A.3.2 Baker-Ericksen inequality

The principal Cauchy stresses σ_i fulfill the Baker-Ericksen inequality $(\sigma_i - \sigma_j)(\lambda_i - \lambda_j) > 0$:

$$\begin{aligned} \sigma_i - \sigma_j &= \frac{\mu}{2} \left(\lambda_i^2 - \lambda_j^2 - \left(\frac{1}{\lambda_i^2} - \frac{1}{\lambda_j^2} \right) \right) = \frac{\mu}{2} \left\{ \lambda_i + \lambda_j + \frac{\lambda_i + \lambda_j}{\lambda_i^2 \lambda_j^2} \right\} (\lambda_i - \lambda_j) \quad \text{for } i \neq j \\ \implies (\sigma_i - \sigma_j)(\lambda_i - \lambda_j) &= \frac{\mu}{2} \left\{ \lambda_i + \lambda_j + \frac{\lambda_i + \lambda_j}{\lambda_i^2 \lambda_j^2} \right\} (\lambda_i - \lambda_j)^2 > 0. \end{aligned} \quad (\text{A.125})$$

A.3.3 Monotonicity of pressure: pressure-compression inequality

The constitutive law (A.122) fulfills the monotonicity of pressure, i.e. $\langle \sigma(\alpha \mathbb{1}) - \sigma(\beta \mathbb{1}), \alpha \mathbb{1} - \beta \mathbb{1} \rangle > 0$ for $\alpha, \beta > 0, \alpha \neq \beta$:

$$\begin{aligned} \langle \sigma(\alpha \mathbb{1}) - \sigma(\beta \mathbb{1}), \alpha \mathbb{1} - \beta \mathbb{1} \rangle &= \left\langle \left\{ \frac{\mu}{2} \left(\alpha - \frac{1}{\alpha} \right) \mathbb{1} + \frac{\lambda}{2} \cdot 3 \log \alpha \cdot \mathbb{1} \right\} - \left\{ \frac{\mu}{2} \left(\beta - \frac{1}{\beta} \right) \mathbb{1} + \frac{\lambda}{2} \cdot 3 \log \beta \cdot \mathbb{1} \right\}, \alpha \mathbb{1} - \beta \mathbb{1} \right\rangle \\ &= \left[\frac{\mu}{2} \cdot \left\{ \alpha - \beta + \frac{\alpha - \beta}{\alpha \beta} \right\} + \frac{3\lambda}{2} \cdot (\log \alpha - \log \beta) \right] \cdot (\alpha - \beta) \langle \mathbb{1}, \mathbb{1} \rangle \\ &= 3 \frac{\mu}{2} \cdot \left\{ (\alpha - \beta)^2 + \frac{(\alpha - \beta)^2}{\alpha \beta} \right\} + 3 \frac{3\lambda}{2} \cdot (\alpha - \beta) \cdot (\log \alpha - \log \beta) > 0. \end{aligned} \quad (\text{A.126})$$

The latter implies the pressure-compression inequality $\frac{1}{3} \text{tr}(\sigma(\alpha \mathbb{1})) \cdot (\alpha - 1) > 0$, by setting $\beta = 1$.

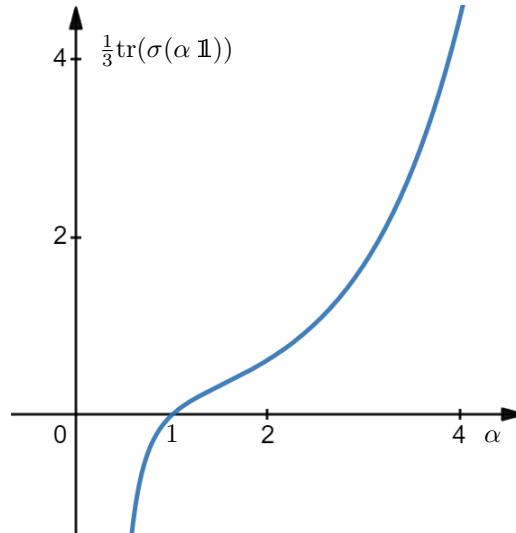


Figure 8: Monotone volume-mean pressure $\alpha \mapsto \frac{1}{3} \text{tr}(\sigma(\alpha \mathbb{1}))$. It is physically reasonable to expect that the mean pressure is an increasing function of the volumetric stretch for a stable material.

A.3.4 Smooth invertibility of $B \mapsto \sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbf{1}$

In an upcoming paper by Martin et al. [45] it will be proven that

$$\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3), \quad B \mapsto \sigma(B) = \frac{\mu}{2}(B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbf{1} \quad (\text{A.127})$$

is smoothly invertible. Hence, there exists an inverse function $\mathcal{F}^{-1} : \text{Sym}(3) \rightarrow \text{Sym}^{++}(3)$, $\sigma \mapsto \mathcal{F}^{-1}(\sigma)$.

A.3.5 Positive definiteness

The induced fourth-order tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$ is positive definite for $\mu > 0, 3\lambda + 2\mu > 0$: Since we have

$$\langle \mathbb{H}^{\text{ZJ}}(\sigma).D, D \rangle = \langle \mathbb{H}^{\text{ZJ}}(\mathcal{F}^{-1}(\sigma)).D, D \rangle = \langle \mathbb{H}^{\text{ZJ}}(B).D, D \rangle \quad \text{for} \quad B = \mathcal{F}^{-1}(\sigma) \in \text{Sym}^{++}(3), \quad (\text{A.128})$$

it follows (cf. (2.81))

$$\begin{aligned} \langle \mathbb{H}^{\text{ZJ}}(B).D, D \rangle &= \left\langle \frac{\mu}{2}(DB + BD + B^{-1}(DB + BD)B^{-1}), D \right\rangle + \frac{\lambda}{2} \langle B^{-1}, DB + BD \rangle \cdot \langle \mathbf{1}, D \rangle \\ &= \frac{\mu}{2}(2\langle BD, D \rangle + 2\langle B^{-1}D, D \rangle) + \frac{\lambda}{2} \langle 2\text{tr}(D) \cdot \mathbf{1}, D \rangle \\ &= \mu(\langle BD, D \rangle + \langle B^{-1}D, D \rangle) + \lambda \text{tr}^2(D) = \mu \langle (B + B^{-1})D, D \rangle + \lambda \text{tr}^2(D) \\ &\geq \underbrace{\mu \lambda_{\min}(B + B^{-1})}_{\geq 2, \text{ see footnote}^3} \cdot \|D\|^2 + \lambda \text{tr}^2(D) \geq \underbrace{2\mu \|D\|^2 + \lambda \text{tr}^2(D)}_{> 0 \text{ for } \mu > 0, 2\mu + 3\lambda > 0} \geq c^+(\mu, \lambda) \cdot \|D\|^2. \end{aligned} \quad (\text{A.129})$$

A.3.6 Minor and major symmetry

Minor symmetry of the induced fourth-order tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$, i.e.

$$D \in \text{Sym}(3) \mapsto \mathbb{H}^{\text{ZJ}}(\sigma).D \in \text{Sym}(3), \quad (\text{A.131})$$

follows directly by using that $B = \mathcal{F}^{-1}(\sigma) \in \text{Sym}^{++}(3)$, B^{-1} and D are symmetric.

For major symmetry of $\mathbb{H}^{\text{ZJ}}(\sigma)$, i.e.

$$\langle \mathbb{H}^{\text{ZJ}}(\sigma).D_1, D_2 \rangle = \langle \mathbb{H}^{\text{ZJ}}(\sigma).D_2, D_1 \rangle, \quad (\text{A.132})$$

we calculate

$$\begin{aligned} \langle \mathbb{H}^{\text{ZJ}}(\sigma).D_1, D_2 \rangle &= \frac{\mu}{2}(\langle BD_1, D_2 \rangle + \langle D_1B, D_2 \rangle + \langle D_1B^{-1}, D_2 \rangle + \langle B^{-1}D_1, D_2 \rangle) + \lambda \text{tr}(D_1) \cdot \text{tr}(D_2) \\ &= \frac{\mu}{2}(\langle D_1, BD_2 \rangle + \langle D_1, D_2B \rangle + \langle D_1, D_2B^{-1} \rangle + \langle D_1, B^{-1}D_2 \rangle) + \lambda \text{tr}(D_2) \cdot \text{tr}(D_1) \\ &= \frac{\mu}{2}(\langle BD_2 + D_2B + B^{-1}D_2 + D_2B^{-1}, D_1 \rangle) + \lambda \text{tr}(D_2) \cdot \text{tr}(D_1) = \langle \mathbb{H}^{\text{ZJ}}(\sigma).D_2, D_1 \rangle. \end{aligned} \quad (\text{A.133})$$

⁴It is

$$\begin{aligned} \langle \xi, (B + B^{-1})\xi \rangle &= \langle \xi, (Q^T \text{diag} Q + Q^T \text{diag}^{-1} Q)\xi \rangle = \langle Q\xi, (\text{diag} + \text{diag}^{-1})Q\xi \rangle \\ &= \left\langle \eta, \text{diag} \left(\lambda_1 + \frac{1}{\lambda_1}, \lambda_2 + \frac{1}{\lambda_2}, \lambda_3 + \frac{1}{\lambda_3} \right) \eta \right\rangle = \sum_{i=1}^3 \eta_i^2 \underbrace{\left(\lambda_i + \frac{1}{\lambda_i} \right)}_{\geq 2} \geq 2\|\eta\|^2 = 2\|\xi\|^2. \end{aligned} \quad (\text{A.130})$$

Example A.1. Consider

$$\sigma(B) = \frac{B}{(\det B)^{\frac{1}{3}}} - \mathbb{1} = (\det B)^{-\frac{1}{3}} \cdot B - \mathbb{1}, \quad \sigma(\mathbb{1}) = 0. \quad (\text{A.134})$$

Hence

$$\begin{aligned} D_B \sigma(B) \cdot H &= -\frac{1}{3} (\det B)^{-\frac{4}{3}} \cdot \langle \text{Cof } B, H \rangle \cdot B + (\det B)^{-\frac{1}{3}} \cdot H \\ &= -\frac{1}{3} (\det B)^{-\frac{4}{3}} \cdot \det B \cdot \langle B^{-1}, H \rangle \cdot B + (\det B)^{-\frac{1}{3}} \cdot H \\ &= -\frac{1}{3} (\det B)^{-\frac{1}{3}} \cdot \langle B^{-1}, H \rangle \cdot B + (\det B)^{-\frac{1}{3}} \cdot H \\ \implies \mathbb{H}^{\text{ZJ}}(\sigma) \cdot D &:= D_B \sigma(B) \cdot [B D + D B] \\ &= -\frac{1}{3} (\det B)^{-\frac{1}{3}} \cdot \langle B^{-1}, B D + D B \rangle B + (\det B)^{-\frac{1}{3}} (B D + D B) \\ &= -\frac{1}{3} (\det B)^{-\frac{1}{3}} \cdot 2 \text{tr}(D) \cdot B + (\det B)^{-\frac{1}{3}} (B D + D B) \\ \implies \langle \mathbb{H}^{\text{ZJ}}(\sigma) \cdot D_1, D_2 \rangle &= -\frac{1}{3} (\det B)^{-\frac{1}{3}} \cdot 2 \underbrace{\text{tr}(D_1) \langle B, D_2 \rangle}_{\text{not interchangeable}} + (\det B)^{-\frac{1}{3}} \cdot \underbrace{\langle B D_1 + D_1 B, D_2 \rangle}_{\text{interchangeable}}. \end{aligned} \quad (\text{A.135})$$

So, this example shows that $\mathbb{H}^{\text{ZJ}}(\sigma)$ is in general not major symmetric (not self-adjoint).

A.3.7 Monotonicity of $\widehat{\sigma}(\log B)$ in $\log B$

It is proven in [53], that even for a larger, more general class of corotational derivatives, the positive definiteness of a minor symmetric fourth-order tangent stiffness tensor $\mathbb{H}^{\text{ZJ}}(\sigma)$ implies the strict monotonicity of $\widehat{\sigma}(\log B)$ in $\log B$, i.e. for all $B_1, B_2 \in \text{Sym}^{++}(3)$ with $B_1 \neq B_2$ we have

$$\text{sym } \mathbb{H}^{\text{ZJ}}(\sigma) \in \text{Sym}_4^{++}(6) \implies \langle \widehat{\sigma}(\log B_1) - \widehat{\sigma}(\log B_2), \log B_1 - \log B_2 \rangle > 0. \quad (\text{A.136})$$

However, for the reader's convenience, we will prove strict monotonicity of $\widehat{\sigma}(\log B)$ in $\log B$ by direct inspection for our constitutive law. For simplicity, let us only consider the case $\mu, \lambda > 0$ (auxetic-like response is excluded, since we assumed Poisson ratio $\nu > 0$).

The term $B - \mathbb{1}$ is monotone in $\log B$, because

$$\begin{aligned} \langle \widehat{\sigma}(\log B_1) - \widehat{\sigma}(\log B_2), \log B_1 - \log B_2 \rangle &= \langle \sigma(B_1) - \sigma(B_2), \log B_1 - \log B_2 \rangle \\ &= \mu \langle B_1 - \mathbb{1} - (B_2 - \mathbb{1}), \log B_1 - \log B_2 \rangle \\ &= \mu \langle B_1 - B_2, \log B_1 - \log B_2 \rangle \\ &= \mu \langle \log B_1 - \log B_2, B_1 - B_2 \rangle > 0, \end{aligned} \quad (\text{A.137})$$

since $B \mapsto \log B$ is a monotone matrix function (cf. the upcoming paper by Martin et al. [45]). Furthermore, $\text{tr}(\log B) \cdot \mathbb{1}$ is monotone in $\log B$, since it is linear.

It remains to check the term $\sigma = \frac{1}{\mu} \sigma_2 = \mathbb{1} - B^{-1}$. We write

$$\begin{aligned} \langle \sigma(B_1) - \sigma(B_2), \log B_1 - \log B_2 \rangle &= \langle \mathbb{1} - B_1^{-1} - (\mathbb{1} - B_2^{-1}), \log B_1 - \log B_2 \rangle \\ &= \langle -B_1^{-1} - (-B_2^{-1}), \log B_1 - \log B_2 \rangle \\ &= \langle -B_1^{-1} + B_2^{-1}, -\log B_1^{-1} - (-\log B_2^{-1}) \rangle \\ &= \langle B_2^{-1} - B_1^{-1}, \log B_2^{-1} - \log B_1^{-1} \rangle > 0 \\ &= \langle \log X - \log Y, X - Y \rangle > 0, \quad X = B_2^{-1}, \quad Y = B_1^{-1}, \end{aligned} \quad (\text{A.138})$$

since $\log(\cdot)$ is monotone in its argument.

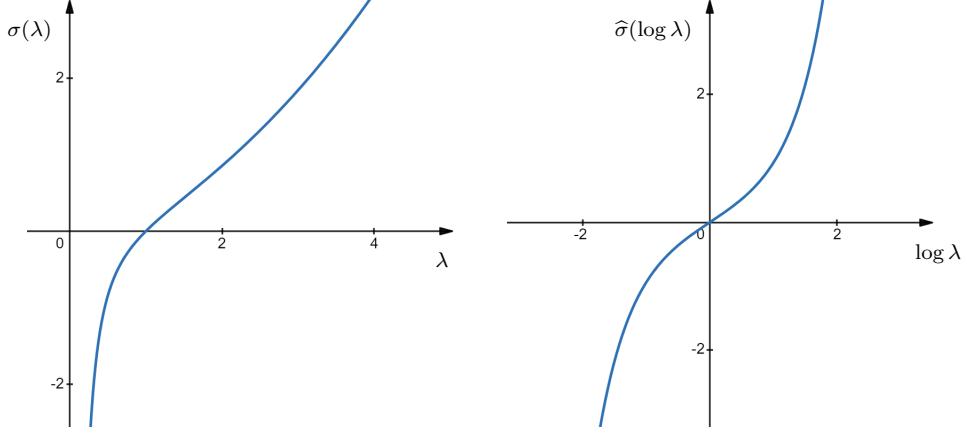


Figure 9: We consider $\sigma : \mathbb{R}^+ \rightarrow \mathbb{R}$ and set $\lambda \mapsto \sigma(\lambda) := \widehat{\sigma}(\log \lambda)$. Depicted is the graph of the one-dimensional forces $\sigma(\lambda) = \frac{1}{6} (\lambda^2 - \frac{1}{\lambda^2} + \log \lambda^2)$ (left) and $\widehat{\sigma}(\log \lambda) = \frac{1}{6} (2 \sinh(2 \log \lambda) + 2 \log \lambda)$ (right), modelled after our Neo-Hooke type law $\sigma(B) = \frac{\mu}{2} (B - B^{-1}) + \frac{\lambda}{2} (\log \det B) \cdot \mathbf{1}$. Here, $\lambda \mapsto \sigma(\lambda)$ and $\log \lambda \mapsto \widehat{\sigma}(\log \lambda)$ are both monotone. In the multi-dimensional setting, $B \mapsto \sigma(B)$ is not monotone, while $\log B \mapsto \widehat{\sigma}(\log B)$ is monotone.

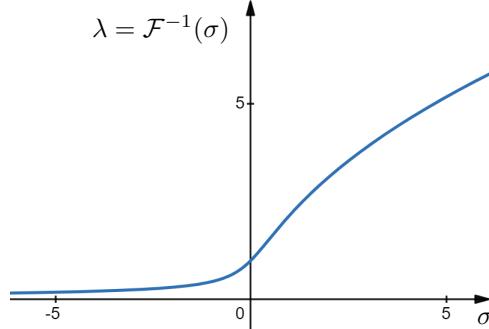


Figure 10: It can be seen that $\mathcal{F}^{-1} : \mathbb{R} \rightarrow \mathbb{R}^+$, $\lambda = \mathcal{F}^{-1}(\sigma)$ is the smooth inverse mapping. The same is true for $\mathcal{F}^{-1} : \text{Sym}(3) \rightarrow \text{Sym}^{++}(3)$ for the constitutive law $\sigma(B) = \frac{\mu}{2} (B - B^{-1}) + \frac{\lambda}{2} (\log \det B) \cdot \mathbf{1}$.

A.4 Further observations for the constitutive law and the rate-formulation

A.4.1 Monotonicity in one dimension

As we have shown in Section A.3.7, the constitutive law

$$\sigma : \text{Sym}^{++}(3) \rightarrow \text{Sym}(3), \quad B \mapsto \sigma(B) = \frac{\mu}{2} (B - B^{-1}) + \frac{\lambda}{2} \log \det B \cdot \mathbf{1} \quad (\text{A.139})$$

fulfills monotonicity of $\log B \mapsto \widehat{\sigma}(\log B) = \sigma(B)$ in $\log B$. Note carefully that in three dimensions this is not equivalent to monotonicity of $B \mapsto \sigma(B)$ in B , as was discussed in Section 1.2 (cf. [53]). However, in one dimension, both types of monotonicity are equivalent, as illustrated by the Figures 9 and 10.

A.4.2 Geometrical meaning of the Eulerian rate of deformation D

As a recapitulation for the reader we provide the following identities

$$\begin{aligned}
\delta(t) &= \|\varphi(x_1, t) - \varphi(x_2, t)\|, \\
\frac{1}{2}\delta^2(t) &= \frac{1}{2}\langle\varphi(x_1, t) - \varphi(x_2, t), \varphi(x_1, t) - \varphi(x_2, t)\rangle, \\
\frac{d}{dt}\frac{1}{2}\delta^2(t) &= \frac{d}{dt}\frac{1}{2}\langle\varphi(x_1, t) - \varphi(x_2, t), \varphi(x_1, t) - \varphi(x_2, t)\rangle = \langle\dot{\varphi}(x_1, t) - \dot{\varphi}(x_2, t), \varphi(x_1, t) - \varphi(x_2, t)\rangle, \quad (\text{A.140}) \\
\delta(t) \cdot \dot{\delta}(t) &= \langle v(\xi_1, t) - v(\xi_2, t), \xi_1 - \xi_2 \rangle \approx \langle D_\xi v(\xi_1) \cdot \delta\xi, \delta\xi \rangle, \quad \text{if } \|\xi_1 - \xi_2\| \ll 1, \quad \xi_2 = \xi_1 + \delta\xi, \\
\dot{\delta}(t) &\cong \frac{1}{\delta(t)} \cdot \langle [\text{sym } D_\xi v(\xi)] \cdot \delta\xi, \delta\xi \rangle, \quad D = \text{sym } D_\xi v(\xi) = \mathbb{S}(\sigma) \cdot \frac{D^{\text{ZJ}}}{Dt}[\sigma(\xi, t)].
\end{aligned}$$

Thus we observe that D describes the change of stretch per unit length, hence the name “stretching” for $D = \text{sym } D_\xi v(\xi)$. Here, $\mathbb{S}(\sigma) = [\mathbb{H}^{\text{ZJ}}(\sigma)]^{-1}$ is the induced fourth-order compliance tensor in the rate-formulation.