Mechanics of Materials 213 (2026) 105561

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mecmat

Research paper

Global existence and uniqueness of weak solutions for a Willis-type model of elastodynamics[☆]

Thomas Blesgen[a,*,1], Patrizio Neff[b,1]

[a] Bingen University of Applied Sciences, Berlinstraße 109, D-55411 Bingen, Germany
[b] Faculty of Mathematics, University of Duisburg–Essen, Thea-Leymann-Straße 9, D-45127 Essen, Germany

ARTICLE INFO

Keywords:
Willis model
Willis coupling
Metamaterials
Existence of weak solutions

ABSTRACT

The existence and uniqueness of weak solutions is shown for a system related to the Willis model of elastodynamics. Both the whole space case and the case of a bounded smooth domain are studied. To this end the equations are reformulated as a linear symmetric hyperbolic system of first order and the existing theory for such systems is applied. If the initial and boundary data is regular enough, classical solutions are obtained. The possibility to transform the problem to a linear symmetric hyperbolic system hinges on a new symmetry condition on the Willis coupling tensor S, not yet considered in the literature. This condition demands that S is a totally symmetric third-order tensor.

1. The Willis model in elastomechanics

The Willis model, see Willis (1981, 2009, 2011) and Milton and Willis (2007), is an extension of classical elastodynamics with the aim to better reproduce wave propagation in metamaterials. The latter topic is of high current interest, see, e.g., Rizzi et al. (2024), Gattin et al. (2025). In fact, metamaterials (or architected materials) show uncommon dispersion relations that are impossible to predict with classical linear elastodynamics. In the Willis type models, the balance of linear momentum is modified, together with a coupling equation. The system reads

$$\begin{aligned} \operatorname{Div}\sigma &= \dot\mu, \\ \sigma &= \mathbb{C}_{\text{eff}}.\operatorname{sym} Du + S_{\text{eff}}\,\dot u, \\ \mu &= S^T_{\text{eff}}.\operatorname{sym} Du + \varrho_{\text{eff}}\,\dot u. \end{aligned} \tag{1}$$

Moreover, $\mathbb{C}_{\text{eff}} : \operatorname{Sym}(3) \to \operatorname{Sym}(3)$, $S_{\text{eff}} : \mathbb{R}^3 \to \operatorname{Sym}(3) \subset \mathbb{R}^{3\times 3}$, $S^T_{\text{eff}} : \operatorname{Sym}(3) \to \mathbb{R}^3$ have a formal character, but should be determined by some 'homogenization' procedure. For $S_{\text{eff}} = 0$, we have just $\dot\mu = \varrho_{\text{eff}}\,\ddot u$. The (symmetric) Cauchy stress tensor is denoted by σ, the displacement is $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and $\mu : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ is the linear momentum density.

In index notation, the system (1) is equivalent to (using summation convention)

$$\partial_j \sigma_{ij} = \dot\mu_i, \tag{2}$$

$$\sigma_{ij} = (\mathbb{C}_{\text{eff}})_{ijkl}\,\varepsilon_{kl} + (S_{\text{eff}})_{ijk}\,\dot u_k, \tag{3}$$

$$\mu_i = (S_{\text{eff}})_{kli}\,\varepsilon_{kl} + \varrho_{\text{eff}}\,\dot u_i. \tag{4}$$

We show existence and uniqueness of (weak) solutions under the two symmetry assumptions on the Willis coupling tensor

$$S_{ijk} = S_{jik} \qquad \text{for } 1 \le i,j,k \le n, \tag{5}$$

$$S_{ijk} = S_{jki} \qquad \text{for } 1 \le i,j,k \le n. \tag{6}$$

While the first symmetry (5) is naturally associated to the symmetry of the Cauchy stress σ, the second condition (6) appears to be new and is related to the possibility to transform the problem to a linear symmetric hyperbolic system. Combined, the conditions (5) and (6) imply that S is a totally symmetric third-order tensor, having at most 10 independent coefficients in dimension 3, see Itin and Reches (2025). Due to the term $(S^T_{\text{eff}})_{lik} = (S_{\text{eff}})_{kli}$ in (4), the material exhibits reciprocity, see Muhlestein et al. (2016), independently of the symmetry assumption (6).

We would like to thank Prof. G. Rosi (University Paris – Est Créteil) for sharing the following observation.

Remark 1. Assume that ϱ, $\mathbb{C}_{ijkl}$ and S_{ijk} are constants independent of (x,t). Differentiating (3) w.r.t. j and (4) w.r.t. t, we obtain for $i \in \{1,2,\dots,n\}$

$$\mathbb{C}_{ijkl}\,\partial_j\partial_k u_l + S_{ijk}\,\partial_t\partial_j u_k = \partial_j\sigma_{ij}, \tag{7}$$

[☆] This article is part of a Special issue entitled: 'Ortiz70' published in Mechanics of Materials.

* Corresponding author.
E-mail addresses: t.blesgen@th-bingen.de (T. Blesgen), patrizio.neff@uni-due.de (P. Neff).

[1] Both authors contributed equally to the present article.

https://doi.org/10.1016/j.mechmat.2025.105561
Received 12 April 2025; Received in revised form 28 October 2025; Accepted 26 November 2025
Available online 1 December 2025
0167-6636/© 2025 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

$$S_{jki}\,\partial_t\partial_j u_k + \varrho\,\partial_t^2 u_i = \partial_t \mu_i. \tag{8}$$

Due to Eq. (2), the right hand sides of (7), (8) are equal such that

$$\varrho\,\partial_t^2 u_i = \mathbb{C}_{ijkl}\,\partial_j\partial_k u_l + (S_{ijk} - S_{jki})\,\partial_t\partial_j u_k. \tag{9}$$

If S satisfies the symmetry (6), the term in brackets on the right disappears and Eq. (9) simplifies to the common linear elasticity equation

$$\varrho\,\partial_t^2 u_i = \mathbb{C}_{ijkl}\,\partial_j\partial_k u_l$$

showing no Willis coupling. Hence, total symmetry of S excludes Willis coupling for constant coefficients. However, in general, the coupling tensor S is space and time dependent in which case the studied Willis problem is non-trivial in the sense that it departs considerably from classical linear elasticity. In addition, even for constant ϱ, $\mathbb{C}$ and S, the Willis coupling does not disappear on $\partial\Omega$ (related to the normal component $\sigma_{ij}n_j$ of the stress). Also note that for materials whose underlying microstructure shows centro-symmetry (inversion symmetry), the third-order Willis coupling tensor S must vanish. This is not at odds with assuming that S is totally symmetric as both conditions are different.

1.1. Background for the Willis approach

Unusual dynamic properties of certain classes of composite materials (architected materials, metamaterials) necessitate to introduce new systems of equations, extending classical linear elasticity. One direction is to explore generalized continua, see. e.g. Madeo et al. (2016), Rizzi et al. (2024), with additional kinematic descriptor fields. Another direction is to change the structure of the equations. In the latter case, the Willis model aims to provide effective constitutive equations for ensemble averages of quantities of interest, i.e. Cauchy stress σ versus displacement u or velocity $v = \dot{u}$. Hence, the usefulness of the Willis approach depends crucially on the assumption that an ensemble average is a reasonable descriptor of the given (periodic) microstructure of the medium.

Let the fully resolved microstructure of the medium obey classical linear elasticity, i.e.

$$\mathrm{Div}\,\sigma \;=\; \dot{\mu} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}(\varrho\,\dot{u}) \tag{10}$$

together with the constitutive law

$$\sigma \;=\; \mathbb{C}(x).\varepsilon, \qquad \varepsilon \;=\; \mathrm{sym}\,\mathrm{D}u. \tag{11}$$

Ensemble averaging (10) we obtain

$$\mathrm{Div}\,\langle\sigma\rangle \;=\; \langle\dot{\mu}\rangle$$

for the averaged quantities $\langle\sigma\rangle$ and $\langle\dot{\mu}\rangle$. The infinitesimal strain tensor is likewise ensemble averaged as

$$\langle\varepsilon\rangle \;=\; \mathrm{sym}\,\mathrm{D}\langle u\rangle. \tag{12}$$

However, ensemble averaging the constitute law $(11)_1$ is not directly achievable since

$$\langle\sigma\rangle \;=\; \langle\mathbb{C}(x).\varepsilon\rangle \;\neq\; \langle\mathbb{C}(x)\rangle.\langle\varepsilon\rangle \tag{13}$$

and $\langle\mu\rangle \neq \langle\varrho\rangle \cdot \langle\dot{u}\rangle$ because in general the product of averages differs from the average of the product. Therefore, the Willis equations provide simple constitutive closure relations in the form

$$\begin{aligned}\langle\sigma\rangle \;&:= \mathbb{C}_{\mathrm{eff}}.\langle\varepsilon\rangle + S_{\mathrm{eff}}.\langle\dot{u}\rangle,\\ \langle\mu\rangle \;&:= S_{\mathrm{eff}}^T.\langle\varepsilon\rangle + \varrho_{\mathrm{eff}}\,\langle\dot{u}\rangle,\end{aligned} \tag{14}$$

where $\mathbb{C}_{\mathrm{eff}}$, S_{eff}, ϱ_{eff} must be determined in an additional step. In the following, we skip the $\langle\cdot\rangle$-notation.

1.2. Invariance considerations

If we adhere to the idea that the Willis system (1) should be the result of some homogenization based on classical linear elastodynamics, then it is natural to require that solutions of (1) should satisfy the same invariance conditions as are satisfied by classic linear elastodynamics

$$\mathrm{Div}\,\sigma \;=\; \varrho\,\partial_{tt}u, \qquad \sigma \;=\; \mathbb{C}.\mathrm{sym}\,\mathrm{D}u. \tag{15}$$

It is easy to see that (15) is *infinitesimal Galilean-invariant*, i.e. if u is a solution, so is

$$u(x) \mapsto u(x) + \overline{A}x + \overline{b}(t), \qquad \overline{b}''(t) = 0, \tag{16}$$

where $\overline{A} \in \mathfrak{so}(3)$ and $t \mapsto \overline{b}(t) \in \mathbb{R}^3$. A direct check reveals that the system (1) likewise admits the invariance (16).[2]

However, the system (15) also admits a lesser-known further invariance condition, the so-called *extended infinitesimal Galilean invariance*

$$u(x) \mapsto u(x) + \overline{A}(t)x + \overline{b}(t), \qquad \overline{b}''(t) = 0, \quad \overline{A}''(t) = 0. \tag{17}$$

This invariance condition has no immediate counterpart in classical nonlinear elasto-dynamics but appears as possibility due to the loss of information inherent in the linearization process of which (15) is the result. Be that as it may, it must be observed that the linear Willis system is not invariant w.r.t. (17) if $S_{\mathrm{eff}} \neq 0$. Thus, whatever process of homogenization is applied, the simplified system (1) cannot entirely capture all effects that are possible in a fully dynamic calculation of a completely resolved microstructure. Nevertheless, we find it worthwhile to look at the mathematical structure presented by the system (1). To the best of our knowledge, no local or global existence proof has yet been given. Due to the linearity, however, this should be possible (but see Lewy, 1957) and indeed, based on the general theory of linear symmetric hyperbolic systems of first order, the abstract Willis system can be cast into a format that permits an existence result.

2. Prerequisites and assumptions

Let $I := \{1, 2, \dots, n\}$ and $\Omega \subset \mathbb{R}^n$ be a domain, $0 < T \leq \infty$ a fixed time, $\Omega_T := \Omega \times (0, T)$ and $\mathcal{D} := \Omega \times [0, T]$. If Ω is bounded we write $\Sigma_T := \partial\Omega \times (0, T)$.

Throughout, we shall employ the following notations. We write ∂_k shortly for $\frac{\partial}{\partial x_k}$ and $\|M\| := \mathrm{tr}(M^T M)$ is the Frobenius norm, where $\mathrm{tr}(M) := \sum_{k\in I} M_{kk}$ is the trace and M^T the transpose of M for $M \in \mathbb{R}^{n\times n}$. We write $\langle v, w\rangle := \sum_{k\in I} v_k w_k$ for the Euclidean inner product of two vectors $v, w \in \mathbb{R}^n$ and $\mathbb{C}.\varepsilon := \mathbb{C}_{ijkl}\,\varepsilon_{kl}$ for the application of a fourth-order tensor $\mathbb{C}$ to a second-order tensor ε. Let $s \in \mathbb{N}$ be a fixed integer. By $W^{s,p}(\Omega)$ we denote the Sobolev space of s-times weakly differentiable functions in $L^p(\Omega)$ and $H^s(\Omega) \equiv W^{s,2}(\Omega)$ is a Hilbert space. By $C_b^m(X)$ we denote the space of m-times bounded differentiable functions of a Banach space X to $\mathbb{R}$. By $\mathrm{Sym}(n)$ we denote the set of symmetric real $n \times n$ matrices.

For $(x, t) \in \Omega_T$, let $u = u(x, t) = (u_i(x, t))_{i\in I}$ denote the displacement, $\mu = \mu(x, t) = (\mu_i(x, t))_{i\in I}$ the momentum density vector, $\sigma = \sigma(x, t) = (\sigma_{ij})_{i,j\in I}$ the symmetric Cauchy stress tensor; $S = S(x, t) = (S_{ijk}(x, t))_{i,j,k\in I}$ is the Willis coupling tensor, $\partial_t u$ is the particle velocity.

One model assumption is that the strain be small. By $\varepsilon = \varepsilon(u) := \mathrm{sym}\,\mathrm{D}u$ we denote the linearized strain tensor, i.e.

$$\varepsilon_{kl} := \frac{1}{2}\Big(\partial_k u_l + \partial_l u_k\Big), \qquad k, l \in I. \tag{18}$$

For the existence proofs below we make the following assumptions on ϱ, $\mathbb{C}$, S, $\overline{u}$ and μ_0.

[2] It is clear that an ensemble average (statistical average) transforms likewise.

(A0) The elasticity tensor $\mathbb{C}$ is a fourth-order tensor with $(\mathbb{C} = \mathbb{C}_{ijkl}(x,t))_{i,j,k,l\in I}$ possibly depending on (x,t) to account for complicated material behavior. We assume

$$\mathbb{C}_{ijkl}(x,t) \in C_b^\infty(\Omega \times [0,T]). \tag{19}$$

The tensor $\mathbb{C}$ satisfies the *major* and *minor symmetry relations*

$$\mathbb{C}_{ijkl}(x,t) = \mathbb{C}_{jikl}(x,t) = \mathbb{C}_{ijlk}(x,t) = \mathbb{C}_{klij}(x,t) \quad \text{for all } i,j,k,l \in I,\ (x,t) \in \mathcal{D}. \tag{20}$$

We assume that $\mathbb{C}$ is uniformly positive definite. This means there exists a constant $c_1 > 0$ such that for all $(x,t) \in \mathcal{D}$

$$\big\langle \mathbb{C}(x,t).\varepsilon, \varepsilon \big\rangle \ \geq \ c_1 \|\varepsilon\|^2 \qquad \text{for all } \varepsilon \in \mathrm{Sym}(n). \tag{21}$$

(A1) The mass density $\varrho = \varrho(x,t)$ of the material is given and satisfies

$$\varrho,\ \partial_t \varrho \in C_b^\infty(\Omega \times [0,T]). \tag{22}$$

There exists a constant $m_0 > 0$ such that

$$\varrho(x,t) \geq m_0 \qquad \text{for all } (x,t) \in \Omega \times [0,T]. \tag{23}$$

(A2) The third-order Willis coupling tensor $S = S_{ikl}(x,t)$ satisfies

$$S_{ikl},\ \partial_t S_{ikl} \in C_b^\infty(\Omega \times [0,T]) \qquad \text{for all } i,k,l \in I. \tag{24}$$

The tensor S satisfies the symmetry relations

$$S_{ijk}(x,t) = S_{jik}(x,t) \qquad \text{for all } i,j,k \in I, (x,t) \in \Omega \times [0,T], \tag{25}$$

$$S_{ijk}(x,t) = S_{jki}(x,t) \qquad \text{for all } i,j,k \in I, (x,t) \in \Omega \times [0,T]. \tag{26}$$

(A3) The initial data u_0 and μ_0 satisfy for an integer $s \geq 1$

$$u_0 \in H^{s+1}(\Omega;\, \mathbb{R}^n), \tag{27}$$

$$\mu_0 \in H^{s}(\Omega;\, \mathbb{R}^n). \tag{28}$$

(A4) The boundary function $\overline{u}$ can be extended to a function on $\overline{\Omega}\times[0,T]$ which satisfies

$$\partial_t^r\, \overline{u}(\cdot,0) \in H^{s+1-r}(\Omega;\, \mathbb{R}^n) \qquad \text{for } 0 \leq r \leq s+1, \tag{29}$$

$$\partial_t^r\, \overline{u} \in L^2(0,T;\, H^{s+2-r}(\Omega;\, \mathbb{R}^n)) \qquad \text{for } 0 \leq r \leq s+2. \tag{30}$$

We write $\varrho_0(x) := \varrho(x,0)$ for the (given) density at time $t = 0$. The condition (25) ensures the symmetry of the Cauchy stress tensor σ. The boundary data $\overline{u}$ in (A4) is introduced below in (40). We assume the compatibility of initial and boundary data, i.e.

$$\overline{u}(\cdot,0) = u_0 \qquad \text{in } \Omega. \tag{31}$$

Due to (20), for $n = 3$, only 36 of the 81 entries of $\mathbb{C}$ are independent. As is well known, see Sommerfeld (1964, pp. 268–269), due to conservation of energy, this reduces further and at most 21 entries may be independent. The material symmetry relation allows to reduce this number even more, see, e.g., Mehrabadi and Cowin (1990), Vannucci (2018).

Using the symmetries (20), we recover the identity

$$\begin{aligned}(\mathbb{C}(x,t).\varepsilon)_{ij} \ &= \ \mathbb{C}_{ijkl}(x,t)\, \frac{1}{2}\big(\partial_k u_l + \partial_l u_k\big)\\ &= \ \frac{1}{2}\mathbb{C}_{ijkl}(x,t)\, \partial_k u_l + \frac{1}{2}\mathbb{C}_{ijkl}(x,t)\, \partial_l u_k\\ &= \ \frac{1}{2}\big(\mathbb{C}_{ijkl}(x,t) + \mathbb{C}_{ijlk}(x,t)\big)\partial_k u_l\\ &= \ \mathbb{C}_{ijkl}(x,t)\, \partial_k u_l \ = \ (\mathbb{C}(x,t).\mathrm{D}u)_{ij}, \qquad i,j \in I,\ (x,t) \in \mathcal{D}.\end{aligned} \tag{32}$$

Subsequently we analyze the following system of equations related to the *Willis model*. Find the solution vector (μ,σ,u) with $\mu \in L^2(0;T;L^2(\Omega;\,\mathbb{R}^n))$, $u \in L^2(0,T;\, H^1(\Omega;\,\mathbb{R}^n))$, $\sigma \in L^2(0;T;\, H^1(\Omega;\,\mathbb{R}^{n\times n}))$ solving in Ω_T

$$\partial_t \mu_i \ = \ \partial_j \sigma_{ij}, \qquad i \in I, \tag{33}$$

$$S_{ijk}\, \partial_t u_k \ = \ \sigma_{ij} - \mathbb{C}_{ijkl}\, \varepsilon_{kl}, \qquad i,j \in I, \tag{34}$$

$$\varrho\, \partial_t u_i \ = \ \mu_i - S_{kli}\, \varepsilon_{kl}, \qquad i \in I \tag{35}$$

subject to the initial and boundary conditions

$$u(\cdot,0) \ = \ u_0, \qquad \text{in } \Omega, \tag{36}$$

$$\mu(\cdot,0) \ = \ \mu_0, \qquad \text{in } \Omega, \tag{37}$$

$$u \ = \ \overline{u}, \qquad \text{on } \partial\Omega \times [0,T] \tag{38}$$

for given initial values $u_0 \in H^{1,2}(\Omega)$ and $\mu_0 \in L^2(\Omega)$. In (32), (33)–(35) and below, we utilize the summation convention and implicitly sum over repeated indices in I unless stated otherwise. In the original Willis model (Willis, 1985), formulated in $n = 3$ space dimensions, S is defined by a convolution. In this article, we do not assume any specific form of S, but consider generic tensors S depending on (x,t).

Remark 2. For $S = \mathbf{0}$, (33)–(35) constitute the classical equations of motion for the propagation of waves in solids and from (34) we recover *Hooke's law*

$$\sigma = \mathbb{C}.\varepsilon. \tag{39}$$

3. Reformulation of the problem

We assume that the boundary data $\overline{u}$ in (38) can be extended to a function $\overline{u} \in H^{s+2}(\overline{\Omega} \times [0,T];\, \mathbb{R}^n)$. We split the deformation vector u by writing

$$u(x,t) \ := \ \tilde{u}(x,t) + \overline{u}(x,t) \tag{40}$$

such that $\tilde{u} = 0$ on $\partial\Omega \times [0,T]$. Due to compatibility of initial and boundary data, $\overline{u}(\cdot,0) - u_0$ has zero trace on $\partial\Omega$. Analogous to the definition of $\varepsilon = \varepsilon(u)$ in (18), we set

$$\widetilde{\varepsilon} \ := \ \varepsilon(\tilde{u}) \in \mathrm{Sym}(n), \qquad \overline{\varepsilon} \ := \ \varepsilon(\overline{u}) \in \mathrm{Sym}(n).$$

With these notations, (33)–(35) rewrites as the following system in Ω_T

$$\partial_t \mu_i \ = \ \partial_j \sigma_{ij}, \qquad i \in I, \tag{41}$$

$$S_{ijk}\, \partial_t(\tilde{u}_k + \overline{u}_k) \ = \ \sigma_{ij} - \mathbb{C}_{ijkl}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}), \qquad i,j \in I, \tag{42}$$

$$\varrho\, \partial_t(\tilde{u}_i + \overline{u}_i) \ = \ \mu_i - S_{kli}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}), \qquad i \in I \tag{43}$$

subject to the initial and boundary conditions (36)–(38).

We reformulate (41)–(43) as a linear hyperbolic system. Using (23), Eqn. (43) becomes

$$\partial_t(\tilde{u}_m + \overline{u}_m) \ = \ \frac{1}{\varrho}\mu_m - \frac{1}{\varrho} S_{klm}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}), \qquad m \in I. \tag{44}$$

Hence

$$\begin{aligned}\sigma_{ij} &\overset{(42)}{=} \mathbb{C}_{ijkl}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}) + S_{ijm}\, \partial_t(\tilde{u}_m + \overline{u}_m)\\ &\overset{(44)}{=} \underbrace{\Big(\mathbb{C}_{ijkl} - \frac{1}{\varrho} S_{ijm} S_{klm}\Big)}_{=:\, H_{ijkl}}(\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}) + \frac{1}{\varrho} S_{ijk}\, \mu_k, \qquad i,j \in I.\end{aligned} \tag{45}$$

Plugging (45) into (41), we obtain

$$\partial_t \mu_i \ = \ \partial_j \Big(H_{ijkl}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}) + \frac{1}{\varrho} S_{ijk}\, \mu_k \Big), \qquad i \in I. \tag{46}$$

By (43), $\mu_i = \varrho\, \partial_t\big(\tilde{u}_i + \overline{u}_i\big) + S_{kli}\, \big(\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}\big)$. Using this relationship on the left of (46) yields

$$\begin{aligned}&\underset{\cdots\cdots}{\partial_t\big(\varrho\, \partial_t(\tilde{u}_i + \overline{u}_i) + S_{kli}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl})\big)} = \underline{\partial_j H_{ijkl}}\, (\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl}) + \underline{\underline{H_{ijkl}\, \partial_j(\widetilde{\varepsilon}_{kl} + \overline{\varepsilon}_{kl})}}\\ &\quad + \underline{\partial_j(\varrho^{-1}) S_{ijk}}\, \mu_k + \frac{1}{\varrho} \partial_j S_{ijk}\, \mu_k + \underline{\frac{1}{\varrho} S_{ijk}}\, \partial_j \mu_k.\end{aligned}$$

Plugging in $\mu_k = \varrho\, \partial_t(\tilde{u}_k + \overline{u}_k) + S_{mnk}\, \big(\widetilde{\varepsilon}_{mn} + \overline{\varepsilon}_{mn}\big)$ on the right together with the definition (45) of H, we obtain

$$
\begin{aligned}
\varrho\,\partial_{tt}\tilde u_i \;=\;& -\partial_j(\varrho^{-1}) S_{ijm} S_{klm}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) - \frac{1}{\varrho}\partial_j S_{ijm} S_{klm}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \\
&- \frac{1}{\varrho} S_{ijm}\partial_j S_{klm}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \\
&+ \partial_j \mathbb{C}_{ijkl}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) + \Big(\mathbb{C}_{ijkl} - \frac{1}{\varrho} S_{ijm} S_{klm}\Big)\partial_j(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \\
&+ \varrho\,\partial_j(\varrho^{-1})\,S_{ijk}\,\partial_t(\tilde u_k+\bar u_k) + \partial_j(\varrho^{-1}) S_{ijk} S_{mnk}\,(\tilde\varepsilon_{mn}+\bar\varepsilon_{mn}) \\
&+ \partial_j S_{ijk}\,\partial_t(\tilde u_k+\bar u_k) + \frac{1}{\varrho}\partial_j S_{ijk} S_{mnk}\,(\tilde\varepsilon_{mn}+\bar\varepsilon_{mn}) \qquad (47)\\
&+ \frac{1}{\varrho} S_{ijk}\,\partial_j\varrho\,\partial_t(\tilde u_k+\bar u_k) + S_{ijk}\,\partial_j(\partial_t\tilde u_k+\partial_t\bar u_k) + \frac{1}{\varrho} S_{ijk}\partial_j S_{mnk}\,(\tilde\varepsilon_{mn}+\bar\varepsilon_{mn}) \\
&+ \frac{1}{\varrho} S_{ijk} S_{mnk}\,\partial_j(\tilde\varepsilon_{mn}+\bar\varepsilon_{mn}) \\
&- \partial_t S_{kli}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) - S_{kli}\,\partial_t(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) - \partial_t\varrho\,\partial_t(\tilde u_i+\bar u_i) - \varrho\,\partial_{tt}\bar u_i, \qquad i\in I.
\end{aligned}
$$

Most terms in (47) cancel out. After simplifications, we are left with

$$
\begin{aligned}
\varrho\,\partial_{tt}\tilde u_i =\;& \mathbb{C}_{ijkl}\,\partial_j(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) + \partial_j S_{ijk}\,\partial_t(\tilde u_k+\bar u_k) + (S_{ijk}-S_{jki})\,\partial_t\partial_j(\tilde u_k+\bar u_k) \\
&+ \partial_j\mathbb{C}_{ijkl}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) - \partial_t S_{kli}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) - \partial_t\varrho\,(\partial_t\tilde u_i+\partial_t\bar u_i) - \varrho\,\partial_{tt}\bar u_i.
\end{aligned} \tag{48}
$$

With (32), we have

$$
\mathbb{C}_{ijkl}\,\partial_j(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \;=\; \mathbb{C}_{ijkl}\,\partial_j\frac{1}{2}(\partial_k\tilde u_l+\partial_l\tilde u_k+\partial_k\bar u_l+\partial_l\bar u_k) \;=\; \mathbb{C}_{ijkl}\,\partial_j\partial_k(\tilde u_l+\bar u_l).
$$

Similarly, as a consequence of (25), the third term on the right of (48) disappears, and

$$
\partial_j\mathbb{C}_{ijkl}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \;=\; \partial_j\mathbb{C}_{ijkl}\,\partial_k(\tilde u_l+\bar u_l), \tag{49}
$$

$$
\partial_t S_{kli}\,(\tilde\varepsilon_{kl}+\bar\varepsilon_{kl}) \;=\; \partial_t S_{kli}\,\partial_k(\tilde u_l+\bar u_l). \tag{50}
$$

Eventually, after introducing $e : \mathcal{D}\to\mathbb{R}^n$ by

$$
\begin{aligned}
\varrho\,e_i \;:=& -\mathbb{C}_{ijkl}\,\partial_j\partial_k\bar u_l - \partial_j S_{ijk}\,\partial_t\bar u_k + (\partial_t S_{kli} - \partial_j\mathbb{C}_{ijkl})\,\partial_k\bar u_l \\
&+ \partial_t\varrho\,\partial_t\bar u_i + \varrho\,\partial_{tt}\bar u_i + (S_{jki}-S_{ijk})\partial_t\partial_j\bar u_k, \qquad i\in I
\end{aligned} \tag{51}
$$

we end up with

$$
\begin{aligned}
\varrho\,\partial_{tt}\tilde u_i =\;& \mathbb{C}_{ijkl}\,\partial_j\partial_k\tilde u_l + (\partial_j\mathbb{C}_{ijkl} - \partial_t S_{kli})\,\partial_k\tilde u_l + \partial_j S_{ijk}\,\partial_t\tilde u_k \\
&- \partial_t\varrho\,\partial_t\tilde u_i + (S_{ijk}-S_{jki})\partial_j\partial_t\tilde u_k - \varrho e_i \qquad \text{in } \Omega_T,\ i\in I.
\end{aligned} \tag{52}
$$

The system (52) is solved subject to the initial and boundary conditions

$$
\varrho_0\,\partial_t\tilde u(x,0) \;=\; g(x), \qquad x\in\Omega, \tag{53}
$$

$$
\tilde u(x,0) \;=\; h(x), \qquad x\in\Omega, \tag{54}
$$

$$
\tilde u(x,t) \;=\; 0, \qquad x\in\partial\Omega,\ t\in[0,T]. \tag{55}
$$

Therein, the initial data h is specified from (36) and (40), g is specified from (43) at $t=0$,

$$
h(x) \;= u_0(x) - \bar u(x,0), \qquad x\in\Omega, \tag{56}
$$

$$
g(x) \;= \Big(\mu_{0i}(x) - S_{kli}(x,0)\,\partial_k\big(u_{0l}(x)+\bar u_l(x,0)\big) - \varrho_0(x)\,\partial_t\bar u_i(x,0)\Big)_{1\le i\le n}, \qquad x\in\Omega. \tag{57}
$$

With (27), it holds $h\in H_0^{s+1}(\Omega)$ due to the compatibility of initial and boundary data.

The Eqs. (52) constitute a linear hyperbolic system and represent the most general form the resulting equations may have under the assumption (25).

We recall that a *first order linear symmetric hyperbolic system* is of the form

$$
Lv := A_0(x,t)\,\partial_t v + \sum_{k=1}^{n} A_k(x,t)\,\partial_k v + B(x,t)\,v = w(x,t), \tag{58}
$$

where $v\;:\;\Omega\times\mathbb{R}_{\ge 0}\to\mathbb{R}^m$, $A_0(x,t), A_1(x,t),\dots,A_n(x,t)\in\mathbb{R}^{m\times m}$ are symmetric matrices for all $(x,t)\in\overline{\Omega}\times[0,T]$, $A_0(x,t)$ is positive definite, $B(x,t)\in\mathbb{R}^{m\times m}$, and $w(x,t)\in\mathbb{R}^m$ is a given right hand side. As a consequence of the symmetry of A_0, it is diagonalizable and all eigenvalues are real.

As a preparation of the following lemma, we introduce the symmetric 3×3 matrices

$$
\begin{aligned}
&\mathbb{C}_1^1 := \begin{pmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1112} & \mathbb{C}_{1131} \\ \mathbb{C}_{2111} & \mathbb{C}_{2112} & \mathbb{C}_{2131} \\ \mathbb{C}_{3111} & \mathbb{C}_{3112} & \mathbb{C}_{3131} \end{pmatrix}, \quad
\mathbb{C}_2^1 := \begin{pmatrix} \mathbb{C}_{1211} & \mathbb{C}_{1221} & \mathbb{C}_{1231} \\ \mathbb{C}_{2211} & \mathbb{C}_{2221} & \mathbb{C}_{2231} \\ \mathbb{C}_{3211} & \mathbb{C}_{3221} & \mathbb{C}_{3231} \end{pmatrix}, \quad
\mathbb{C}_3^1 := \begin{pmatrix} \mathbb{C}_{1311} & \mathbb{C}_{1321} & \mathbb{C}_{1331} \\ \mathbb{C}_{2311} & \mathbb{C}_{2321} & \mathbb{C}_{2331} \\ \mathbb{C}_{3311} & \mathbb{C}_{3321} & \mathbb{C}_{3331} \end{pmatrix}, \\
&\mathbb{C}_1^2 := \begin{pmatrix} \mathbb{C}_{1112} & \mathbb{C}_{1122} & \mathbb{C}_{1132} \\ \mathbb{C}_{2112} & \mathbb{C}_{2122} & \mathbb{C}_{2132} \\ \mathbb{C}_{3112} & \mathbb{C}_{3122} & \mathbb{C}_{3132} \end{pmatrix}, \quad
\mathbb{C}_2^2 := \begin{pmatrix} \mathbb{C}_{1212} & \mathbb{C}_{1222} & \mathbb{C}_{1232} \\ \mathbb{C}_{2212} & \mathbb{C}_{2222} & \mathbb{C}_{2232} \\ \mathbb{C}_{3212} & \mathbb{C}_{3222} & \mathbb{C}_{3232} \end{pmatrix}, \quad
\mathbb{C}_3^2 := \begin{pmatrix} \mathbb{C}_{1312} & \mathbb{C}_{1322} & \mathbb{C}_{1332} \\ \mathbb{C}_{2312} & \mathbb{C}_{2322} & \mathbb{C}_{2332} \\ \mathbb{C}_{3312} & \mathbb{C}_{3322} & \mathbb{C}_{3332} \end{pmatrix},
\end{aligned} \tag{59}
$$

$$
\mathbb{C}_1^3 := \begin{pmatrix} \mathbb{C}_{1113} & \mathbb{C}_{1123} & \mathbb{C}_{1233} \\ \mathbb{C}_{2113} & \mathbb{C}_{2123} & \mathbb{C}_{2233} \\ \mathbb{C}_{3113} & \mathbb{C}_{3123} & \mathbb{C}_{3233} \end{pmatrix}, \quad
\mathbb{C}_2^3 := \begin{pmatrix} \mathbb{C}_{1213} & \mathbb{C}_{1223} & \mathbb{C}_{1233} \\ \mathbb{C}_{2213} & \mathbb{C}_{2223} & \mathbb{C}_{2233} \\ \mathbb{C}_{3213} & \mathbb{C}_{3223} & \mathbb{C}_{3233} \end{pmatrix}, \quad
\mathbb{C}_3^3 := \begin{pmatrix} \mathbb{C}_{1313} & \mathbb{C}_{1323} & \mathbb{C}_{1333} \\ \mathbb{C}_{2313} & \mathbb{C}_{2323} & \mathbb{C}_{2333} \\ \mathbb{C}_{3313} & \mathbb{C}_{3323} & \mathbb{C}_{3333} \end{pmatrix}.
$$

It holds $C_i^j = C_i^j(x,t)$ with the symmetry $\mathbb{C}_i^j(x,t) = \mathbb{C}_j^i(x,t)$ for $i,j\in\{1,2,3\}$.

For the second term on the right of (52), we introduce the shorthand notation

$$
D_{kl}^i(x,t) := \partial_t S_{kli}(x,t) - \partial_j\mathbb{C}_{ijkl}(x,t), \qquad i,k,l\in\{1,2,3\}. \tag{60}
$$

If Ω is bounded and $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{R}^3$ is the unit outer normal vector at a point in $\partial\Omega$, let

$$
\mathbb{C}_\nu := \left(\begin{array}{c:c:c} \nu_1\mathbb{C}_1^1 & \nu_1\mathbb{C}_2^1 & \nu_1\mathbb{C}_3^1 \\ \hdashline \nu_2\mathbb{C}_1^2 & \nu_2\mathbb{C}_2^2 & \nu_2\mathbb{C}_3^2 \\ \hdashline \nu_3\mathbb{C}_1^3 & \nu_3\mathbb{C}_2^3 & \nu_3\mathbb{C}_3^3 \end{array}\right) \in\mathbb{R}^{9\times 9}. \tag{61}
$$

Lemma 1. *Assume that the Willis coupling tensor satisfies the further symmetry relation*

$$
S_{ijk}(x,t) = S_{jki}(x,t) \qquad \textit{for all } i,j,k\in I,\ (x,t)\in\Omega\times[0,T]. \tag{62}
$$

Then the Eqs. (52)–(55) constitute a linear symmetric hyperbolic system of first order, i.e. they can be written as the mixed initial boundary value problem

$$
Lv = w \qquad \textit{in } \Omega\times[0,T], \tag{63}
$$

$$
M(x)v = 0 \qquad \textit{in } \Gamma\times[0,T], \tag{64}
$$

$$
v(\cdot,0) = v_0 \qquad \textit{in } \Omega, \tag{65}
$$

where L is given by (58), $M(x)\in\mathbb{R}^{m\times m}$ for $x\in\Gamma$, and $w=w(x,t)\in\mathbb{R}^m$, $v_0\in\mathbb{R}^m$ are suitable vectors. In $n=3$ dimensions, Eqs. (66)–(68) hold, see Box I below.
The boundary matrix and the vector of the right hand side are given by

$$
M = \left(\begin{array}{c:c:c} \mathbb{C}_\nu\in\mathbb{R}^{9\times 9} & 0\in\mathbb{R}^{9\times 3} & 0\in\mathbb{R}^{9\times 3} \\ \hdashline 0\in\mathbb{R}^{3\times 9} & 0\in\mathbb{R}^{3\times 3} & 0\in\mathbb{R}^{3\times 3} \\ \hdashline 0\in\mathbb{R}^{3\times 3} & 0\in\mathbb{R}^{3\times 3} & \mathbb{1}\in\mathbb{R}^{3\times 3} \end{array}\right), \qquad
w(x,t) = \left(\begin{array}{c} \vec 0\in\mathbb{R}^9 \\ \hdashline \varrho(x,t)\,e_1(x,t) \\ \varrho(x,t)\,e_2(x,t) \\ \varrho(x,t)\,e_3(x,t) \\ \hdashline \vec 0\in\mathbb{R}^3 \end{array}\right). \tag{69}
$$

Proof. The following procedure is a modification of an example in John (1982, p.163) for a scalar hyperbolic equation. Subsequently we restrict to the case $n=3$ which allows us to explicitly write down all matrices and explain the necessary transformations. However, the method is valid for any dimension $n\in\mathbb{N}$. Let for $n=3$

$$
\begin{aligned}
v &= (v_1,\dots,v_{15})^T\in\mathbb{R}^{15} \\
&:= (\partial_1\tilde u_1,\partial_1\tilde u_2,\partial_1\tilde u_3,\partial_2\tilde u_1,\partial_2\tilde u_2,\partial_2\tilde u_3,\partial_3\tilde u_1,\partial_3\tilde u_2,\partial_3\tilde u_3,\partial_t\tilde u_1,\partial_t\tilde u_2,\partial_t\tilde u_3,\tilde u_1,\tilde u_2,\tilde u_3)^T.
\end{aligned} \tag{70}
$$

Note that the last 3 components $\tilde u_1$, $\tilde u_2$ and $\tilde u_3$ of v are only necessary to incorporate the Dirichlet boundary conditions (55). We have a first set of compatibility equations

$$
\begin{aligned}
&\partial_t v_1 - \partial_1 v_{10} = 0, \qquad \partial_t v_4 - \partial_2 v_{10} = 0, \qquad \partial_t v_7 - \partial_3 v_{10} = 0, \\
&\partial_t v_2 - \partial_1 v_{11} = 0, \qquad \partial_t v_5 - \partial_2 v_{11} = 0, \qquad \partial_t v_8 - \partial_3 v_{11} = 0,
\end{aligned} \tag{71}
$$

$$A_0(x,t)=\left(\begin{array}{c:c:c:c:c} \mathbb{C}^1_1 & \mathbb{C}^1_2 & \mathbb{C}^1_3 & 0 & 0 \\ \hdashline \mathbb{C}^2_1 & \mathbb{C}^2_2 & \mathbb{C}^2_3 & 0 & 0 \\ \hdashline \mathbb{C}^3_1 & \mathbb{C}^3_2 & \mathbb{C}^3_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \varrho\,\mathbb{1}_{3\times 3} & 0 \\ \hdashline 0 & 0 & 0 & 0 & \mathbb{1}_{3\times 3} \end{array}\right)(x,t)\in\mathbb{R}^{15\times 15}, \tag{66}$$

$$A_k(x,t)=\left(\begin{array}{c:c:c:c:c} 0 & 0 & 0 & -\mathbb{C}^k_1 & 0 \\ \hdashline 0 & 0 & 0 & -\mathbb{C}^k_2 & 0 \\ \hdashline 0 & 0 & 0 & -\mathbb{C}^k_3 & 0 \\ \hdashline -\mathbb{C}^k_1 & -\mathbb{C}^k_2 & -\mathbb{C}^k_3 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \mathbb{1}_{3\times 3} \end{array}\right)(x,t)\in\mathbb{R}^{15\times 15},\quad k=1,2,3, \tag{67}$$

$$B=\left(\begin{array}{ccc:ccc:ccc:ccc:c} & 0 & & & 0 & & & 0 & & & 0 & & 0 \\ \hdashline & 0 & & & 0 & & & 0 & & & 0 & & 0 \\ \hdashline & 0 & & & 0 & & & 0 & & & 0 & & 0 \\ \hdashline D^1_{11} & D^1_{12} & D^1_{13} & D^1_{21} & D^1_{22} & D^1_{23} & D^1_{31} & D^1_{32} & D^1_{33} & \partial_j S_{1j1}-\partial_t\varrho & \partial_j S_{1j2} & \partial_j S_{1j3} & \\ D^2_{11} & D^2_{12} & D^2_{13} & D^2_{21} & D^2_{22} & D^2_{23} & D^2_{31} & D^2_{32} & D^2_{33} & \partial_j S_{2j1} & \partial_j S_{2j2}-\partial_t\varrho & \partial_j S_{2j3} & 0 \\ D^3_{11} & D^3_{12} & D^3_{13} & D^3_{21} & D^3_{22} & D^3_{23} & D^3_{31} & D^3_{32} & D^3_{33} & \partial_j S_{3j1} & \partial_j S_{3j2} & \partial_j S_{3j3}-\partial_t\varrho & \\ \hdashline & -\mathbb{1} & & & -\mathbb{1} & & & -\mathbb{1} & & & -\mathbb{1} & & 0 \end{array}\right). \tag{68}$$

Box I.

$$\partial_t v_3-\partial_1 v_{12}=0,\qquad \partial_t v_6-\partial_2 v_{12}=0,\qquad \partial_t v_9-\partial_3 v_{12}=0.$$

The Eqs. (52) determine $\partial_t v_{10}$, $\partial_t v_{11}$ and $\partial_t v_{12}$. For the last variables, there is a second set of compatibility relations

$$\partial_t v_{13}-v_{10}=0,\qquad \partial_t v_{14}-v_{11}=0,\qquad \partial_t v_{15}-v_{12}=0 \tag{72}$$

together with

$$\begin{aligned} &\partial_1 v_{13}-v_1=0, &&\partial_2 v_{13}-v_4=0, &&\partial_3 v_{13}-v_7=0,\\ &\partial_1 v_{14}-v_2=0, &&\partial_2 v_{14}-v_5=0, &&\partial_3 v_{14}-v_8=0,\\ &\partial_1 v_{15}-v_3=0, &&\partial_2 v_{15}-v_6=0, &&\partial_3 v_{15}-v_9=0. \end{aligned} \tag{73}$$

With (70), we write the modified linear elasticity equation

$$\varrho(x,t)\,\partial_{tt}\tilde{u}_i-\mathbb{C}_{ijkl}(x,t)\,\partial_j\partial_k\tilde{u}_l+(S_{jki}-S_{ijk})\partial_j\partial_t\tilde{u}_k=0 \tag{74}$$

and (71), (72), (73) in the matrix form

$$\tilde{A}_0\,\partial_t v+\sum_{k=1}^{3}\tilde{A}_k\,\partial_k v+Bv=0.$$

Direct inspection yields Eqs. (75)–(78) given in Box II for the 3×3-matrices

$$F_1:=\begin{pmatrix} 0 & S_{121}-S_{112} & S_{131}-S_{113}\\ S_{112}-S_{211} & S_{122}-S_{212} & S_{132}-S_{213}\\ S_{113}-S_{311} & S_{123}-S_{312} & S_{133}-S_{313}\end{pmatrix},\quad F_2:=\begin{pmatrix} S_{211}-S_{121} & S_{221}-S_{122} & S_{231}-S_{123}\\ S_{212}-S_{221} & 0 & S_{232}-S_{223}\\ S_{213}-S_{321} & S_{223}-S_{322} & S_{233}-S_{323}\end{pmatrix},$$
$$F_3:=\begin{pmatrix} S_{311}-S_{131} & S_{321}-S_{132} & S_{331}-S_{133}\\ S_{312}-S_{231} & S_{322}-S_{232} & S_{332}-S_{233}\\ S_{313}-S_{331} & S_{323}-S_{332} & 0\end{pmatrix}. \tag{79}$$

However, $\tilde{A}_1(x,t)$, $\tilde{A}_2(x,t)$, $\tilde{A}_3(x,t)\in\mathbb{R}^{15\times 15}$ are not symmetric. The matrices F_i are on the diagonal of $\tilde{A}_i$, $i=1,2,3$. They are not symmetric and cannot be rearranged without destroying the symmetry and positive definiteness of A_0. Imposing the strong assumption (62), we obtain $F_1=F_2=F_3=0\in\mathbb{R}^{3\times 3}$.

Now, to symmetrize $\tilde{A}_i$, we form suitable linear combinations of the compatibility Eqs. (71). This idea is first presented in Sfyris (2024).

Exemplary, to get the first line of A_1, we use (cf. the first column of the matrix $\tilde{A}_1$)

$$\mathbb{C}_{1111}(\partial_t v_1-\partial_1 v_{10})+\mathbb{C}_{2111}(\partial_t v_2-\partial_1 v_{11})+\mathbb{C}_{3111}(\partial_t v_3-\partial_1 v_{12})=0, \tag{80}$$

to get the second line of A_1, we use (cf. the second column of $\tilde{A}_1$)

$$\mathbb{C}_{1112}(\partial_t v_1-\partial_1 v_{10})+\mathbb{C}_{2112}(\partial_t v_2-\partial_1 v_{11})+\mathbb{C}_{3112}(\partial_t v_3-\partial_1 v_{12})=0, \tag{81}$$

and eventually to get the ninth line of A_1, we use (cf. the ninth column of $\tilde{A}_1$)

$$\mathbb{C}_{1331}(\partial_t v_1-\partial_1 v_{10})+\mathbb{C}_{2331}(\partial_t v_2-\partial_1 v_{11})+\mathbb{C}_{3331}(\partial_t v_3-\partial_1 v_{12})=0. \tag{82}$$

So we obtain A_1, see (67) with $k=1$. In the same way, $\tilde{A}_2$ is symmetrized. For the first line of A_2, we use (cf. 1. column of $\tilde{A}_2$)

$$\mathbb{C}_{1112}(\partial_t v_4-\partial_2 v_{10})+\mathbb{C}_{2112}(\partial_t v_5-\partial_2 v_{11})+\mathbb{C}_{3112}(\partial_t v_6-\partial_2 v_{12})=0, \tag{83}$$

for the second line

$$\mathbb{C}_{1122}(\partial_t v_4-\partial_2 v_{10})+\mathbb{C}_{2122}(\partial_t v_5-\partial_2 v_{11})+\mathbb{C}_{3122}(\partial_t v_6-\partial_2 v_{12})=0, \tag{84}$$

and so forth. Finally, to symmetrize $\tilde{A}_3$, we use for the first two lines

$$\mathbb{C}_{1113}(\partial_t v_7-\partial_3 v_{10})+\mathbb{C}_{2113}(\partial_t v_8-\partial_3 v_{11})+\mathbb{C}_{3113}(\partial_t v_9-\partial_3 v_{12})=0, \tag{85}$$

$$\mathbb{C}_{1123}(\partial_t v_7-\partial_3 v_{10})+\mathbb{C}_{2123}(\partial_t v_8-\partial_3 v_{11})+\mathbb{C}_{3123}(\partial_t v_9-\partial_3 v_{12})=0, \tag{86}$$

and similar operations for lines 3 to 9. With these operations we obtain A_2, A_3, see (67) with $k=2,3$. The linear combinations (80)–(82), (83)–(86) lead as well to changes in $\tilde{A}_0$, resulting in (66). For instance, (80) modifies the first line of A_0 which results in three non-zero entries, (81) changes the second line of A_0. In total, the coefficients in (80)–(82) constitute the first 3×3-block $\mathbb{C}^1_1$ in A_0.

The major and minor symmetry (20) of $\mathbb{C}$ imposes the symmetry of the matrices $\mathbb{C}^i_j$ in (59) as well as $\mathbb{C}^i_j=\mathbb{C}^j_i$ for $i,j\in\{1,2,3\}$, implying the symmetry of $A_0(x,t)$ for all (x,t). Due to $\varrho(x,t)>0$ in $\mathcal{D}$ and the positive definiteness of $\mathbb{C}$, $A_0(x,t)$ is positive definite. The components $-v_{10}$ in $(72)_1$, $-v_{11}$ in $(72)_2$ and $-v_{12}$ in $(72)_3$ lead to the last $-\mathbb{1}\in\mathbb{R}^{3\times 3}$ block of B in (68). The other $-\mathbb{1}$ blocks in B are a consequence of the terms $-v_1,\ldots,-v_9$ in (73). The $\mathbb{1}\in\mathbb{R}^{3\times 3}$ block in (76)–(78) is due to $\partial_1 v_{13},\ldots,\partial_3 v_{15}$ in (73). This completes the reformulation of (71)–(74) in form of a linear symmetric first-order hyperbolic system.

The remaining terms of (52) can be incorporated in B, leading to (68). The boundary matrix M and the right hand side w are specified by (69).

We want to comment on the matrix $\mathbb{C}_\nu$ appearing in M. Due to Dirichlet boundary conditions, it holds $\tilde{u}=0$ on Γ. As a consequence, all tangential derivatives of $\tilde{u}$ must vanish along Γ. Exemplary, consider the special case of the local coordinates introduced before Definition 2 below, where $\Gamma=\partial\Omega$ is straight and corresponds to $x_1=0$. Here, the tangential derivatives $\partial_2\tilde{u}_i$ and $\partial_3\tilde{u}_i$, $i\in\{1,2,3\}$ vanish on Γ. In matrix

$$\tilde{A}_0(x,t) = \left(\begin{array}{c:c:c} \mathbb{1} \in \mathbb{R}^{9\times 9} & 0 \in \mathbb{R}^{9\times 3} & 0 \in \mathbb{R}^{9\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & \varrho(x,t)\,\mathbb{1} \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & \mathbb{1} \in \mathbb{R}^{3\times 3} \end{array}\right) \in \mathbb{R}^{15\times 15}, \tag{75}$$

$$\tilde{A}_1 = \left(\begin{array}{c:c:c} 0 \in \mathbb{R}^{3\times 9} & -\mathbb{1} \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline \begin{matrix} -\mathbb{C}_{1111} & -\mathbb{C}_{1112} & -\mathbb{C}_{1131} & -\mathbb{C}_{1211} & -\mathbb{C}_{1221} & -\mathbb{C}_{1231} & -\mathbb{C}_{1311} & -\mathbb{C}_{1321} & -\mathbb{C}_{1331} \\ -\mathbb{C}_{2111} & -\mathbb{C}_{2112} & -\mathbb{C}_{2131} & -\mathbb{C}_{2211} & -\mathbb{C}_{2221} & -\mathbb{C}_{2231} & -\mathbb{C}_{2311} & -\mathbb{C}_{2321} & -\mathbb{C}_{2331} \\ -\mathbb{C}_{3111} & -\mathbb{C}_{3112} & -\mathbb{C}_{3131} & -\mathbb{C}_{3211} & -\mathbb{C}_{3221} & -\mathbb{C}_{3231} & -\mathbb{C}_{3311} & -\mathbb{C}_{3321} & -\mathbb{C}_{3331} \end{matrix} & F_1 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & \mathbb{1} \in \mathbb{R}^{3\times 3} \end{array}\right), \tag{76}$$

$$\tilde{A}_2 = \left(\begin{array}{c:c:c} 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & -\mathbb{1} \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline \begin{matrix} -\mathbb{C}_{1112} & -\mathbb{C}_{1122} & -\mathbb{C}_{1132} & -\mathbb{C}_{1212} & -\mathbb{C}_{1222} & -\mathbb{C}_{1232} & -\mathbb{C}_{1312} & -\mathbb{C}_{1322} & -\mathbb{C}_{1332} \\ -\mathbb{C}_{2112} & -\mathbb{C}_{2122} & -\mathbb{C}_{2132} & -\mathbb{C}_{2212} & -\mathbb{C}_{2222} & -\mathbb{C}_{2232} & -\mathbb{C}_{2312} & -\mathbb{C}_{2322} & -\mathbb{C}_{2332} \\ -\mathbb{C}_{3112} & -\mathbb{C}_{3122} & -\mathbb{C}_{3132} & -\mathbb{C}_{3212} & -\mathbb{C}_{3222} & -\mathbb{C}_{3232} & -\mathbb{C}_{3312} & -\mathbb{C}_{3322} & -\mathbb{C}_{3332} \end{matrix} & F_2 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & \mathbb{1} \in \mathbb{R}^{3\times 3} \end{array}\right), \tag{77}$$

$$\tilde{A}_3 = \left(\begin{array}{c:c:c} 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & -\mathbb{1} \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline \begin{matrix} -\mathbb{C}_{1113} & -\mathbb{C}_{1123} & -\mathbb{C}_{1233} & -\mathbb{C}_{1213} & -\mathbb{C}_{1223} & -\mathbb{C}_{1233} & -\mathbb{C}_{1313} & -\mathbb{C}_{1323} & -\mathbb{C}_{1333} \\ -\mathbb{C}_{2113} & -\mathbb{C}_{2123} & -\mathbb{C}_{2133} & -\mathbb{C}_{2213} & -\mathbb{C}_{2223} & -\mathbb{C}_{2233} & -\mathbb{C}_{2313} & -\mathbb{C}_{2323} & -\mathbb{C}_{2333} \\ -\mathbb{C}_{3113} & -\mathbb{C}_{3123} & -\mathbb{C}_{3233} & -\mathbb{C}_{3213} & -\mathbb{C}_{3223} & -\mathbb{C}_{3233} & -\mathbb{C}_{3313} & -\mathbb{C}_{3323} & -\mathbb{C}_{3333} \end{matrix} & F_3 \in \mathbb{R}^{3\times 3} & 0 \in \mathbb{R}^{3\times 3} \\ \hdashline 0 \in \mathbb{R}^{3\times 9} & 0 \in \mathbb{R}^{3\times 3} & \mathbb{1} \in \mathbb{R}^{3\times 3} \end{array}\right) \tag{78}$$

Box II.

form this reads

$$P(\, v_1\, v_2\, v_3 \,\vdots\, v_4\, v_5\, v_6 \,\vdots\, v_7\, v_8\, v_9\,)^T := \left(\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & \mathbb{1} & 0 \\ \hdashline 0 & 0 & \mathbb{1} \end{array}\right) \times (\, v_1\, v_2\, v_3 \,\vdots\, v_4\, v_5\, v_6 \,\vdots\, v_7\, v_8\, v_9\,)^T = 0. \tag{87}$$

In symmetrizing $\tilde{A}_k$ for $k = 1,2,3$, the linear combinations (80)–(86) are applied, converting the matrix $P \in \mathbb{R}^{9\times 9}$ to $\mathbb{C}_\nu$ with $\nu = (-1,0,0)^T$ in the example. This is in line with the transformation of the upper left 9×9 block $\mathbb{1}$ of $\tilde{A}_0$ to the upper left 9×9 block of A_0 in (66).

The vector v_0 for the initial values can be directly read off from (53), (54),

$$\begin{aligned} v_0(x) = \Big(\, & \partial_1 u_{0,1}(x) - \partial_1 \bar{u}_1(x,0), \partial_1 u_{0,2}(x) - \partial_1 \bar{u}_2(x,0), \partial_1 u_{0,3}(x) - \partial_1 \bar{u}_3(x,0), \ldots, \\ & \partial_3 u_{0,3}(x) - \partial_3 \bar{u}_3(x,0), \frac{\mu_{01}(x) - S_{1kl}(x,0)\partial_k(u_{0l}(x) + \bar{u}_l(x,0))}{\varrho_0(x)} - \partial_t \bar{u}_1(x,0), \\ & \frac{\mu_{02}(x) - S_{2kl}(x,0)\partial_k(u_{0l}(x) + \bar{u}_l(x,0))}{\varrho_0(x)} - \partial_t \bar{u}_2(x,0), \\ & \frac{\mu_{03}(x) - S_{3kl}(x,0)\partial_k(u_{0l}(x) + \bar{u}_l(x,0))}{\varrho_0(x)} - \partial_t \bar{u}_3(x,0), \\ & u_{0,1}(x) - \bar{u}_1(x,0), u_{0,2}(x) - \bar{u}_2(x,0), u_{0,2}(x) - \bar{u}_2(x,0)\,\Big)^T. \end{aligned} \tag{88}$$

With (66), (67), (68) and (69), the equivalence of (63)–(65) with (52)–(55) has been shown. □

4. Existence and uniqueness of weak solutions

In this section we apply the existence theory for linear symmetric hyperbolic systems of first order to the Willis system. A good general introduction and overview of mathematical methods for hyperbolic systems can be found in Evans (2010, Chapter 7). An early L^2-theory for linear symmetric hyperbolic systems in bounded domains is developed in Friedrichs (1958), see also (Friedrichs, 1954). The case where the boundary is non-characteristic (see Definition 1 below for explanations) is covered in Rauch and Massey (1974) and Lax and Phillips (1960). In the situation studied here, Γ is characteristic of constant multiplicity. This has been analyzed for tangential regularity in Rauch (1985) and more generally in Ohno et al. (1995). We also learned a lot from the seminal paper (Hughes and Marsden, 1977). An alternative approach would be the use of semigroup theory, see, e.g. Xin and Sha (2009), which however appears to be less flexible.

4.1. Existence theory for $\Omega = \mathbb{R}^n$

As long as no boundary conditions are involved, the proof of existence and uniqueness of solutions to linear symmetric hyperbolic first-order systems is straightforward and we begin with this case. The following theorem is taken from Hughes and Marsden (1977).

Theorem 1 (*Existence and Uniqueness for $\Omega = \mathbb{R}^n$*). *Consider the linear symmetric first-order hyperbolic system (58) on $\mathbb{R}^n$ with initial data v_0. Let $s \in \mathbb{N}$ and assume that*

(i) *A_0, A_i and B are in $C_b^\infty(\mathbb{R}^n \times [0,T]; \mathbb{R}^{m\times m})$.*
(ii) *A_0 and A_i, $1 \le i \le n$ are symmetric.*
(iii) *A_0 is uniformly positive definite, i.e. there exists a constant $\delta > 0$ with*

$$\langle \xi, A_0(x,t)\xi \rangle \ge \delta\, \|\xi\|^2 \qquad \text{for all } \xi \in \mathbb{R}^m \backslash \{0\} \text{ and all } x \in \mathbb{R}^n,\ t \in [0,T]. \tag{89}$$

(iv) *$w \in H^s(\mathbb{R}^n \times [0,T]; \mathbb{R}^m)$.*
(v) *$v_0 \in H^s(\mathbb{R}^n; \mathbb{R}^m)$.*

Then there exists a unique solution v of (58) in $\mathbb{R}^n$ belonging to $C^r([0,T]; H^{s-r}(\mathbb{R}^n; \mathbb{R}^m))$ for $0 \le r \le s$, such that $v(\cdot,0) = v_0$. The solution varies continuously with the initial data in $H^s(\mathbb{R}^n; \mathbb{R}^m)$. Finally, the equations are hyperbolic in the sense that if v_0 and w have compact support then so does $v(\cdot,t)$ for each t.

An immediate consequence is

Corollary 1. *Let the assumptions (A0)–(A2) and (26) hold, and let (A3) be satisfied for an integer $s \geq 1$. Then there exists a unique solution*

$$v \in \bigcap_{r=0}^{s} C^r([0,T];\; H^{s-r}(\mathbb{R}^n;\; R^m)) \tag{90}$$

to the symmetric linear hyperbolic system (63), (65) with $\Omega = \mathbb{R}^n$. Consequently, there exists a unique solution vector (μ, σ, u) to (33)–(37) of the Willis system in $\Omega = \mathbb{R}^n$ satisfying

$$u \in \bigcap_{r=0}^{s+1} C^r([0,T];\; H^{s+1-r}(\Omega;\; \mathbb{R}^n)), \tag{91}$$

$$\mu,\, \sigma \in \bigcap_{r=0}^{s} C^r([0,T];\; H^{s-r}(\Omega;\; \mathbb{R}^n)). \tag{92}$$

Proof. First we verify that the matrices A_0, A_i and B introduced in Lemma 1, (66)–(68) satisfy the requirements (i)–(iii) of Theorem 1. Evidently, (i) follows from (A0), (A1) and (A2). The symmetry (ii) is a direct consequence of (20) and (66), (67). By the uniform positive definiteness of $\mathbb{C}$, and Sylvester's criterion, all principal minors of $\mathbb{C}$ are strictly positive. Together with the positivity condition (23) on ϱ, this yields that A_0 is uniformly positive definite. In the case $\Omega = \mathbb{R}^n$, one can formally set $\bar{u} := 0$ in the derivation of the system leading to $u = \tilde{u}$ and $w(x,t) \equiv 0$ such that (iv) holds. Finally, due to (88) and (A0)–(A3), we obtain $v_0 \in H^s(\mathbb{R}^n; \mathbb{R}^m)$. Hence, (i) to (v) of Theorem 1 are satisfied, yielding the existence of a unique solution $v \in W := \bigcap_{r=0}^{s} C^r([0,T];\, H^{s-r}(\mathbb{R}^n;\, \mathbb{R}^m))$. By (70), it holds $\varepsilon(u)$, Du, $\partial_t u \in W$, implying $u \in \bigcap_{r=0}^{s} C^r([0,T];\, H^{s+1-r}(\mathbb{R}^n;\, \mathbb{R}^m))$ and $u \in \bigcap_{r=0}^{s} C^{r+1}([0,T];\, H^{s-r}(\mathbb{R}^n;\, \mathbb{R}^m))$. This can be subsumed in one formula (91). With the smoothness (19), (24) of $\mathbb{C}$ and S, Eq. (34) yields $\sigma \in W$. With the smoothness (22) of ϱ and (24) of S, Eq. (35) yields $\mu \in W$. This proves (92). □

Remark 3. We can use the embedding

$$W^{s,p}(\mathbb{R}^n) \hookrightarrow C^{r,1-\frac{n}{p}}(\mathbb{R}^n)$$

with $ps > n$, $r + \alpha = s - n/p$ for $\alpha \in (0,1)$, see Brezis (2011, Theorem 9.12), setting $p = 2$.

In $n = 3$ dimensions we obtain the continuity of the embeddings

$$H^2(\mathbb{R}^3) \hookrightarrow C^{0,\frac{1}{2}}(\mathbb{R}^3), \quad H^3(\mathbb{R}^3) \hookrightarrow C^{1,\frac{1}{2}}(\mathbb{R}^3), \quad H^4(\mathbb{R}^3) \hookrightarrow C^{2,\frac{1}{2}}(\mathbb{R}^3). \tag{93}$$

For $\mu_0 \in H^2(\mathbb{R}^3; \mathbb{R}^3)$, $u_0 \in H^3(\mathbb{R}^3; \mathbb{R}^3)$, (92) and (93) yield

$$u \in C^0([0,T];\; C^{1,1/2}(\mathbb{R}^3;\; \mathbb{R}^3)) \cap C^1([0,T];\; C^{0,1/2}(\mathbb{R}^3;\; \mathbb{R}^3)), \tag{94}$$

$$\mu,\, \sigma \in C^0([0,T];\; C^{0,1/2}(\mathbb{R}^3;\; \mathbb{R}^3)) \cap C^1([0,T];\; H^1(\mathbb{R}^3;\; \mathbb{R}^3)), \tag{95}$$

while for $\mu_0 \in H^3(\mathbb{R}^3; \mathbb{R}^3)$, $u_0 \in H^4(\mathbb{R}^3; \mathbb{R}^3)$ we even obtain

$$u \in C^0([0,T]; C^{2,1/2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T]; C^{1,1/2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^0([0,T]; C^{0,1/2}(\mathbb{R}^3;\mathbb{R}^3)), \tag{96}$$

$$\mu,\, \sigma \in C^0([0,T]; C^{1,1/2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T]; C^{0,1/2}(\mathbb{R}^3;\mathbb{R}^3)) \tag{97}$$

for the unique solution vector (μ, σ, u) of (33)–(37).

4.2. Existence theory in a bounded domain

Now we turn to the boundary value problem (63)–(65) for a bounded domain $\Omega \subset \mathbb{R}^n$.

Corollary 2 (*Classical Dirichlet Boundary Conditions*)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume*

$$\mathrm{supp}(u_0),\ \mathrm{supp}(\mu_0),\ \mathrm{supp}(\bar{u}_i(t)),\ \mathrm{supp}(\partial_j \bar{u}_i(t)) \subset \Omega \qquad \text{for } t \in [0,T],\ i,j \in I. \tag{98}$$

Let the assumptions (A0)–(A2) and (26) hold and let (A3) be satisfied for an integer $s \geq 1$. Then, Corollary 1 remains true, i.e. there exists a unique solution (μ, σ, u) to (33)–(38) and (90)–(92) hold for a bounded domain Ω. In $n = 3$ dimensions, if Ω has Lipschitz boundary, for $\mu_0 \in H^2(\Omega; \mathbb{R}^3)$, $u_0 \in H^3(\Omega; \mathbb{R}^3)$, it holds

$$u \in C^0([0,T];\; C^{1,1/2}(\Omega;\; \mathbb{R}^3)) \cap C^1([0,T];\; C^{0,1/2}(\Omega;\; \mathbb{R}^3)), \tag{99}$$

$$\mu,\, \sigma \in C^0([0,T];\; C^{0,1/2}(\Omega;\; \mathbb{R}^3)), \quad \mu \in C^1([0,T];\; H^1(\Omega;\; \mathbb{R}^3)) \tag{100}$$

for the unique solution vector (μ, σ, u) of (33)–(38).

Proof. The condition (98) implies $\mathrm{supp}(\partial_t \bar{u})$, $\mathrm{supp}(\partial_{tt} \bar{u}) \subset \Omega$ and $\mathrm{supp}(\partial_j u_0) \subset \Omega$. So, by (51), $\mathrm{supp}((\varrho e_i)(t)) \subset \Omega$ for $t \in [0,T]$ and $i \in \{1,2,3\}$. Using (69), (88) and (98), we find

$$\mathrm{supp}(v_0),\ \mathrm{supp}(w(t)) \subset \Omega \qquad \text{for } t \in [0,T].$$

With Theorem 1, this ensures $\mathrm{supp}\,(v(t)) \subset \Omega$ for $t \in [0,T]$. If $\partial\Omega$ is Lipschitz continuous, there is a compact embedding $H^s(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega})$ for $s > 3/2$, see Evans (2010, Theorem 6 p.270), leading to (99), (100). □

Note that (98) implies $\bar{u} = 0$ on $\partial\Omega$, i.e. classical Dirichlet boundary conditions.

Now we discuss general, possibly non-regular Dirichlet boundary conditions. The following analysis is based on the methods developed in Ohno et al. (1995) and Rauch (1985).

Definition 1. Let $\nu(x) = (\nu_1, \ldots, \nu_n)(x)$ be the unit outer normal to Ω in $x \in \Gamma = \partial\Omega$. The *boundary matrix* is given by

$$A_{\nu(x)} := \sum_{k=1}^{n} \nu_k(x) A_k(x,t). \tag{101}$$

If A_ν is invertible everywhere on Γ, the boundary is called *non-characteristic*. If A_ν is not invertible but has constant rank on Γ, the boundary is called *characteristic of constant multiplicity*.

A particular difficulty for the theory of boundary value problems is the possible loss of derivatives in normal direction. For that reason we need to introduce new spaces.

Let $(\mathcal{O}_i, \chi_i)_{1 \leq i \leq l}$ be a partition of unity of Γ. For some $\delta > 0$, let

$$\Omega_\delta := \{x \in \Omega \mid \mathrm{dist}(x, \Gamma) > \delta\}$$

and let χ_0 be a smooth function with $\chi_0 = 1$ in Ω_δ and $\chi_0 = 0$ in a neighborhood of Γ. Assume $\sum_{i=0}^{l} \chi_i^2 = 1$ in $\overline{\Omega}$. Introduce local coordinates and consider a family of diffeomorphisms $(\tau_i)_{1 \leq i \leq l}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that Γ corresponds to $x_1 = 0$ and Ω corresponds to

$$B_1^+ := \{x \in \mathbb{R}^n \mid |x| < 1,\ x_1 > 0\}.$$

We recall that $\Lambda \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ is a *tangential vector field* if

$$\langle \Lambda(x), \nu(x) \rangle = 0 \qquad \text{for all } x \in \Gamma. \tag{102}$$

The following three definitions are taken from Ohno et al. (1995).

Definition 2. Let $s \geq 0$ be an integer. We introduce $H^s_*(\Omega)$ as the set of functions $f \in L^2(\Omega; \mathbb{R})$ with the following property:
Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_j$ be tangential vector fields and $\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_k$ be non-tangential vector fields. Then

$$\Lambda_1 \Lambda_2 \ldots \Lambda_j \Lambda'_1 \Lambda'_2 \ldots \Lambda'_k f \in L^2(\Omega) \qquad \text{for } j + 2k \leq s. \tag{103}$$

The norm in $H^s_*(\Omega)$ is given by

$$\|f\|^2_{H^s_*(\Omega)} := \|\chi_0 f\|^2_{H^s(\Omega)} + \sum_{i=1}^{m} \sum_{|\alpha|+2k \leq s} \|\partial^\alpha_{\tan} \partial_1^k f^{(i)}\|^2_{L^2(B_1^+)}, \tag{104}$$

where $f^{(i)} := (\chi_i f) \circ \tau_i^{-1}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and

$$\partial^\alpha_{\tan} := (x_1 \partial_1)^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}.$$

The key to understanding this definition is the observation that in local coordinates for any point in a neighborhood of Γ, $x_1\partial_1, \partial_2, \dots, \partial_n$ span the tangential vector fields. The normal vector field ∂_ν corresponds to $-\partial_1$ in local coordinates. The space $H^1_*(\Omega)$ is identical to $H^1_{\tan}(\Omega)$ introduced in Rauch (1985).

Definition 3. Let $s \ge 0$ be an integer. We introduce $H^s_{**}(\Omega)$ as the set of all functions $f \in L^2(\Omega;\mathbb{R})$ with the following property:
Let $\Lambda_1, \Lambda_2, \dots, \Lambda_j$ be tangential vector fields and let $\Lambda'_1, \Lambda'_2, \dots, \Lambda'_k$ be non-tangential vector fields. Then

$$\Lambda_1\Lambda_2\cdots\Lambda_j\Lambda'_1\Lambda'_2\cdots\Lambda'_k f \in L^2(\Omega) \qquad \text{for } j+2k \le s+1 \text{ and } j+k \le m. \tag{105}$$

The norm on $H^s_{**}(\Omega)$ is given by

$$\|f\|^2_{H^s_{**}(\Omega)} := \|\chi_0 f\|^2_{H^s(\Omega)} + \sum_{i=1}^{l} \sum_{\substack{|\alpha|+2k\le s+1\\ |\alpha|+k\le s}} \|\partial^\alpha_{\tan}\partial_1^k f^{(i)}\|^2_{L^2(B_1^+)}. \tag{106}$$

In (104), (106), different choices of $(\mathcal{O}_i,\chi_i)_{0\le i\le l}$ and $(\tau_i)_{1\le i\le l}$ lead to equivalent norms. For any $s \ge 0$, there is a continuous embedding $H^s(\Omega) \hookrightarrow H^s_{**}(\Omega) \hookrightarrow H^s_*(\Omega)$. Therefore, $H^s_*(\Omega)$ and $H^s_{**}(\Omega)$ may be regarded as subspaces of $H^s(\Omega)$.

Definition 4. For an integer $s \ge 0$, let $X^s([0,T];\Omega)$ be the space of functions f with

$$\partial_t^r f \in C^0([0,T];\, H^{m-r}(\Omega)), \qquad 0 \le r \le s.$$

Here, $\partial_t^r f$, $0 \le r \le s$ are the derivatives of f in the sense of distributions. The space $X^s([0,T];\Omega)$ is a Banach space with the norm

$$|||f|||_{X^s([0,T];\Omega)} := \max_{t\in[0,T]} |||f(t)|||_s, \tag{107}$$

$$|||f(t)|||_s^2 := \sum_{r=0}^{s} \|\partial_t^r f(t)\|^2_{H^{s-r}(\Omega)}. \tag{108}$$

Analogously, let $X^s_*([0,T];\Omega)$ be the space of functions f with

$$\partial_t^r f \in C^0([0,T];\, H^{s-r}_*(\Omega)), \qquad 0 \le r \le s.$$

The space $X^s_*([0,T];\Omega)$ is a Banach space with the norm

$$|||f|||^2_{X^s_*([0,T];\Omega)} := \max_{t\in[0,T]} |||f(t)|||_{s,*}, \tag{109}$$

$$|||f(t)|||^2_{s,*} := \sum_{r=0}^{s} \|\partial_t^r f(t)\|^2_{H^{s-r}_*(\Omega)}. \tag{110}$$

By $W^s_*(0,T;\Omega)$ we denote the space of functions f such that

$$\partial_t^r f \in L^2(0,T;\, H^{s-r}_*(\Omega)), \qquad 0 \le r \le s.$$

For vector-valued functions $f = (f_1,\dots,f_m)$ with $f_i \in X^s([0,T];\Omega)$ for $1 \le i \le m$ we write $f \in X^s([0,T];\Omega)^m$ and $W^s_*(0,T;\Omega)^m$ is defined analogously.

The following theorem is a reformulation of Ohno et al. (1995, Theorem 2.1) which also investigates certain quasi-linear problems where $A_0, A_1, \dots, A_n$ may depend on a further function. Theorem 2.1 in Ohno et al. (1995) is an improved version of Theorem 10 in Rauch (1985) which requires slightly stronger assumptions, only proves tangential regularity, and does not prove uniqueness of the solution. Note that several assumptions of Theorem 10 in Rauch (1985) are not formulated explicitly, but are mentioned elsewhere in the article.

Theorem 2. *Let $s \ge 1$ be an integer. Then an initial boundary value problem (63)–(65) has a unique solution $v \in X^s_*([0,T];\Omega)^m$ provided the following conditions are satisfied:*

(i) *$\Omega \subset \mathbb{R}^n$ is a bounded open set with boundary Γ of class C^∞.*
(ii) *$M(x)$ is a real matrix-valued function with $M \in \mathbb{C}^\infty(\Gamma;\mathbb{R}^{m\times m})$.*
(iii) *$A_k(x,t) \in \mathbb{R}^{m\times m}$, $0 \le k \le n$ are real symmetric matrices for every $(x,t) \in \overline{\Omega}\times[0,T]$. The matrix $A_0(x,t)$ is positive definite for $(x,t) \in \overline{\Omega}\times[0,T]$.*
(iv) *The dimension of $\mathcal{N}(x) := \ker A_{\nu(x)}$ and the dimension of $\ker M(x)$ are constant on each component of Γ and it holds $0 < \dim \mathcal{N}(x) < m$.*
(v) *$\ker M(x)$ is a maximal non-negative subspace of $A_{\nu(x)}$ for $x \in \Gamma$.*
(vi) *It holds $w \in W^s_*(0,T;\Omega)^m$, $\partial_t^r w(0) \in H^{s-1-r}(\Omega;\mathbb{R}^m)$ for $0 \le r \le s-1$ and the initial values fulfill $v_0 \in H^s(\Omega;\mathbb{R}^m)$.*
(vii) *The data w, v_0 fulfill the compatibility conditions of order $s-1$ for the initial boundary value problem (63)–(65).*

Then the solution v obeys for positive constants c_1, c_2 the estimate

$$|||v(t)|||_{s,*} \le c_1\Big(\|v_0\|_{H^s(\Omega;\mathbb{R}^m)} + |||w(0)|||_{s-1}\Big)e^{c_2 t} + c_2\int_0^t e^{c_2(t-\tau)}|||w(\tau)|||_{s,*}\,d\tau, \quad t\in[0,T]. \tag{111}$$

Remark 4. The condition (v) in Theorem 2 means that

$$\langle A_\nu(x,t)z, z\rangle \ge 0 \qquad \text{for all } (x,t)\in\Gamma\times[0,T],\ z\in\ker M(x) \tag{112}$$

and $\ker M$ as a subspace of $\mathbb{C}^m$ cannot be enlarged while (112) remains valid.

The compatibility conditions on w and v_0 up to order $s-1$ are canonical as in (31). One takes spatial derivatives up to order $s-1$ of the system, resolves $\partial_t^p v$ for $0 \le p \le s-1$ and evaluates the result at $t = 0$. In detail, the procedure is as follows, cf. Ohno et al. (1995, p.169). For $p = 0$, set $v_{0,0} := v_0$. For $p = 1,2,\dots,s-1$, set iteratively

$$v_{0,p} := \sum_{i=0}^{p-1}\binom{p-1}{i} G_i(0)v_{0,p-1-i} + \partial_t^{p-1}(A_0^{-1}w)(0) \qquad \text{in } \Omega,$$

where

$$G_0(t) := -\sum_{k=1}^{n} A_0^{-1}A_k\partial_k - A_0^{-1}B,$$

$$G_i(t) := -\sum_{k=1}^{n} \partial_t^i(A_0^{-1}A_k)\partial_k - \partial_t^i(A_0^{-1}B), \qquad i \ge 1.$$

Then, the *compatibility conditions* up to order $s-1$ are

$$Mv_{0,p} = 0 \qquad \text{on } \Gamma \text{ for } 0 \le p \le s-1. \tag{113}$$

Corollary 3. *Let $n = 3$ and $\Omega \subset \mathbb{R}^3$ be a bounded open set with boundary Γ of class C^∞. Assume (26) and let (A0)–(A4) be satisfied for an integer $s \ge 1$. Let u_0, $\bar{u}$ satisfy the compatibility conditions up to order $s+1$. Let $\mathbb{C}_\nu$ be defined by (61) and assume that*

$$\dim\ker(\mathbb{C}_\nu) = \text{const} > 0 \qquad \text{on every component of } \Gamma. \tag{114}$$

Then there exists a unique solution (μ,σ,u) of the Willis system (33)–(37) which satisfies

$$u \in \bigcap_{r=0}^{s+1} C^r([0,T];\, H^{s+1-r}_*(\Omega;\mathbb{R}^n)), \tag{115}$$

$$\mu,\sigma \in \bigcap_{r=0}^{s} C^r([0,T];\, H^{s-r}_*(\Omega;\mathbb{R}^n)). \tag{116}$$

Proof. We need to validate (i)–(vii) of Theorem 2 for $n = 3$. Clearly (i) holds by assumption and (ii) follows from (A0) and (69). Condition (iii) follows as in Corollary 1 from Lemma 1, the symmetry (20) and the positivity condition (23) on ϱ.

The condition $v_0 \in H^s(\Omega;\mathbb{R}^m)$ in (vi) follows from (A0)–(A3). By (29) and (51) the condition on $\partial_t^r w(0)$ in (vi) is ensured. Finally, due to (30), the remaining condition $w \in W^s_*(0,T;\Omega)^m$ from (vi) is also satisfied.

Writing $\mathbb{C}_k^\nu := \nu_1\mathbb{C}_k^1 + \nu_2\mathbb{C}_k^2 + \nu_3\mathbb{C}_k^3$ for $k = 1,2,3$, due to (67) and (101), we have

$$A_\nu = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{C}_1^\nu & 0 \\ 0 & 0 & 0 & -\mathbb{C}_2^\nu & 0 \\ 0 & 0 & 0 & -\mathbb{C}_3^\nu & 0 \\ -\mathbb{C}_1^\nu & -\mathbb{C}_2^\nu & -\mathbb{C}_3^\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & (\nu_1+\nu_2+\nu_3)\mathbb{1} \end{pmatrix} \in \mathbb{R}^{15\times 15}. \tag{117}$$

Let $z := (z^1, z^2, \dots, z^5)^T \in \mathbb{R}^{15}$ be a vector consisting of blocks $z^k \in \mathbb{R}^3$ for $k = 1,\dots,5$. Evaluating $A_\nu z = 0$ leads to $z^4 = z^5 = 0$ and

$$\mathbb{C}_1^\nu z^1 + \mathbb{C}_2^\nu z^2 + \mathbb{C}_3^\nu z^3 = 0 \in \mathbb{R}^3$$

which is equivalent to $z^4 = z^5 = 0$ and

$$(z^1, z^2, z^3)^T \in \ker(\mathbb{C}_\nu) \tag{118}$$

with $\mathbb{C}_\nu \in \mathbb{R}^{9\times 9}$ defined in (61). So we obtain

$$\ker A_{\nu(x)} = \Big\{ z = (z^1,\dots,z^5)^T \in \mathbb{R}^{15} \;\Big|\; z^4 = z^5 = 0,\ (z^1,z^2,z^3)^T \in \ker(\mathbb{C}_{\nu(x)}) \Big\}, \quad x \in \Gamma. \tag{119}$$

This demonstrates that $0 < \dim\ker A_{\nu(x)} < m$ for $x \in \Gamma$. Due to (69), we find

$$\ker M(x) = \Big\{ z = (z^1,\dots,z^5)^T \in \mathbb{R}^{15} \;\Big|\; z^5 = 0,\ (z^1,z^2,z^3)^T \in \ker(\mathbb{C}_{\nu(x)}) \Big\}, \quad x \in \Gamma. \tag{120}$$

With Assumption (114), this implies (iv) of Theorem 2.

For a vector $z = (z^1,\dots,z^5)^T \in \ker(M(x))$, the condition (112) becomes with $\nu = \nu(x)$

$$\begin{aligned}\langle A_\nu z, z\rangle = & -\langle \mathbb{C}_1^\nu z^4, z^1\rangle - \langle \mathbb{C}_2^\nu z^4, z^2\rangle - \langle \mathbb{C}_3^\nu z^4, z^3\rangle \\ & - \langle \mathbb{C}_1^\nu z^1, z^4\rangle - \langle \mathbb{C}_2^\nu z^2, z^4\rangle - \langle \mathbb{C}_3^\nu z^3, z^4\rangle \;\geq\; 0.\end{aligned}$$

Due to the symmetry of $\mathbb{C}_k^\nu$, this is equivalent to

$$\langle \mathbb{C}_1^\nu z^1 + \mathbb{C}_2^\nu z^2 + \mathbb{C}_3^\nu z^3, z^4\rangle \leq 0.$$

Since z^4 for $z = (z^1, z^2, z^3, z^4, z^5) \in \ker(M)$ is arbitrary, we must have $\mathbb{C}_1^\nu z^1 + \mathbb{C}_2^\nu z^2 + \mathbb{C}_3^\nu z^3 = 0$ or

$$\nu_1(\mathbb{C}_1^1 z^1 + \mathbb{C}_2^1 z^2 + \mathbb{C}_3^1 z^3) + \nu_2(\mathbb{C}_1^2 z^1 + \mathbb{C}_2^2 z^2 + \mathbb{C}_3^2 z^3) + \nu_3(\mathbb{C}_1^3 z^1 + \mathbb{C}_2^3 z^2 + \mathbb{C}_3^3 z^3) = 0. \tag{121}$$

This last condition (121) is equivalent to $\mathbb{C}_\nu(z^1, z^2, z^3) = 0$ which holds due to (120). Geometrically, Eq. (121) ensures that all tangential derivatives of $\tilde{u}$ vanish along Γ. In summary, (112) holds with equality and enlarging $\ker(M)$ violates (121). This proves the remaining condition (v) in Theorem 2. □

5. Concluding remarks

In this article, three existence and uniqueness results for (weak) solutions to a system of partial differential equations related to the Willis model have been derived, both for the whole space case and for bounded domains. The investigated system (33)–(35) differs from the original Willis equations in that no explicit form of S, e.g. no convolution expression, is postulated. In addition to the natural symmetry condition (25) on S which guarantees the symmetry of the Cauchy stress tensor σ, a second symmetry condition (26) has to be imposed for the analysis. This condition appears to be necessary mathematically and admits to write (33)–(35) as a linear symmetric hyperbolic system of first order. Combined, (25) and (26) impose strong restrictions on S: the third-order Willis coupling tensor must be totally symmetric. It has to be checked experimentally whether these conditions are satisfied for a real-world material. At this point we have no rigorous physical justification for (26). We refer to Muhlestein et al. (2016) for a discussion of necessary conditions on the quantities of the Willis system in order to have a physically correct model. Since inhomogeneous dynamic linear elasticity constitutes a linear symmetric hyperbolic system, Hughes and Marsden (1977), Sfyris (2024), so should be any homogenized problem based on the former, here the Willis equations. The condition (26) seems to be a welcome novel restriction on the Willis coupling tensor S introduced by this requirement.

In summary, with the additional symmetry condition (26), after establishing suitable structural assumptions, mostly on S and ϱ, the existence and uniqueness of a (weak) solution follows from established existence results for linear symmetric first-order hyperbolic systems. If the initial and boundary data is regular enough, even a unique classical solution is obtained. Finally, in bounded domains, the condition (114) is essential for the existence theory. Its validity depends crucially on material properties.

CRediT authorship contribution statement

Thomas Blesgen: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Patrizio Neff:** Validation, Supervision, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

Brezis, H., 2011. Functional analysis. In: Sobolev Spaces and Partial Differential Equations. Springer, New York.

Evans, L.C., 2010. Partial differential equations, Second ed. In: Graduate studies in Mathematics, vol. 19, American Mathematical Society.

Friedrichs, K.O., 1954. Symmetric hyperbolic linear differential equations. Pure Appl. Math. (Xi'an) 7 (2), 345–392.

Friedrichs, K.O., 1958. Symmetric positive linear differential equations. Commun. Pure Appl. Math. XI 33, 3–418.

Gattin, M., Bochud, N., Rosi, G., Cotterill, P.A., Parnell, W.J., Naili, S., 2025. Reflection and transmission of elastic waves from an immersed willis slab at oblique incidence. Int. J. Solids Struct. 318, 113394.

Hughes, T.J.R., Marsden, J.E., 1977. Classical elastodynamics as a linear symmetric hyperbolic system. J. Elasticity 8 (1), 97–110.

Itin, Y., Reches, S., 2025. Decomposition of third-order constitutive tensors. Math. Mech. Solids 25 (2), 222–249.

John, F., 1982. Partial Differential Equations, Fourth ed. Springer.

Lax, P.D., Phillips, R.S., 1960. Local boundary conditions for dissipative symmetric linear differential operator. Commun. Pure Appl. Math. XIII 42, 7–455.

Lewy, H., 1957. An example of a smooth linear partial differential equation without solution. Ann. Math. 66 (1), 155–158.

Madeo, A., Neff, P., Ghiba, I.-D., Rosi, G., 2016. Reflection and transmission of elastic waves in non-local band-gap meta-materials: A comprehensive study via the relaxed micromorphic model. J. Mech. Phys. Solids 95, 441–479.

Mehrabadi, M.M., Cowin, S.C., 1990. Eigenvectors of linear anisotropic elastic materials. Quaterly J. Mech. Appl. Math. 43 (1), 15–41.

Milton, G.W., Willis, J.R., 2007. On modifications of Newton's second law and linear continuum elastodynamics. Proc.: Math. Phys. Eng. Sci. 463 (855).

Muhlestein, M.B., Sieck, C.F., Alù, A., Haberman, M.R., 2016. Reciprocity, passivity and causality in willis materials. Proc. R. Soc. A 472, 20160604.

Ohno, M., Shizuta, Y., Yanagisawa, T., 1995. The initial boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity. J. Math. Kyoto Univ 35 (2), 143–210.

Rauch, J., 1985. Symmetric positive systems with boundary characteristic of constant multiplicity. Trans. Amer. Math. Soc. 291 (1), 167–187.

Rauch, J.B., Massey, F.J., 1974. Differentiability of solutions to hyperbolic initial–boundary value problems. Trans. Amer. Math. Soc. 189, 303–318.

Rizzi, G., d'Agostino, M.V., Voss, J., Bernardini, D., Neff, P., Madeo, A., 2024. From frequency-dependent models to frequency-independent enriched continua for mechanical metamaterials. Eur. J. Mech. A Solids 106, 105269.

Sfyris, D., 2024. Classical elastodynamics as a linear symmetric hyperbolic system in terms of (u_x, u_t). J. Elasticity 156, 525–548.
Sommerfeld, A., 1964. Mechanik der deformierbaren Medien. Akademische Verlagsgesellschaft, Leipzig.
Vannucci, P., 2018. Anisotropic elasticity. In: Lecture Notes in Applied and Computational Mechanics, vol. 85, Springer.
Willis, J.R., 1981. Variational principles for dynamic problems for inhomogeneous elastic media. Wave Motion 3 (1).
Willis, J.R., 1985. The nonlocal influence of density variations in a composite. Int. J. Solids Struct. 21 (7), 805–817.
Willis, J.R., 2009. Exact effective relations for dynamics of a laminated body. Mech. Mater. 41 (385).
Willis, J.R., 2011. Effective constitutive relations for waves in composites and metamaterials. Proc. R. Soc. A 467, 1865–1879.
Xin, J., Sha, X., 2009. Existence of solutions for hyperbolic system of second order outside a domain. J. Inequal. Appl. (2009), 15. http://dx.doi.org/10.1155/2009/489061.